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# SOME RESULTS ON CONVERGENCE IN DISTRIBUTION FOR FUZZY RANDOM SETS

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ABSTRACT. In this paper, we first establish some characterization of tightness for a sequence of random elements taking values in the space of normal and upper- semicontinuous fuzzy sets with compact support in $R^p$. As a result, we give some sufficient conditions for a sequence of fuzzy random sets to converge in distribution.

## 1. Introduction

The theory of fuzzy sets introduced by Zadeh [17] has been extensively studied and applied in the area of probability theory in recent years. Since Puri and Ralescu [12] introduced the concept of a fuzzy random variable as a natural generalization of random set, there have been increasing interests in limit theorems for fuzzy stochastic processes because of its usefulness in several applied fields (e.g., [5], [9], [10], [14-16] and so on). In order to obtain limit theorems for fuzzy random variables, many authors have used the theory of Banach space-valued random variables by embedding the space of fuzzy numbers(i.e., normal, upper semicontinuous and convex fuzzy sets with compact support) into a proper Banach space. But this approach may not be valid any more if we exclude the convexity condition.

The theory of convergence in distribution for random sets was given by Artstein [1], Salinetti and Wets [13]. Recently, Joo and Kim [7] gave some characterizations of convergence in distribution and tightness for fuzzy random variables. We distinguish between a fuzzy random variable

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and a fuzzy random set; a fuzzy random variable is a random element taking values in the space of fuzzy numbers, whereas a fuzzy random set is a random element taking values in the space of more general fuzzy sets without convexity.

In this paper, we first establish another characterization of tightness for fuzzy random sets and apply it to obtain some sufficient conditions for a sequence of fuzzy random sets to converge in distribution.

## 2. Preliminaries

Let $\mathcal{K}(R^p)$ denote the family of non-empty compact subsets of the Euclidean space $R^p$. Then $\mathcal{K}(R^p)$ is metrizable by the Hausdorff metric $h$ defined by

$$h(A,B) = \max\{\sup_{a\in A}\inf_{b\in B} |a-b|, \sup_{b\in B}\inf_{a\in A} |a-b|\}.$$

A norm of $A \in \mathcal{K}(R^p)$ is defined by

$$\|A\| = h(A,\{0\}) = \sup_{a\in A}|a|.$$

It is well-known that $\mathcal{K}(R^p)$ is complete and separable with respect to the Hausdorff metric $h$.

In what follows, $clA$ denotes the closure of a set $A \subset R^p$. Let $\mathcal{F}(R^p)$ denote the family of all fuzzy sets $\tilde{u} : R^p \to [0,1]$ with the following properties;

(1) $\tilde{u}$ is normal, i.e., there exists $x \in R^p$ such that $\tilde{u}(x) = 1$;
(2) $\tilde{u}$ is upper semicontinuous;
(3) $\operatorname{supp}\tilde{u} = cl\{x \in R^p : \tilde{u}(x) > 0\}$ is compact.

For a fuzzy set $\tilde{u}$ in $R^p$, the following notations will be used frequently;

$$L_\alpha \tilde{u} = \begin{cases} \{x : \tilde{u}(x) \geq \alpha\}, & \text{if } 0 < \alpha \leq 1 \\ \operatorname{supp} \tilde{u}, & \text{if } \alpha = 0, \end{cases}$$

$$L_{\alpha^+}\tilde{u} = cl\{x \in R^p : \tilde{u}(x) > \alpha\}.$$

Then, it follows immediately that

$$\tilde{u} \in \mathcal{F}(R^p) \text{ if and only if } L_\alpha\tilde{u} \in \mathcal{K}(R^p) \text{ for each } \alpha \in [0,1],$$

and

$$\lim_{\beta \downarrow \alpha} h(L_\beta \tilde{u}, L_{\alpha^+} \tilde{u}) = 0 \text{ for each } \alpha \in [0,1].$$

Now if we define

$$(2.1) \qquad j_{\tilde{u}}(\alpha) \;=\; h(L_\alpha \tilde{u}, L_{\alpha^+} \tilde{u})$$

then it is known that $\{\alpha : j_{\tilde{u}}(\alpha) > \epsilon\}$ is finite for each $\epsilon > 0$ and so $\{\alpha : j_{\tilde{u}}(\alpha) > 0\}$ is countable for all $\tilde{u} \in \mathcal{F}(R^p)$.

The uniform metric on $\mathcal{F}(R^p)$ is usually defined as follows:

$$(2.2) \qquad d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \le \alpha \le 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}).$$

Also, the norm of $\tilde{u}$ is defined as

$$\|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \sup_{x \in L_0 \tilde{u}} |x|,$$

where $\tilde{0} = I_{\{0\}}$ is the indicator function of $\{0\}$.

Then it is well-known that $\mathcal{F}(R^p)$ is complete but is not separable with respect to the metric $d_\infty$. Joo and Kim [6] introduced a metric $d_s$ on $\mathcal{F}(R^p)$ which makes it a separable space as follows:

Definition 2.1. Let $T$ denote the class of strictly increasing, continuous mapping of $[0,1]$ onto itself. For $\tilde{u}, \tilde{v} \in \mathcal{F}(R^p)$, we define

$$d_s(\tilde{u}, \tilde{v}) = \inf\{\epsilon > 0 : \text{there exists a } t \in T \text{ such that } \sup_{0 \le \alpha \le 1} |t(\alpha) - \alpha| \le \epsilon \text{ and } d_\infty(\tilde{u}, t(\tilde{v})) \le \epsilon\},$$

where $t(\tilde{v})$ denotes the composition of $\tilde{v}$ and $t$.

This metric $d_s$ will be called the Hausdorff-Skorohod metric. It is well-known that the metric space $(\mathcal{F}(R^p), d_s)$ is separable and topologically complete.

For $\tilde{u} \in \mathcal{F}(R^p)$ and $0 < \delta < 1$, we define,

$$(2.3) \qquad \tau(\tilde{u}, \delta) = \inf_{\{\alpha_i\}} \max_{1 \le i \le r} h(L_{\alpha_i} \tilde{u}, L_{\alpha_{i-1}^+} \tilde{u}),$$

where the infimum is taken over all partitions $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of $[0,1]$ satisfying $\min_{1 < i < r} (\alpha_i - \alpha_{i-1}) > \delta$.

THEOREM 2.2. *Let $K$ be a subset of the metric space $(\mathcal{F}(R^p), d_s)$. Then $K$ is relatively compact if and only if*

$$\sup_{\tilde{u}\in K} \|\tilde{u}\| < \infty \tag{2.4}$$

*and*

$$\lim_{\delta\to 0}\sup_{\tilde{u}\in K} \tau(\tilde{u}, \delta) = 0. \tag{2.5}$$

## 3. Main results

In this section, we assume that $\mathcal{K}(R^p)$ and $\mathcal{F}(R^p)$ are the metric spaces endowed with the metrics $h$ and $d_s$, respectively. Also, it is assumed that the Cartesian product $\mathcal{K}^k(R^p)$ of $k$-copies of $\mathcal{K}(R^p)$ is endowed with the product topology.

Let $(\Omega, \mathcal{A}, P)$ be a probability space. A set valued function $X : \Omega \to \mathcal{K}(R^p)$ is called a random set if it is measurable. Also, A fuzzy set valued function $\tilde{X} : \Omega \to \mathcal{F}(R^p)$ is called a fuzzy random set if it is measurable. It is well-known that $\tilde{X}$ is a fuzzy random set if and only if for each $\alpha \in [0,1], L_\alpha\tilde{X}$ is a random set (For details, see Butnariu [3], Colubi et al. [4] and Kim [8]).

Since $\mathcal{F}(R^p)$ is separable and topologically complete, we can apply the notions of tightness and convergence in distribution for random elements in a complete separable metric space which can be found in Billingsley [2], Prohorov [11].

DEFINITION 3.1. Let $\tilde{X}_n, \tilde{X}$ be fuzzy random sets.

(1) $\{\tilde{X}_n\}$ is said to be tight if for every $\epsilon > 0$, there exists a compact subset $K$ of $\mathcal{F}(R^p)$ such that

$$P(\tilde{X}_n \notin K) < \epsilon \text{ for all } n.$$

(2) $\{\tilde{X}_n\}$ is said to converge in distribution to $\tilde{X}$ and write $\tilde{X}_n \Rightarrow \tilde{X}$ if for any bounded continuous function $f : \mathcal{F}(R^p) \to R$,

$$\int f(\tilde{X}_n)\, dP \to \int f(\tilde{X})\, dP.$$

Joo and Kim [7] gave some relationships between tightness and convergence in distribution for fuzzy random sets. For easy reference, we enumerate the results of it.

THEOREM 3.2. *$\{\tilde{X}_n\}$ is tight if and only if the following two conditions are satisfied.*

(1) *For each $\eta > 0$, there exists a $\lambda > 0$ such that for all $n$,*

$$P(\|\tilde{X}_n\| > \lambda) \leq \eta.$$

(2) *For each $\epsilon > 0$ and $\eta > 0$, there exists a $\delta \in (0,1)$ such that for all $n$,*

$$P\{\tau(\tilde{X}_n, \delta) \geq \epsilon\} \leq \eta.$$

Now we define, for each fuzzy random set $\tilde{X}$,

$$I_{\tilde{X}} = \{\alpha \in [0,1] : L_\alpha \text{ is continuous almost surely } [P_{\tilde{X}}]\},$$

where $P_{\tilde{X}}$ is the probability distribution of $\tilde{X}$. Then $\alpha \in I_{\tilde{X}}$ if and only if $P\{\omega : j_{\tilde{X}(\omega)}(\alpha) > 0\} = 0$. Also, it is known that $I_{\tilde{X}}$ contains 0 and 1, and $[0,1] \setminus I_{\tilde{X}}$ is at most countable.

For $\alpha_1, \dots, \alpha_k \in [0,1]$, if we define $L_{\alpha_1,\dots,\alpha_k} : \mathcal{F}(R^p) \to \mathcal{K}^k(R^p)$ by

$$L_{\alpha_1,\dots,\alpha_k}(\tilde{u}) = (L_{\alpha_1}\tilde{u}, \dots, L_{\alpha_k}\tilde{u}),$$

then $L_{\alpha_1,\dots,\alpha_k}$ is Borel measurable. If $\alpha_1, \dots, \alpha_k \in I_{\tilde{X}}$, then $L_{\alpha_1,\dots,\alpha_k}$ is continuous a.s. $[P_{\tilde{X}}]$, and so

$$\tilde{X}_n \Rightarrow \tilde{X} \text{ implies } L_{\alpha_1,\dots,\alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1,\dots,\alpha_k}(\tilde{X}).$$

But the converse is not true.

THEOREM 3.3. *If $\{\tilde{X}_n\}$ is tight and if for all $\alpha_1, \dots, \alpha_k \in I_{\tilde{X}}$ with $k$ arbitrary,*

$$L_{\alpha_1,\dots,\alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1,\dots,\alpha_k}(\tilde{X}),$$

*then $\tilde{X}_n \Rightarrow \tilde{X}$.*

Now we wish to establish another characterization of tightness and apply it to convergence in distribution for fuzzy random sets. To this end, we first have to obtain another characterization of compactness on the space $\mathcal{F}(R^p)$. This will be proceeded with replacing (2.5) by other conditions.

For $0 < \delta < 1$ and $\tilde{u} \in \mathcal{F}(R^p)$, we define

$$\rho(\tilde{u}, \delta) = \sup_{\substack{\beta \leq \alpha \leq \gamma \\ \gamma - \beta \leq \delta}} h(L_\alpha \tilde{u}, L_\beta \tilde{u}) \wedge h(L_\alpha \tilde{u}, L_\gamma \tilde{u}). \tag{3.1}$$

LEMMA 3.4. *For $0 < \delta < 1$ and $\tilde{u} \in \mathcal{F}(R^p)$, we have*

$$\rho(\tilde{u}, \delta) \leq \tau(\tilde{u}, \delta).$$

*Proof.* Let $\tau(\tilde{u}, \delta) < \epsilon$. Then there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ satisfying $\alpha_i - \alpha_{i-1} > \delta$ such that

$$h(L_{\alpha_i}\tilde{u}, L_{\alpha_{i-1}^+}\tilde{u}) < \epsilon \text{ for all } i.$$

Let $\alpha, \beta, \gamma \in [0, 1]$ such that $\gamma - \beta \leq \delta$ and $\beta \leq \alpha \leq \gamma$. Then either $\alpha_{i-1} < \beta \leq \gamma \leq \alpha_i$ for some $i$, or $\alpha_{i-1} < \beta \leq \alpha_i < \gamma \leq \alpha_{i+1}$ for some $i$.

If $\alpha_{i-1} < \beta \leq \gamma \leq \alpha_i$, then

$$h(L_\alpha\tilde{u}, L_\beta\tilde{u}) \wedge h(L_\alpha\tilde{u}, L_\gamma\tilde{u}) \leq h(L_{\alpha_i}\tilde{u}, L_{\alpha_{i-1}^+}\tilde{u}) < \epsilon.$$

If $\alpha_{i-1} < \beta \leq \alpha_i < \gamma \leq \alpha_{i+1}$ and $\alpha \leq \alpha_i$, then

$$h(L_\alpha\tilde{u}, L_\beta\tilde{u}) \leq h(L_{\alpha_i}\tilde{u}, L_{\alpha_{i-1}^+}\tilde{u}) < \epsilon.$$

If $\alpha_{i-1} < \beta \leq \alpha_i < \gamma \leq \alpha_{i+1}$ and $\alpha > \alpha_i$, then

$$h(L_\alpha\tilde{u}, L_\gamma\tilde{u}) \leq h(L_{\alpha_{i+1}}\tilde{u}, L_{\alpha_i^+}\tilde{u}) < \epsilon.$$

In any cases, we have $h(L_\alpha\tilde{u}, L_\beta\tilde{u}) \wedge h(L_\alpha\tilde{u}, L_\gamma\tilde{u}) < \epsilon$. This implies $\rho(\tilde{u}, \delta) \leq \epsilon$ and so we complete the proof. □

The following example shows that the inequality in the above Lemma may be strict.

EXAMPLE 1. Let $0 < \lambda < 1$ and define

$$\tilde{u}_\lambda(x) = \begin{cases} 1, & \text{if } x = 0 \\ \lambda, & \text{if } 0 < |x| \leq 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Then,

$$L_\alpha\tilde{u}_\lambda = \begin{cases} \{x : |x| \leq 1\}, & \text{if } 0 \leq \alpha \leq \lambda \\ \{0\}, & \text{if } \lambda < \alpha \leq 1. \end{cases}$$

Thus, $\rho(\tilde{u}_\lambda, \delta) = 0$, but $\tau(\tilde{u}_\lambda, \delta) = 1$ for $\delta \geq \lambda$.

The above example shows that there can be no compactness condition if we replace (2.4) only by a condition in terms of $\rho(\tilde{u}, \delta)$. However, we can formulate a compactness condition by assuming additional conditions for the behavior of $L_\alpha\tilde{u}$ near 0 and 1.

LEMMA 3.5. *If $\beta \leq \alpha_1 \leq \alpha_2 \leq \gamma$ and $\gamma - \beta \leq \delta$, then*

$$h(L_{\alpha_1}\tilde{u}, L_\beta\tilde{u}) \wedge h(L_{\alpha_2}\tilde{u}, L_\gamma\tilde{u}) \leq \rho(\tilde{u}, \delta).$$

*Proof.* If $h(L_{\alpha_1}\tilde{u}, L_\beta\tilde{u}) > \rho(\tilde{u}, \delta)$, then by definition of $\rho(\tilde{u}, \delta)$,

$$h(L_{\alpha_1}\tilde{u}, L_\gamma\tilde{u}) \leq \rho(\tilde{u}, \delta).$$

Thus,

$$h(L_{\alpha_1}\tilde{u}, L_{\alpha_2}\tilde{u}) \leq \rho(\tilde{u}, \delta)$$

and

$$h(L_{\alpha_2}\tilde{u}, L_\gamma\tilde{u}) \leq \rho(\tilde{u}, \delta).$$

Similarly, it can be proved that if $h(L_{\alpha_2}\tilde{u}, L_\gamma\tilde{u}) > \rho(\tilde{u}, \delta)$, then $h(L_{\alpha_1}\tilde{u}, L_\beta\tilde{u}) \leq \rho(\tilde{u}, \delta)$. This completes the proof. □

LEMMA 3.6. *For each $\tilde{u} \in \mathcal{F}(R^p)$, we have*

$$\tau(\tilde{u}, \delta) \leq 3[\rho(\tilde{u}, 2\delta) \vee h(L_{2\delta}\tilde{u}, L_0\tilde{u}) \vee h(L_{1-2\delta}\tilde{u}, L_1\tilde{u})].$$

*Proof.* Let $\epsilon > [\rho(\tilde{u}, 2\delta) \vee h(L_{2\delta}\tilde{u}, L_0\tilde{u}) \vee h(L_{1-2\delta}\tilde{u}, L_1\tilde{u})]$. First we note that if $0 < \alpha_1 < \alpha_2 < 1$ and $j_{\tilde{u}}(\alpha_i) > \epsilon, i = 1, 2$, then

$$\alpha_2 - \alpha_1 \geq 2\delta.$$

For, if $\alpha_2 - \alpha_1 < 2\delta$, then there exist $\beta_i, i = 1, 2$ such that

$$\beta_1 < \alpha_1 < \alpha_2 < \beta_2 \text{ and } \beta_2 - \beta_1 < 2\delta.$$

Then since $j_{\tilde{u}}(\alpha_i) > \epsilon$, we have for each $\gamma \in (\alpha_1, \alpha_2)$,

$$h(L_{\beta_i}\tilde{u}, L_\gamma\tilde{u}) > \epsilon, \ i = 1, 2.$$

But this contradicts to the fact that

$$h(L_{\beta_1}\tilde{u}, L_\gamma\tilde{u}) \wedge h(L_{\beta_2}\tilde{u}, L_\gamma\tilde{u}) \leq \rho(\tilde{u}, 2\delta) < \epsilon.$$

Also, since $\epsilon > h(L_{2\delta}\tilde{u}, L_0\tilde{u}) \vee h(L_{1-2\delta}\tilde{u}, L_1\tilde{u})$, we have

$$j_{\tilde{u}}(\alpha) < \epsilon \text{ for each } \alpha \in [0, 2\delta) \cup (1 - 2\delta, 1].$$

Now let us define $\beta_0 = 0$ and

$$\beta_1 = \begin{cases} 1, & \text{if } \{\alpha : j_{\tilde{u}}(\alpha) > \epsilon\} = \emptyset, \\ \inf\{\alpha : j_{\tilde{u}}(\alpha) > \epsilon\}, & \text{otherwise,} \end{cases}$$

$$\beta_2 = \begin{cases} 1, & \text{if } \{\alpha > \beta_1 : j_{\tilde{u}}(\alpha) > \epsilon\} = \emptyset, \\ \inf\{\alpha > \beta_1 : j_{\tilde{u}}(\alpha) > \epsilon\}, & \text{otherwise.} \end{cases}$$

Repeating this process, we obtain a partition $0 = \beta_0 < \beta_1 < \cdots < \beta_k = 1$ of $[0,1]$ such that $\beta_i - \beta_{i-1} \geq 2\delta$ and

$$\{\alpha : j_{\tilde{u}}(\alpha) > \epsilon\} = \{\beta_1, \dots, \beta_{k-1}\}.$$

If $\beta_j - \beta_{j-1} > 2\delta$ for some $j$, we enlarge $\{\beta_i\}$ by including their midpoint $(\beta_{j-1} + \beta_j)/2$. Continuing in this way, we obtain a new partition $0 = \beta_0 < \beta_1 < \cdots < \beta_r = 1$ of $[0,1]$ such that $\delta < \beta_i - \beta_{i-1} \leq 2\delta$ and any point $\alpha$ satisfying $j_{\tilde{u}}(\alpha) > \epsilon$ is one of the $\{\beta_i : i = 1, \dots, r\}$.

Now it suffices to show that

$$h(L_{\beta_i}\tilde{u}, L_{\beta_{i-1}^+}\tilde{u}) \leq 3\epsilon \text{ for each } i.$$

Let us define

$$\gamma_{i1} = \sup\{\gamma \in (\beta_{i-1}, \beta_i] : h(L_\gamma \tilde{u}, L_{\beta_{i-1}^+}\tilde{u}) \leq \epsilon\},$$
$$\gamma_{i2} = \inf\{\gamma \in (\beta_{i-1}, \beta_i] : h(L_\gamma \tilde{u}, L_{\beta_i}\tilde{u}) \leq \epsilon\}.$$

If $\gamma_{i1} < \gamma_{i2}$, then there exist $\alpha_{i1}, \alpha_{i2}$ satisfying $\gamma_{i1} < \alpha_{i1} < \alpha_{i2} < \gamma_{i2}$ such that

$$h(L_{\alpha_{i1}}\tilde{u}, L_{\beta_{i-1}^+}\tilde{u}) > \epsilon \text{ and } h(L_{\alpha_{i2}}\tilde{u}, L_{\beta_i}\tilde{u}) > \epsilon.$$

But by Lemma 3.5,

$$h(L_{\alpha_{i1}}\tilde{u}, L_{\beta_{i-1}}\tilde{u}) \wedge h(L_{\alpha_{i2}}\tilde{u}, L_{\beta_i}\tilde{u}) \leq \rho(\tilde{u}, 2\delta) < \epsilon,$$

which leads to the contradiction. Thus, $\gamma_{i1} \geq \gamma_{i2}$. Now it is obvious that

$$h(L_{\gamma_{i1}}\tilde{u}, L_{\beta_{i-1}^+}\tilde{u}) \leq \epsilon \text{ and } h(L_{\gamma_{i2}^+}\tilde{u}, L_{\beta_i}\tilde{u}) \leq \epsilon.$$

Also, since $\beta_{i-1} < \gamma_{i2} < \beta_i$, we have by construction of $\{\beta_i\}$,

$$j_{\tilde{u}}(\gamma_{i2}) \leq \epsilon.$$

Thus, it follows from the fact $\gamma_{i1} \geq \gamma_{i2}$ that

$$\begin{aligned} h(L_{\beta_i}\tilde{u}, L_{\beta_{i-1}^+}\tilde{u}) &\leq h(L_{\beta_i}\tilde{u}, L_{\gamma_{i2}^+}\tilde{u}) + j_{\tilde{u}}(\gamma_{i2}) + h(L_{\gamma_{i2}}\tilde{u}, L_{\beta_{i-1}^+}\tilde{u}) \\ &\leq 3\epsilon. \end{aligned}$$

This completes the proof. □

THEOREM 3.7. *Let $K$ be a subset of $(\mathcal{F}(R^p), d_s)$. Then $K$ is relatively compact if and only if* (2.4) *holds and*

$$\lim_{\delta\to 0}\sup_{\tilde{u}\in K}\rho(\tilde{u},\delta)=0, \tag{3.2}$$

$$\lim_{\delta\to 0}\sup_{\tilde{u}\in K}h(L_\delta\tilde{u},L_0\tilde{u})=0, \tag{3.3}$$

$$\lim_{\delta\to 0}\sup_{\tilde{u}\in K}h(L_{1-\delta}\tilde{u},L_1\tilde{u})=0. \tag{3.4}$$

*Proof.* It suffices to show that (2.5) is equivalent to (3.2), (3.3) and (3.4). Let (2.5) hold. Then (3.2) follows from Lemma 3.4. Also, (3.3) and (3.4) follow immediately from the inequality

$$h(L_\delta\tilde{u},L_0\tilde{u})\vee h(L_{1-\delta}\tilde{u},L_1\tilde{u})\le\tau(\tilde{u},\delta).$$

The reverse implication is obvious from Lemma 3.6. □

Now we are in a position to characterize tightness of fuzzy random sets.

THEOREM 3.8. *$\{\tilde{X}_n\}$ is tight if and only if the following two conditions hold:*

(1) *For each $\epsilon>0$, there exists a $\lambda>0$ such that for all $n$,*

$$P\{\omega:\|\tilde{X}_n(\omega)\|\ge\lambda\}\le\epsilon. \tag{3.5}$$

(2) *For each $\epsilon>0$ and $\eta>0$, there exists a $\delta\in(0,1)$ such that for all $n$,*

$$P\{\omega:\rho(\tilde{X}_n(\omega),\delta)\ge\eta\}\le\epsilon \tag{3.6}$$

$$P\{\omega:h(L_0\tilde{X}_n(\omega),L_\delta\tilde{X}_n(\omega))\ge\eta\}\le\epsilon \tag{3.7}$$

$$P\{\omega:h(L_{1-\delta}\tilde{X}_n(\omega),L_1\tilde{X}_n(\omega))\ge\eta\}\le\epsilon. \tag{3.8}$$

*Proof.* (Necessity): Suppose that $\{\tilde{X}_n\}$ is tight. Then for given $\epsilon>0$ and $\eta>0$, there exists a compact subset $K$ of $\mathcal{F}(R^p)$ such that

$$P(\tilde{X}_n\notin K)<\epsilon \text{ for all } n.$$

By Theorem 3.7, we have that

$$K \subset \{\tilde{u} : \|\tilde{u}\| \leq \lambda\} \text{ for sufficiently large } \lambda,$$

and for sufficiently small $\delta$,

$$K \subset \{\tilde{u} : \rho(\tilde{u}, \delta) \vee h(L_\delta \tilde{u}, L_0 \tilde{u}) \vee h(L_{1-\delta}\tilde{u}, L_1 \tilde{u}) < \eta\}.$$

Therefore, (1) and (2) are satisfied.

(Sufficiency): Suppose that (1) and (2) are satisfied. For given $\epsilon > 0$, we can choose $\lambda > 0$ so that

$$P(\|\tilde{X}_n\| > \lambda) \leq \frac{\epsilon}{2} \text{ for all } n.$$

Then for each natural number $m$, we can choose $\delta_m$ so that

$$P(\rho(\tilde{X}_n, \delta_m) \vee h(L_{\delta_m}\tilde{X}_n, L_0\tilde{X}_n) \vee h(L_{1-\delta_m}\tilde{X}_n, L_1\tilde{X}_n) \geq \frac{1}{m}) \leq \frac{\epsilon}{2^{m+1}}$$

for all $n$. Let $A = \{\tilde{u} : \|\tilde{u}\| \leq \lambda\}$ and

$$A_m = \{\tilde{u} : \rho(\tilde{u}, \delta_m) \vee h(L_{\delta_m}\tilde{u}, L_0\tilde{u}) \vee h(L_{1-\delta_m}\tilde{u}, L_1\tilde{u}) < \frac{1}{m}\}.$$

If $K$ is the closure of $A \cap (\cap_{m=1}^{\infty} A_m)$, then $K$ is compact by Theorem 3.7. Thus the tightness of $\{\tilde{X}_n\}$ follows since

$$P(\tilde{X}_n \notin K) \leq P(\tilde{X}_n \notin A) + \sum_{m=1}^{\infty} P(\tilde{X}_n \notin A_m) < \epsilon.$$

□

Since $\mathcal{F}(R^p)$ is separable and topologically complete, a single fuzzy random set is tight. Thus we have the following modified form.

THEOREM 3.9. *$\{\tilde{X}_n\}$ is tight if and only if*

$$\lim_{\lambda \to \infty} \limsup_{n \to \infty} P\{\|\tilde{X}_n\| > \lambda\} = 0 \tag{3.9}$$

*and for each $\eta > 0$,*

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\{\rho(\tilde{X}_n, \delta) \geq \eta\} = 0 \tag{3.10}$$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\{h(L_0\tilde{X}_n, L_\delta\tilde{X}_n) \geq \eta\} = 0 \tag{3.11}$$

$$\lim_{\delta \to 0} \limsup_{n \to \infty} P\{h(L_{1-\delta}\tilde{X}_n, L_1\tilde{X}_n) \geq \eta\} = 0. \tag{3.12}$$

THEOREM 3.10. *Suppose that for each $\epsilon > 0$ and $\eta > 0$, there exist a $\delta \in (0, 1)$ and an integer $n_0$ such that*

$$P\{\omega : \rho(\tilde{X}_n(\omega), \delta) \geq \eta\} \leq \epsilon \quad \text{for } n \geq n_0. \tag{3.13}$$

*If $L_{\alpha_1,\dots,\alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1,\dots,\alpha_k}(\tilde{X})$, whenever the $\alpha_i$ all lie in $I_{\tilde{X}}$ with $k$ arbitrary, then*

$$\tilde{X}_n \Rightarrow \tilde{X}.$$

*Proof.* It suffices to prove that $\{\tilde{X}_n\}$ is tight. Since (3.13) is equivalent to (3.10), we need to check (3.9), (3.11) and (3.12).

(3.9) follows easily from the fact that $\{L_0\tilde{X}_n\}$ is tight since it converges in distribution to $L_0\tilde{X}$.

For (3.11), we note that by right continuity at 0 of $L_\alpha\tilde{X}$,

$$P(h(L_\alpha\tilde{X}, L_0\tilde{X}) \geq \eta) \leq \epsilon$$

for sufficiently small $\alpha$. By hypothesis,

$$(L_0\tilde{X}_n, L_\alpha\tilde{X}_n) \Rightarrow (L_0\tilde{X}, L_\alpha\tilde{X}) \text{ if } \alpha \in I_{\tilde{X}}.$$

By the well-known mapping theorem,

$$h(L_\alpha\tilde{X}_n, L_0\tilde{X}_n) \Rightarrow h(L_\alpha\tilde{X}, L_0\tilde{X}) \text{ if } \alpha \in I_{\tilde{X}}.$$

Hence we have for sufficiently small $\alpha \in I_{\tilde{X}}$,

$$\limsup_{n\to\infty} P(h(L_\alpha\tilde{X}_n, L_0\tilde{X}_n) \geq \eta) \leq P(h(L_\alpha\tilde{X}, L_0\tilde{X}) \geq \eta) \leq \epsilon.$$

Therefore, (3.11) is satisfied. Similarly, (3.12) can be proved by left continuity at 1 of $L_\alpha\tilde{X}$. □

The next example shows that the condition (3.13) cannot be removed.

EXAMPLE 2. For each $n \geq 2$, let

$$\tilde{u}_n(x) = \begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{2} + \frac{1}{n}(1 - |x|), & \text{if } 0 < |x| \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\tilde{u}(x) = \begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{2}, & \text{if } 0 < |x| \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$L_\alpha \tilde{u}_n = \begin{cases} \{x : |x| \le 1\}, & \text{if } 0 \le \alpha \le 1/2 \\ \{x : |x| \le 1 - (\alpha - \frac{1}{2})n\}, & \text{if } 1/2 < \alpha \le 1/2 + 1/n \\ \{0\}, & \text{if } 1/2 + 1/n < \alpha \le 1, \end{cases}$$

and

$$L_\alpha \tilde{u} = \begin{cases} \{x : |x| \le 1\}, & \text{if } 0 \le \alpha \le 1/2 \\ \{0\}, & \text{if } 1/2 < \alpha \le 1. \end{cases}$$

First we note that $\{\tilde{u}_n\}$ does not converge. For, if $\{\tilde{u}_n\}$ converge to $\tilde{v}$ for some $\tilde{v} \in \mathcal{F}(R^p)$, then there exists a sequence of functions $\{t_n\}$ in $T$ such that

$$\lim_{n\to\infty} t_n(\alpha) = \alpha \text{ uniformly in } [0,1]$$

and

$$\lim_{n\to\infty} d_\infty(t_n(\tilde{u}_n), \tilde{v}) = 0.$$

For each $\alpha \in [0,1]$, if we take $\alpha_n = t_n^{-1}(\alpha)$, then

$$\alpha_n \to \alpha \text{ and } L_{\alpha_n} \tilde{u}_n \to L_\alpha \tilde{v}.$$

Then, we should have $\tilde{v} = \tilde{u}$. But for each $t \in T$, $d_\infty(\tilde{u}_n, t(\tilde{u})) \ge 1/2$ and so $d_s(\tilde{u}_n, \tilde{u}) \ge 1/2$. This implies that $\{\tilde{u}_n\}$ does not converge.

Thus if we take $\tilde{X}_n = \tilde{u}_n$ and $\tilde{X} = \tilde{u}$, then

$$\tilde{X}_n \not\Rightarrow \tilde{X}.$$

But if $\alpha \le 1/2$, then

$$L_\alpha \tilde{X}_n = L_\alpha \tilde{X} = \{x : |x| \le 1\}.$$

If $\alpha > 1/2$, then for $1/n < \alpha - 1/2$,

$$L_\alpha \tilde{X}_n = L_\alpha \tilde{X} = \{0\}.$$

Thus

$$L_{\alpha_1,\dots,\alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1,\dots,\alpha_k}(\tilde{X}) \text{ for all } \alpha_1, \dots, \alpha_k \in [0,1].$$

We note that $\{\tilde{X}_n\}$ does not satisfy the condition (3.13). For, by definition of $\rho(\tilde{u}_n, \delta)$, we have

$$\rho(\tilde{u}_n, \delta) = \begin{cases} \delta, & \text{if } \delta < 1/n, \\ 1, & \text{if } \delta \ge 1/n. \end{cases}$$

Thus, for each $\eta > 0, P\{\rho(\tilde{X}_n, \delta) \ge \eta\} = 1$ for large $n$.

THEOREM 3.11. *Suppose that there exist a non-decreasing continuous function $g$ on $[0,1]$ and $a \geq 0, b > 1$ such that for $\alpha \leq \gamma \leq \beta$ , $\epsilon > 0$ and for all large $n$,*

$$(3.14)\quad P\{h(L_\alpha \tilde{X}_n, L_\gamma \tilde{X}_n) \wedge h(L_\gamma \tilde{X}_n, L_\beta \tilde{X}_n) \geq \epsilon\} \leq \frac{1}{\epsilon^a}[g(\beta) - g(\alpha)]^b.$$

*If $L_{\alpha_1,\dots,\alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1,\dots,\alpha_k}(\tilde{X})$, whenever the $\alpha_i$ all lie in $I_{\tilde{X}}$ with $k$ arbitrary, then*

$$\tilde{X}_n \Rightarrow \tilde{X}.$$

To prove this, we need some lemmas. First for a fixed fuzzy random set $\tilde{X}$ and $\alpha, \delta \in [0,1]$ such that $\alpha + \delta \in [0,1]$, we define

$$(3.15)\qquad \xi_m = \max_{0 \leq i \leq m} h(L_\alpha \tilde{X}, L_{\alpha + \frac{i}{m}\delta} \tilde{X}) \wedge h(L_{\alpha + \frac{i}{m}\delta} \tilde{X}, L_{\alpha+\delta} \tilde{X})$$

where $m$ is a positive integer.

LEMMA 3.12. *Suppose that for $\alpha \leq \gamma \leq \beta$ , $\epsilon > 0$,*

$$(3.16)\qquad P\{h(L_\alpha \tilde{X}, L_\gamma \tilde{X}) \wedge h(L_\gamma \tilde{X}, L_\beta \tilde{X}) \geq \epsilon\} \leq \frac{1}{\epsilon^a}[g(\beta) - g(\alpha)]^b$$

*where $a \geq 0, b > 1$ and $g$ is a non-decreasing continuous function on $[0,1]$. Then*

$$(3.17)\qquad P\{\xi_m \geq \epsilon\} \leq \frac{M}{\epsilon^a}[g(\alpha+\delta) - g(\alpha)]^b,$$

*where $M = M_{a,b}$ is a constant depending only on $a$ and $b$.*

*Proof.* We shall prove by induction on $m$. The case $m = 1$ or $m = 2$ is trivial.

Assume that (3.17) holds for all integers smaller than $m$. We can choose an integer $q, 1 \leq q \leq m$ so that

$$(3.18)\qquad g(\alpha + \frac{q-1}{m}\delta) \leq \frac{g(\alpha+\delta) + g(\alpha)}{2} \leq g(\alpha + \frac{q}{m}\delta).$$

Let us define

$$(3.19)\qquad \eta_1 = \max_{0 \leq i \leq q-1} h(L_\alpha \tilde{X}, L_{\alpha + \frac{i}{m}\delta} \tilde{X}) \wedge h(L_{\alpha + \frac{i}{m}\delta} \tilde{X}, L_{\alpha + \frac{q-1}{m}\delta} \tilde{X}),$$

$$(3.20)\qquad \eta_2 = \max_{q\le i\le m} h(L_{\alpha+\frac{q}{m}}\tilde{X}, L_{\alpha+\frac{i}{m}\delta}\tilde{X}) \wedge h(L_{\alpha+\frac{i}{m}\delta}\tilde{X}, L_{\alpha+\delta}\tilde{X}).$$

Then by the induction hypothesis, we have

$$P\{\eta_1 \ge \epsilon\} \le \frac{M}{\epsilon^a}[g(\alpha+\frac{q-1}{m}\delta) - g(\alpha)]^b$$

and

$$P\{\eta_2 \ge \epsilon\} \le \frac{M}{\epsilon^a}[g(\alpha+\delta) - g(\alpha+\frac{q}{m}\delta)]^b.$$

By (3.18), we have

$$(3.21)\qquad P\{\eta_1 \ge \epsilon\} \le \frac{M}{2^b\epsilon^a}[g(\alpha+\delta) - g(\alpha)]^b$$

and

$$(3.22)\qquad P\{\eta_2 \ge \epsilon\} \le \frac{M}{2^b\epsilon^a}[g(\alpha+\delta) - g(\alpha)]^b.$$

Now for $0 \le i \le m$, let

$$\psi(i) = h(L_\alpha\tilde{X}, L_{\alpha+\frac{i}{m}\delta}\tilde{X}) \wedge h(L_{\alpha+\frac{i}{m}\delta}\tilde{X}, L_{\alpha+\delta}\tilde{X})$$

and define

$$\eta_3 = \max\{\psi(q-1), \psi(q)\}.$$

Then by (3.16),

$$(3.23)\qquad P\{\eta_3 \ge \epsilon\} \le \frac{2}{\epsilon^a}[g(\alpha+\delta) - g(\alpha)]^b.$$

Now we show that $\xi_m \le \max(\eta_1, \eta_2) + \eta_3$. If $0 \le i \le q-1$, then

$$\begin{aligned}&h(L_{\alpha+\frac{i}{m}\delta}\tilde{X}, L_{\alpha+\delta}\tilde{X})\\ &\le h(L_{\alpha+\frac{i}{m}\delta}\tilde{X}, L_{\alpha+\frac{q-1}{m}\delta}\tilde{X}) + h(L_{\alpha+\frac{q-1}{m}\delta}\tilde{X}, L_{\alpha+\delta}\tilde{X}),\end{aligned}$$

and so

$$\begin{aligned}&h(L_\alpha\tilde{X}, L_{\alpha+\frac{i}{m}\delta}\tilde{X}) \wedge h(L_{\alpha+\frac{i}{m}\delta}\tilde{X}, L_{\alpha+\delta}\tilde{X})\\ &\le h(L_\alpha\tilde{X}, L_{\alpha+\frac{i}{m}\delta}\tilde{X}) \wedge h(L_{\alpha+\frac{i}{m}\delta}\tilde{X}, L_{\alpha+\frac{q-1}{m}\delta}\tilde{X})\\ &\quad + h(L_\alpha\tilde{X}, L_{\alpha+\frac{q-1}{m}\delta}\tilde{X}) \wedge h(L_{\alpha+\frac{q-1}{m}\delta}\tilde{X}, L_{\alpha+\delta}\tilde{X})\\ &\le \eta_1 + \eta_3.\end{aligned}$$

Similarly, if $q \le i \le m$, then

$$h(L_\alpha \tilde{X}, L_{\alpha+\frac{i}{m}\delta}\tilde{X}) \le h(L_\alpha \tilde{X}, L_{\alpha+\frac{q}{m}\delta}\tilde{X}) + h(L_{\alpha+\frac{q}{m}\delta}\tilde{X}, L_{\alpha+\frac{i}{m}\delta}\tilde{X}),$$

and so

$$\begin{aligned} &h(L_\alpha \tilde{X}, L_{\alpha+\frac{i}{m}\delta}\tilde{X}) \wedge h(L_{\alpha+\frac{i}{m}\delta}\tilde{X}, L_{\alpha+\delta}\tilde{X}) \\ &\le h(L_\alpha \tilde{X}, L_{\alpha+\frac{q}{m}\delta}\tilde{X}) \wedge h(L_{\alpha+\frac{q}{m}\delta}\tilde{X}, L_{\alpha+\delta}\tilde{X}) \\ &\quad + h(L_{\alpha+\frac{q}{m}\delta}\tilde{X}, L_{\alpha+\frac{i}{m}\delta}\tilde{X}) \wedge h(L_{\alpha+\frac{i}{m}\delta}\tilde{X}, L_{\alpha+\delta}\tilde{X}) \\ &\le \eta_3 + \eta_2. \end{aligned}$$

In any cases, we have $\xi_m \le \max(\eta_1, \eta_2) + \eta_3$. Therefore, by (3.21), (3.22) and (3.23), we obtain for $0 < \epsilon_1 < \epsilon$,

$$\begin{aligned} &P\{\xi_m \ge \epsilon\} \\ &\le P\{\eta_1 \ge \epsilon_1\} + P\{\eta_2 \ge \epsilon_1\} + P\{\eta_3 \ge \epsilon - \epsilon_1\} \\ &\le [\frac{2^{1-b}M}{\epsilon_1^a} + \frac{2}{(\epsilon-\epsilon_1)^a}][g(\alpha+\delta) - g(\alpha)]^b. \end{aligned}$$

Note that if $c_0$ and $c_1$ are positive numbers, then a function

$$f(t) = \frac{c_0}{t^a} + \frac{c_1}{(\epsilon - t)^a}, \quad 0 < t < \epsilon$$

has the minimum value $\frac{1}{\epsilon^a}[c_0^{1/(a+1)} + c_1^{1/(a+1)}]^{a+1}$ at $t = \frac{\epsilon c_0^{1/(a+1)}}{c_0^{1/(a+1)} + c_1^{1/(a+1)}}$. Thus, we have by choosing $\epsilon_1$ properly,

$$\begin{aligned} &P\{\xi_m \ge \epsilon\} \\ &\le \frac{1}{\epsilon^a}[(M2^{1-b})^{1/(a+1)} + 2^{1/(a+1)}]^{a+1}[g(\alpha+\delta) - g(\alpha)]^b. \end{aligned}$$

Since $b > 1$, it follows that for sufficiently large $M$,

$$(M2^{1-b})^{1/(a+1)} + 2^{1/(a+1)} \le M^{1/(a+1)}.$$

Therefore, we obtain the desired result. □

LEMMA 3.13. *For fixed* $\tilde{u} \in \mathcal{F}(R^p)$ *and* $\alpha, \delta \in [0,1]$ *such that* $\alpha+\delta \in [0,1]$, *let*

$$\rho(\tilde{u}, [\alpha, \alpha+\delta]) = \sup_{\alpha \le \beta \le \alpha+\delta} h(L_\alpha \tilde{u}, L_\beta \tilde{u}) \wedge h(L_\beta \tilde{u}, L_{\alpha+\delta} \tilde{u}).$$

*Then the followings hold:*

(1) $\rho(\tilde{u}, \delta) = \sup_{0 \le \alpha \le 1-\delta} \rho(\tilde{u}, [\alpha, \alpha+\delta])$.

(2) *For each positive integer* $r$,

$$\rho(\tilde{u}, \frac{1}{2r}) \le \max_{0 \le i \le r-1} \rho(\tilde{u}, [\frac{i}{r}, \frac{i+1}{r}]) \vee \max_{0 \le i \le r-2} \rho(\tilde{u}, [\frac{i}{r} + \frac{1}{2r}, \frac{i+1}{r} + \frac{1}{2r}]).$$

*Proof.* (1) is trivial. To prove (2), let $M$ be the right-hand side. If $\delta = \frac{1}{2r}$, then both $\alpha$ and $\alpha+\delta$ lie in one of $2r-1$ intervals listed in the right-hand side. Thus, $\rho(\tilde{u}, [\alpha, \alpha + \frac{1}{2r}]) \le M$ and so

$$\rho(\tilde{u}, \frac{1}{2r}) = \sup_{0 \le \alpha \le 1-\frac{1}{2r}} \rho(\tilde{u}, [\alpha, \alpha + \frac{1}{2r}]) \le M.$$

This completes the proof. □

*Proof of Theorem 3.11:*. It suffices to prove (3.13). For fixed $\alpha, \delta \in [0,1]$ such that $\alpha + \delta \in [0,1]$, let

$$\xi_m^{(n)} = \max_{0 \le i \le m} h(L_\alpha \tilde{X}_n, L_{\alpha+\frac{i}{m}\delta} \tilde{X}_n) \wedge h(L_{\alpha+\frac{i}{m}\delta} \tilde{X}_n, L_{\alpha+\delta} \tilde{X}_n).$$

Then by Lemma 3.12 , for large $n$,

$$P\{\xi_m^{(n)} \ge \epsilon\} \le \frac{M}{\epsilon^a} [g(\alpha+\delta) - g(\alpha)]^b, \tag{3.24}$$

where $M = M_{a,b}$ is a constant depending only on $a$ and $b$.

Since $L_\alpha \tilde{X}$ is left continuous on $[0,1]$ as a function of $\alpha$, letting $m \to \infty$, (3.24) yields that for large $n$,

$$P\{\rho(\tilde{X}_n, [\alpha, \alpha+\delta]) \ge \epsilon\} \le \frac{M}{\epsilon^a} [g(\alpha+\delta) - g(\alpha)]^b.$$

By Lemma 3.13, it follows that for large $n$,

$$\begin{aligned}
&P\{\rho(\tilde{X}_n, 1/2r) \geq \epsilon\} \\
&\leq \sum_{i=0}^{r-1} P\{\rho(\tilde{X}_n, [\frac{i}{r}, \frac{i+1}{r}]) \geq \epsilon\} \\
&\quad + \sum_{i=0}^{r-2} P\{\rho(\tilde{X}_n, [\frac{i}{r} + \frac{1}{2r}, \frac{i+1}{r} + \frac{1}{2r}]) \geq \epsilon\} \\
&\leq \frac{M}{\epsilon^a} \sum_{i=0}^{r-1} [g(\frac{i+1}{r}) - g(\frac{i}{r})]^b + \frac{M}{\epsilon^a} \sum_{i=0}^{r-2} [g(\frac{i+1}{r} + \frac{1}{2r}) - g(\frac{i}{r} + \frac{1}{2r})]^b \\
&\leq \frac{2M}{\epsilon^a} [g(1) - g(0)] \sup_{|\alpha-\beta| \leq 1/r} |g(\alpha) - g(\beta)|^{b-1}.
\end{aligned}$$

Since $b > 1$ and $g$ is continuous, we have

$$\sup_{|\alpha-\beta| \leq 1/r} |g(\alpha) - g(\beta)|^{b-1} \to 0 \quad \text{as } r \to \infty,$$

which proves the desired result. □

COROLLARY 3.14. *Suppose that there exist a non-decreasing continuous function $g$ on $[0, 1]$ and $a \geq 0, b > 1$ such that for all large $n$,*

$$P\{h(L_\alpha \tilde{X}_n, L_\beta \tilde{X}_n) \geq \epsilon\} \leq \frac{1}{\epsilon^a} [g(\beta) - g(\alpha)]^b \tag{3.25}$$

*for $\alpha \leq \beta, \epsilon > 0$. If $L_{\alpha_1, \dots, \alpha_k}(\tilde{X}_n) \Rightarrow L_{\alpha_1, \dots, \alpha_k}(\tilde{X})$, whenever the $\alpha_i$ all lie in $I_{\tilde{X}}$ with $k$ arbitrary, then*

$$\tilde{X}_n \Rightarrow \tilde{X}.$$

*Proof.* It follows from Theorem 3.11 and the inequality that for $\alpha \leq \gamma \leq \beta$,

$$h(L_\alpha \tilde{X}_n, L_\beta \tilde{X}_n) \geq h(L_\alpha \tilde{X}_n, L_\gamma \tilde{X}_n) \wedge h(L_\gamma \tilde{X}_n, L_\beta \tilde{X}_n).$$

□

## References


[1] Z. Artstein, *Weak convergence of set-valued functions and control*, SIAM J. Control Optim. **13** (1975), 865–878.

[2] P. Billingsley, *Convergence of probability measures, Second Edition*, Wiley, New York, 1999.

[3] D. Butnariu, *Measurability concepts for fuzzy mappings*, Fuzzy Sets and Systems **31** (1989), 77–82.

[4] A. Colubi, J. S. Domingeuz-Menchero, M. Lopez-Diaz and D. Ralescu, *A $D_E$ [0, 1] representation of random upper semicontinuous functions*, Proc. Amer. Math. Soc. **130** (2002), 3237–3242.

[5] Y. Feng, *Mean-square integral and differential of fuzzy stochastic processes*, Fuzzy Sets and Systems **102** (1999), 271–280.

[6] S. Y. Joo and Y. K. Kim, *Topological properties on the space of fuzzy sets*, J. Math. Anal. Appl. **246** (2000), 576–590.

[7] ———, *Weak convergence and tightness for fuzzy random variables (submitted).*

[8] Y. K. Kim, *Measurability for fuzzy valued functions*, Fuzzy Sets and Systems **129** (2002), 105–109.

[9] S. Li and Y. Ogura, *Fuzzy random variables, conditional expectations and fuzzy martingales*, J. Fuzzy Math. **4** (1996), 905–927.

[10] S. Li, Y. Ogura and H. T. Nguyen, *Gaussian processes and martingales for fuzzy-valued variables with continuous parameter*, Inform. Sci. **133** (2001), 7–21.

[11] Yu. V. Prokhorov, *Convergence of random processes and limit theorems in probability theory*, Theory Probab. Appl. **1** (1956), 157–214.

[12] M. L. Puri and D. A. Ralescu, *Fuzzy random variables*, J. Math. Anal. Appl. **114** (1986), 402–422.

[13] G. Salinetti and R. Wets, *On the convergence in distribution of measurable multifunctions(random sets), normal integrands, stochastic processes and stochastic infima*, Math. Oper. Res. **11** (1986), 385–419.

[14] G. Wang and Y. Zhang, *The theory of fuzzy stochastic processes*, Fuzzy Sets and Systems **51** (1992), 161–178.

[15] H. C. Wu, *The central limit theorems for fuzzy random variables*, Inform. Sci. **120** (1999), 239–256.

[16] Y. Yoshida, M. Yasuda, J. Nakagami and M. Kurano,, *Optimal stopping problem in a stochastic and fuzzy system*, J. Math. Anal. Appl. **246** (2000), 135–149.

[17] L. A. Zadeh, *Fuzzy sets*, Inform. and Control **8** (1965), 338–353.



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