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콤팩트 집합치 쇼케이적분에 관한 연구

A note on compact set-valued Choquet integrals

*장이채 · **김현미
*Lee Chae Jang · **Hyun Mee Kim

*Dept. of Mathematics and Computer Science, Konkuk University
**Dept. of Mathematics, Kyunghee University

요 약

퍼지측도와 관련된 폐집합치 쇼케이적분에 대해 장에 의해 연구되어 왔음을 알 수 있다. 본 논문에서는 컴팩트 집합치 함수의 쇼케이적분을 생각하고 이와 관련된 성질들을 조사한다. 특히, 구간치 함수 대신에 컴팩트 집합치 함수를 이용하여 컴팩트 집합치 쇼케이적분의 특성들을 조사한다.

Abstract

We note that Jang et al. studied closed set-valued Choquet integrals with respect to fuzzy measures. In this paper, we consider Choquet integrals of compact set-valued functions, and prove some properties of them. In particular, using compact set-valued functions instead of interval valued, we investigate characterization of compact set-valued Choquet integrals.

Key Words : fuzzy measures, Choquet integrals, Compact set-valued functions.

1. Introduction

In this paper, we consider Choquet integrals of compact set-valued functions. We note that Jang et al. [1] studied closed set-valued Choquet integrals with respect to fuzzy measures. In Section 2, we define Choquet integrals of compact set-valued functions and discuss their basis properties. In Section 3, using these definitions and properties, we investigate characterization of compact set-valued Choquet integrals.

2. Preliminaries and definitions

Throughout this paper, we assume that X is a locally compact Hausdorff space, K is the class of continuous functions on X with compact support, Ω is the class of Borel sets, C is the class of compact sets, and O is the class open set. The class of measurable functions is denoted by M and the class of non-negative measurable functions is denoted by M^+.

A non-additive measure on a measurable space (X,Ω) is an extended real-valued function $\mu:\Omega\rightarrow[0,\infty]$ satisfying

(1) $\mu(\emptyset)=0$,

(2) $\mu(A)\le\mu(B)$,

whenever $A,B\in\Omega, A\subset B$. A fuzzy measure μ is said to be lower semi-continuous if for every increasing sequence $\{A_n\}$ of measurable sets, we have

$$\mu(\cup_{n=1}^{\infty}A_n)=\operatorname{Lim}_{n\rightarrow\infty}\mu(A_n).$$

A fuzzy measure μ is said to be upper semi-continuous if for every decreasing sequence $\{A_n\}$ of measurable sets and $\mu(A_1)<\infty$, we have

$$\mu(\cap_{n=1}^{\infty}A_n)=\operatorname{Lim}_{n\rightarrow\infty}\mu(A_n).$$

If μ is lower and upper semi-continuous, it is said to be continuous(see [6]).

We note that if $\mu(X)<\infty$, we can define the conjugate μ^C of μ as in the following

$$\mu^C(A)=\mu(X)-\mu(A^C),$$

where A^C is thee complement of $A\in\Omega$.

Recall that a function $f:X\rightarrow[0,\infty]$ is said to be measurable if $\{x|f(x)>\alpha\}\in\Omega$ for all $\alpha\in(-\infty,\infty)$ ([1,2,3]).

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Definition 2.1 [4,5,7] (1) The Choquet integral of measurable function $f \in M^+$ with respect to a fuzzy measure μ is defined by

$$(C)\int f\,d\mu = \int_0^\infty \mu_f(r)dr,$$

where the integral on the right-hand side is an ordinary one.

(2) A measurable function f is called integrable if the Choquet of f can be defined and its value is finite.

Throughout this paper, R^+ will denote the interval $[0, \infty)$,

$$I(R^+) = \{[a,b] | a, b \in R^+ \text{and } a \le b\}.$$

Then an element in $I(R^+)$ is called an interval number. On the interval number set, we consider as in the following three operations and one order relation;

for each pair $[a,b], [c,d] \in I(R^+)$ and $k \in R^+$,

(1) Addition:

$$[a,b] + [c,d] = [a+c, b+d],$$

(2) Multiplication:

$$[a,b] \cdot [c,d] = [a \cdot c, b \cdot d],$$

(3) Scalar multiplication:

$$k[a,b] = [ka, kb],$$

(4) Order:

$$[a,b] \le [c,d] \text{ if and only if } a \le c \text{ and } b \le d.$$

Then $(I(R^+), d_H)$ is a metric space, where of the Hausdorff metric defined by

$$d_H(A,B) = max\{sup_{x \in A} inf_{y \in B}|x-y|, sup_{y \in B} inf_{x \in A}|x-y|\}$$

for all $A, B \in I(R^+)$.

By the definition of the Hausdorff metric, we have immediately the following proposition.

Proposition 2.2 [4,5,7] For each pair $[a,b]$, $[c,d] \in I(R^+)$,

$$d_H([a,b],[c,d]) = max\{|a-c|, |b-d|\}$$

Let $C(R^+)$ be the class of compact subsets of R^+. Throughout this paper, we consider a compact set-valued function $F: X \to C(R^+) \setminus \{\emptyset\}$ and an interval number-valued function $F: X \to I(R^+) \setminus \{\emptyset\}$. We denote that

$$d_H - \lim_{n \to \infty} A_n = A \text{ if and only if } \lim_{n \to \infty} d_H(A_n, A) = 0,$$

where $A \in I(R^+)$ and $\{A_n\} \subset I(R^+)$.

Definition 2.3 A compact set-valued function F is said to be measurable if for each open set $O \subset R^+$,

$$F^{-1}(O) = \{x | F(x) \cap O\} (\ne \emptyset) \in \Omega.$$

Definition 2.4 Let F be a compact set-valued function. A measurable function $f: X \to R^+$ satisfying

$$f(x) \in F(x), \forall x \in X$$

is called a measurable selection of F.

We say $f: X \to R^+$ is in $L_1(\mu)$ if and only if f is measurable and $(C)\int f\,d\mu < \infty$. We note that "$x \in X$ $\mu - a.e.$" stand for "$x \in X$ μ − almost everywhere". The property $P(x)$ holds for $x \in X$ $\mu - a.e.$ means that there is a measurable set A such that $\mu(A) = 0$ and the property $P(x)$ holds for all $x \in A^C$, where A^C is the complement of A.

Definition 2.5 Let F be a compact set-valued function, let μ be a non-additive measure and $A \in \Omega$.

(1) The Choquet integral of F on A is defined by

$$(C)\int F d\mu = \{(C)\int_A f d\mu | f \in S(F)\},$$

where $S(F)$ is the family of $\mu - a.e.$ measurable selections of F, that is,

$$S(F) = \{f \in M^+ | f(x) \in F(x), x \in X, \mu - a.e\}.$$

(2) A compact set-valued function F is said to be Choquet integrable if $(C)\int F d\mu \ne \emptyset$,

and it is said to be Choquet integrable if $(C)\int F d\mu$ exists and dose not include ∞.

(3) A compact set-valued function F is said to be Choquet integrably bounded if there is a function $g \in M^+$ such that

$$F(x) = sup_{r \in F(x)}|r| \le g(x), \forall x \in X.$$

3. Main results

Throughout this paper, μ is a continuous fuzzy measure. In this section, we consider the following classes of closed set-valued functions and compact set-valued functions;

$\mathbb{T} = \{F|F: X \to C(R^+)$ is measurable closed set- valued function and Choquet integrably bounded $\}$,

$\mathbb{T}_1 = \{F|F: X \to C(R^+)$ is measurable compact set- valued function and Choquet integrably bounded $\}$.

We recall that $cl(A)$ means the closure of a subset A in R^+.

Theorem 3.1 If $F \in \mathbb{T}_1$, then $(C)\int Fd\mu$ is compact.

Proof. We assume that F is measurable compact set-valued function on X and shall show that $(C)\int Fd\mu$ is closed and bounded on $C(R^+)\setminus\{\varnothing\}$.

Let F be closed. Then we have

$$(C)\int_A Fd\mu = \{(C)\int_A fd\mu | f \in S(F)\}$$

is closed. Hence, Let F be Choquet integrably bounded. Then there is $g(x)$ such that $F(x) \le g(x)$. Thus we obtain

$$F(x) \subset [0, g(x)].$$

For every $f \in S(F)$,

$$0 \le (C)\int f d\mu \le (C)\int gd\mu. \tag{3.1}$$

Since $f^- \le |f|$, we have

$$0 \le (C)\int f^- d\mu \le (C)\int |f|d\mu. \tag{3.2}$$

Since $f^+ \le |f| = f^+ + f^-$, we have

$$0 \le (C)\int f^+ d\mu \le (C)\int |f|d\mu. \tag{3.3}$$

By (3.1),(3.2) and (3.3),

$$(C)\int f d\mu \le (C)\int gd\mu$$

for every $f \in S(F)$ such that $f(x) \le g(x)$.

So $(C)\int Fd\mu$ is bounded. Thus $(C)\int Fd\mu$ is compact. □

Corollary 3.2 If $F \in \mathbb{T}$ and $G \in \mathbb{T}_1$, then $(C)\int (F \cap G)d\mu$ is compact.

Definition 3.3 Let $G, H \in \mathbb{T}$. We define $(G \cup H)(x) = G(x) \cup H(x)$ for all $x \in X$.

Theorem 3.4 (1) If $F, G \in \mathbb{T}$, then we have $F \cup G \in \mathbb{T}$ and $F \cap G \in \mathbb{T}$.
(2) If $F, G \in \mathbb{T}_1$, then we have $F \cup G \in \mathbb{T}_1$ and $F \cap G \in \mathbb{T}_1$.

Theorem 3.5 If $F, G \in \mathbb{T}_1$, then we have

$$(C)\int (F \cup G)d\mu = (C)\int Fd\mu \cup (C)\int Gd\mu.$$

Theorem 3.6 If $F, G \in \mathbb{T}_1$, then we have

$$(C)\int (F \cap G)d\mu \subset (C)\int Fd\mu \cap (C)\int Gd\mu.$$

Definition 3.7 Let $G, H \in \mathbb{T}$. Then an addition operation $\uplus$ on $\mathbb{T}$ is defined by

$$G \uplus H = cl(G \cup H).$$

We easily obtain the following property of this operation $\uplus$ on $\mathbb{T}$.

Theorem 3.8 If $F, G \in \mathbb{T}_1$, then $(C)\int (F \uplus G)d\mu = (C)\int Fd\mu \uplus (C)\int Gd\mu.$

Proof. By Theorem 3.1, we have

$$cl\{(C)\int Fd\mu\} = (C)\int Fd\mu.$$

Since F and G are measurable compact set-valued functions on X, by Theorem 3.5 , we have

$$(C)\int (F\uplus G)d\mu=(C)\int d(F\cup G)d\mu$$
$$=(C)[\int (d(F)\cup d(G))d\mu]$$
$$=(C)\int d(F)d\mu \cup (C)\int d(G)d\mu$$
$$=d\,((C)\int Fd\mu \cup (C)\int Gd\mu\,)$$
$$=(C)\int Fd\mu \uplus (C)\int Gd\mu. \qquad \square$$

Clearly, Theorem 3.8 implies the following corollary.

Corollary 3.9 Let $\{F_k \mid, k=1,2,\cdots\}$ be a sequnce of compact set-valued functions in $\mathbb{T}_1$. Then we have

$$(C)\int \uplus_{k=1}^{n}F_k d\mu=\uplus_{k=1}^{n}(C)\int F_k d\mu.$$

We define

$aA=\{ax|x \in A\}$,

$A+B=\{x+y|x \in A, y \in B\}$

where $A,\ B \in C(R^+)$ and $a \in R^+$.

Using these operations, we can define another addition operation on $\mathbb{T}$.

Definition 3.10 Let $G, H \in \mathbb{T}$. Then an addition operation $\oplus$ on $\mathbb{T}$ is defined by

$$G\oplus H=cl(G+H).$$

Definition 3.11 [4,5] Let $F, G \in \mathbb{T}$. We say that F and G are comonotonic, in symbol, $F\sim G$ if and only if

(1) $f^*(x) < f^*(x') \rightarrow g^*(x) \le g^*(x')$ for all $x,\ x' \in X$, and

(2) $f_*(x) < f_*(x') \rightarrow g_*(x) \le g_*(x')$ for all $x,\ x' \in X$, where $f^*(x)=sup\{r|r \in F(x)\}$,

$f_*(x)=inf\{r|r \in F(x)\}$,

$g^*(x)=sup\{r|r \in G(x)\}$,

and $g_*(x)=inf\{r|r \in G(x)\}$.

We recall that every convex compact set in R is a closed bounded interval set and a closed bounded interval means interval number. Thus we can see that every interval-valued function is an element of $\mathbb{T}_1$. Using these facts, in the papers([6,7]), we proved the following important theorem.

Theorem 3.12 ([6,7]) Let F, G be interval-valued Choquet integrably bounded functions. If $F\sim G$, then we have

$$(C)\int (F+G)d\mu$$
$$=(C)\int Fd\mu+(C)\int Gd\mu.$$

Thus we can prove the following important theorem.

Theorem 3.13 Let F, G be interval-valued Choquet integrably bounded functions. If $F\sim G$, then we have

$$(C)\int (F\oplus G)d\mu$$
$$=(C)\int Fd\mu\oplus(C)\int Gd\mu.$$

Proof. Let F and G be measurable compact set-valued functions on X and $F\sim G$.

By Definition 3.10 and Theorem 3.1, $(C)\int (F\oplus G)d\mu$

$$=(C)\int d(F+G)d\mu=d[(C)\int (F+G)d\mu]$$
$$=d\,[(C)\int Fd\mu+(C)\int Gd\mu]$$
$$=(C)\int Fd\mu\oplus(C)\int Gd\mu. \qquad \square$$

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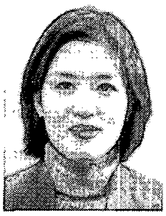
저 자 소 개

장이채(Lee Chae Jang)
1979년 2월 : 경북대 수학과(이학사)
1981년 2월 : 경북대 대학원 수학과(이학석사)
1987년 2월 : 경북대 대학원 수학과(이학박사)
1987년 6월-1998년 6월 : 미국 Cincinnati 대(교환교수)
1987년 3월-현재 : 건국대학교 컴퓨터응용과학부 전산수학전공 교수

관심분야 : 해석학, 퍼지측도와 쇼케이적분, 퍼지이론과 지식공간, 수학교육 등
E-mail : jang@kku.ac.kr



김현미(Hyun Mee Kim)
1987년 2월 : 상명대 수학교육과(이학사)
1991년 8월 : 경희대 대학원 수학과(이학석사)
1999년 8월 : 경희대 대학원 수학과(이학박사)
1995~현재 : 한국외국어대, 건국대, 상명대 강사

관심분야 : 퍼지측도론, 함수해석학 등
E-mail : love4g@smu.ac.kr