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REMARKS ON THE KKM PROPERTY FOR OPEN-VALUED MULTIMAPS ON GENERALIZED CONVEX SPACES

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ABSTRACT. Let $(X, D; \Gamma)$ be a G-convex space and Y a Hausdorff space. Then $\mathfrak{A}_c^\kappa(X, Y) \subset \mathfrak{KO}(X, Y)$, where $\mathfrak{A}_c^\kappa$ is an admissible class (due to Park) and $\mathfrak{KO}$ denotes the class of multimaps having the KKM property for open-valued multimaps. This new result is used to obtain a KKM type theorem, matching theorems, a fixed point theorem, and a coincidence theorem.

1. Introduction

The KKM theory of generalized convex spaces (or G-convex spaces) has been developed mainly by the second author and followed by a number of other authors; for the literature, see the references at the end of the present paper.

In this paper, our main aims are to improve one of our earlier results [11, Theorem 11] and to obtain some of its applications. In fact, let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, and $G : D \multimap Y$ an open-valued multimap. If an admissible map $F \in \mathfrak{A}_c^\kappa(X, Y)$ (due to Park [2-4]) satisfies $F(\Gamma_A) \subset G(A)$ for all finite subset A of D, then the family $\{G(z)\}_{z \in D}$ of values of G has the finite intersection property. In [11], this was proved under the restriction that Y is T_1 and regular.

Section 2 deals with preliminaries taken from a recent work of the second author [9]. In Section 3, we prove our main result and apply it to obtain a generalized form of a KKM theorem, matching theorems for closed [resp. open] valued multimaps, a fixed point theorem, and a coincidence theorem.

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2. The KKM theorem for G-convex spaces

A *multimap* (or simply, a *map*) $F : X \multimap Y$ is a function from a set X into the power set of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^{-}(y) := \{x \in X \,|\, y \in F(x)\}$ for $y \in Y$. For $A \subset X$, let $F(A) := \bigcup\{F(x) \,|\, x \in A\}$. Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context.

For topological spaces X and Y, a multimap $F : X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) [resp. *lower semicontinuous* (l.s.c.)] if for each closed [resp. open] set $B \subset Y$, $F^{-}(B) := \{x \in X | \, F(x) \cap B \neq \emptyset\}$ is closed [resp. open] in X.

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with its cardinal $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma_A := \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J := \Gamma(J)$. In certain cases, we may assume $\phi_A(\Delta_n) = \Gamma_A$.

Note that Δ_n is an n-simplex with vertices $v_0, v_1, \dots, v_n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \dots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \dots, a_{i_k}\} \subset A$, then $\Delta_J = \text{co}\{v_{i_0}, v_{i_1}, \dots, v_{i_k}\}$.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$.

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$.

Examples of G-convex spaces can be found in [5, 6, 8] and references therein.

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if

$$\Gamma_N \subset F(N) \qquad \text{for each } N \in \langle D \rangle.$$

The following is a KKM theorem for G-convex spaces [5, 6]:

THEOREM 2.1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a map such that*

(1) *F has closed [resp. open] values; and*

(2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property.

Further, if

(3) *$\bigcap_{z \in M} \overline{F(z)}$ is compact for some $M \in \langle D \rangle$,*

then we have

$$\bigcap_{z \in D} \overline{F(z)} \neq \emptyset.$$

REMARK. There have appeared several variations of Theorem 2.1; see [6, 12].

Let $(X, D; \Gamma)$ be a G-convex space and Y a topological space. A multimap $F : X \multimap Y$ is said to have *the KKM property* if, for any map $G : D \multimap Y$ with closed [resp. open] values satisfying

$$F(\Gamma_A) \subset G(A) \qquad \text{for all } \; A \in \langle D \rangle,$$

the family $\{G(z)\}_{z \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(X, Y) := \{F : X \multimap Y \mid F \; \text{ has the KKM property}\}.$$

Some authors use the notation $KKM(X, Y)$. Note that $1_X \in \mathfrak{K}(X, X)$ by Theorem 2.1. Moreover, if $F : X \to Y$ is a continuous single-valued map or if $F : X \multimap Y$ has a continuous selection, then it is easy to check that $F \in \mathfrak{K}(X, Y)$. Note that there are many known selection theorems due to Michael and others.

From now on, $\mathfrak{KC}$ denote S the class $\mathfrak{K}$ for closed-valued maps G, and $\mathfrak{KO}$ for open-valued maps G.

From Theorem 2.1, we derived the following basic coincidence theorem in [7]:

THEOREM 2.2. *Let $(X, D; \Gamma)$ be a G-convex space, Y a topological space, $S : D \multimap Y$, $T : X \multimap Y$, and $F \in \mathfrak{KC}(X, Y)$. Suppose that*

(1) *S has open values;*
(2) *for each $y \in F(X)$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$; and*
(3) *$\overline{F(X)} \subset S(N)$ for some $N \in \langle D \rangle$.*

Then F and T have a coincidence point $x_* \in X$; that is, $F(x_*) \cap T(x_*) \neq \emptyset$.

Theorem 2.2 is applied in [7] to the Fan-Browder theorem, Φ-spaces, and ω-connected spaces.

Similarly, for the class $\mathfrak{KO}(X, Y)$, we have the following basic coincidence theorem in [9]:

THEOREM 2.2′. *Let $(X, D; \Gamma)$ be a G-convex space, Y a topological space, $S : D \multimap Y$, $T : X \multimap Y$, and $F \in \mathfrak{KO}(X, Y)$. Suppose that*

(1) *S has closed values;*
(2) *for each $y \in F(X)$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$; and*
(3) *$Y = S(N)$ for some $N \in \langle D \rangle$.*

Then F and T have a coincidence point $x_* \in X$; that is, $F(x_*) \cap T(x_*) \neq \emptyset$.

REMARK. It would be possible to replace the class $\mathfrak{K}$ in this paper by the so-called S-KKM class introduced by some authors. However, we will not do this in the present paper.

By putting $X = Y$ and $F = 1_X$ in Theorems 2.2 and 2.2$'$, we have a general form of the Fan–Browder theorem for G-convex spaces:

THEOREM 2.3. *Let $(X, D; \Gamma)$ be a G-convex space, and $S : D \multimap X$, $T : X \multimap X$ two maps satisfying*

(1) *for each $z \in D$, $S(z)$ is open [resp. closed];*
(2) *for each $y \in X$, $M \in \langle S^-(y) \rangle$ implies $\Gamma_M \subset T^-(y)$; and*
(3) *$X = S(N)$ for some $N \in \langle D \rangle$.*

Then T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$.

Theorem 2.3 is obtained in [7] and applied to various forms of the Fan–Browder theorem, the Ky Fan intersection theorem, and the Nash equilibrium theorem for G-convex spaces.

From Theorem 2.3, we deduced the following in [9]:

THEOREM 2.3$'$. *Let $(X \supset D; \Gamma)$ be a G-convex space and $A : X \multimap X$ be a multimap such that $A(x)$ is Γ-convex for each $x \in X$. If there exist $z_1, z_2, \dots, z_n \in D$ and nonempty open [resp. closed] subsets $G_i \subset A^-(z_i)$ for $i = 1, 2, \dots, n$ such that $X = \bigcup_{i=1}^n G_i$, then A has a fixed point.*

A *polytope* is a finite dimensional compact convex subset of a t.v.s. Let X and Y be topological spaces. An *admissible class* $\mathfrak{A}_c^\kappa(X, Y)$ of maps $T : X \multimap Y$ is one such that, for each compact subset K of X, there exists a map $S \in \mathfrak{A}_c(K, Y)$ satisfying $S(x) \subset T(x)$ for all $x \in K$; where $\mathfrak{A}_c$ is consisting of finite composites of maps in $\mathfrak{A}$, and $\mathfrak{A}$ is a class of maps satisfying the following properties:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is upper semicontinuous and compact-valued; and
(iii) for each polytope P, each $F \in \mathfrak{A}_c(P, P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathfrak{A}$.

Examples of $\mathfrak{A}$ are continuous functions $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values and codomains are convex spaces), the Aronszajn maps $\mathbb{M}$ (with R_δ values), the acyclic maps $\mathbb{V}$ (with acyclic values), the Powers maps $\mathbb{V}_c$, the O'Neil maps $\mathbb{N}$ (continuous with values of one or m acyclic components, where m is fixed), the approachable maps $\mathbb{A}$ (whose domains and codomains are subsets of uniform spaces), admissible maps of Górniewicz, σ-selectional maps of Haddad and Lasry, permissible maps of Dzedzej, and others. Further, $\mathbb{K}_c^+$ due to Lassonde, $\mathbb{V}_c^+$ due to Park *et al.*, and approximable maps $\mathbb{A}^\kappa$ due to Ben-El-Mechaiekh and Idzik are examples of $\mathfrak{A}_c^\kappa$. For the literature, see [2-4]. Many other careless authors mistook $\mathfrak{A}$ for $\mathcal{U}$.

LEMMA 2.4. *Let $(X, D; \Gamma)$ be a G-convex space and Y a Hausdorff space. Then $\mathfrak{A}_c^\kappa(X, Y) \subset \mathfrak{KC}(X, Y)$.*

This is given as [11, Corollary]. In the same paper, we showed that $\mathfrak{A}_c^\kappa(X, Y) \subset \mathfrak{KO}(X, Y)$ whenever Y is T_1 regular [11, Theorem 11] in view of the following [11, Lemma]:

LEMMA 2.5. *Let $(X, D; \Gamma)$ be a G-convex space, $|D| = n + 1$, Y a regular space, and $F : X \multimap Y$ a compact-valued u.s.c. map. If $G : D \multimap Y$ is an open-valued map such that*

(1) *for each $J \in \langle D \rangle$, $F(\Gamma_J) \subset G(J)$ [or $F(\phi_D(\Delta_J)) \subset G(J)$];*

then there is a closed-valued map $H : D \multimap Y$ such that $H(x) \subset G(x)$ for all $x \in D$; and

(2) *$F(\phi_D(\Delta_J)) \subset H(J)$ for each $J \subset D$;*

where Δ_J is the face of Δ_n corresponding to J and $\phi_D : \Delta_n \to \Gamma_D$ a continuous map such that $\phi_D(\Delta_J) \subset \Gamma_J$.

In [11], it was assumed thoroughly that $D \subset X$, which is now redundant in our present definition of $(X, D; \Gamma)$.

3. Main results

Now we show that regularity of Y in [11, Theorem 11] can be eliminated, or that $\mathfrak{KC}$ can be replaced by $\mathfrak{KO}$ in Lemma 2.4, as follows:

THEOREM 3.1. *Let $(X, D; \Gamma)$ be a G-convex space and Y a Hausdorff space. Then $\mathfrak{A}_c^\kappa(X, Y) \subset \mathfrak{KO}(X, Y)$.*

Proof. Let $F \in \mathfrak{A}_c^\kappa(X, Y)$ and $G : D \multimap Y$ an open-valued multimap satisfying $F(\Gamma_A) \subset G(A)$ for all $A \in \langle D \rangle$. Let A have $n + 1$ elements. Then $\phi_A(\Delta_n) \subset \Gamma_A$, where $\phi_A : \Delta_n \to \Gamma_A$ is a continuous function such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma_J$ in the definition of G-convex spaces. Since $\phi_A(\Delta_n)$ is compact and $F \in \mathfrak{A}_c^\kappa(X, Y)$, there exists a map $F' \in \mathfrak{A}_c(\phi_A(\Delta_n), Y)$ such that $F'(x) \subset F(x)$ for $x \in \phi_A(\Delta_n)$. Note that

$F'(\phi_A(\Delta_n)) \subset F(\Gamma_A) \subset G(A)$. Since F' is u.s.c. and compact-valued, $F'(\phi_A(\Delta_n))$ is Hausdorff and compact, hence is normal and regular. Moreover,

$$F'(\phi_A(\Delta_n)) \subset \bigcup_{a\in A} G(a) \cap F'(\phi_A(\Delta_n)) = G'(A),$$

where $G' : A \multimap F'(\phi_A(\Delta_n))$ is defined by $G'(a) := G(a) \cap F'(\phi_A(\Delta_n))$ for all $a \in A$. Note that G' has (relatively) open values such that, for each $J \in \langle A \rangle$,

$$F'(\phi_A(\Delta_J)) \subset F(\Gamma_J) \subset G(J) \Rightarrow F'(\phi_A(\Delta_J)) \subset G'(J).$$

Define a map $\Gamma' : \langle A \rangle \multimap \phi_A(\Delta_n)$ by $\Gamma'(J) := \phi_A(\Delta_J)$ for each $J \in \langle A \rangle$. Then $(\phi_A(\Delta_n), A; \Gamma')$ is a G-convex space. Therefore, by Lemma 2.5, there is a closed-valued map $H : A \multimap F'(\phi_A(\Delta_n))$ such that $H(a) \subset G'(a)$ for all $a \in A$ and

$$F'(\phi_A(\Delta_J)) \subset H(J) \text{ for each } J \in \langle A \rangle.$$

Note that $F'\phi_A \in \mathfrak{A}_c(\Delta_n, F'(\phi_A(\Delta_n))) \subset \mathfrak{KC}(\Delta_n, F'(\phi_A(\Delta_n)))$ and hence, by the definition of the class $\mathfrak{KC}$, $\{H(a)\}_{a\in A}$ has the finite intersection property. Since

$$\emptyset \neq \bigcap_{a\in A} H(a) \subset \bigcap_{a\in A} G'(a) = [\bigcap_{a\in A} G(a)] \cap F'(\phi_A(\Delta_n)),$$

we conclude that $\bigcap_{a\in A} G(a) \neq \emptyset$. Since $A \in \langle D \rangle$ is arbitrary, $\{G(z)\}_{z\in D}$ has the finite intersection property. Therefore, we conclude that $F \in \mathfrak{KD}(X, Y)$. □

Theorem 3.1 can be restated as follows:

THEOREM 3.2. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $F \in \mathfrak{A}_c^\kappa(X, Y)$, and $G : D \multimap Y$ such that*

(1) *for each $x \in D$, $G(x)$ is open; and*
(2) *for any $N \in \langle D \rangle$, $F(\Gamma_N) \subset G(N)$.*

Then $\{G(x) \mid x \in D\}$ has the finite intersection property.

Theorem 3.2 improves [11, Theorem 11], where Y is assumed to be T_1 and regular.

Some of other results in earlier works of the authors mentioned in the end of [11, Theorem 11] can be also improved. For example, a KKM type theorem [10, Theorem 6] can be stated as follows:

THEOREM 3.3. *Let $(X, D; \Gamma)$ be a G-convex space, $A \in \langle D \rangle$, Y a topological space, $G : A \multimap Y$ a map, and $F \in \mathfrak{KO}(X, Y)$. Suppose that*

(1) *for each $x \in A$, $G(x)$ is open in Y; and*
(2) *for any $N \in \langle A \rangle$, $F(\Gamma_N) \subset G(N)$.*

Then $F(\Gamma_A) \cap \bigcap\{G(a) \mid a \in A\} \neq \emptyset$.

Proof. Suppose the conclusion does not hold. Then $F(\Gamma_A) \subset S(A)$ where $S(x) = F(\Gamma_A) \backslash G(x)$ for $x \in A$. Define a map $\Gamma' : A \multimap \Gamma_A$ by $\Gamma'(J) := \Gamma_J \cap \Gamma_A$ for each $J \in \langle A \rangle$, then $(\Gamma_A, A; \Gamma')$ is a G-convex space. Then conditions (1) and (3) in Theorem 2.2′ are satisfied for $((\Gamma_A, A; \Gamma'), F(\Gamma_A))$ instead of $((X, D; \Gamma), Y)$. Let $H : F(\Gamma_A) \multimap \Gamma_A$ and $T : \Gamma_A \multimap F(\Gamma_A)$ be defined by $H(y) := \bigcup\{\Gamma'_M \mid M \in \langle S^-(y) \rangle\}$ for $y \in F(\Gamma_A)$ and $T(x) := H^-(x)$ for $x \in \Gamma_A$. Then (2) in Theorem 2.2′ is satisfied, hence T and F have a coincidence point $x_0 \in \Gamma_A$; that is, $T(x_0) \cap F(x_0) \neq \emptyset$. For $y \in T(x_0) \cap F(x_0)$, we have $x_0 \in T^-(y) = \bigcup\{\Gamma'_M \mid M \in \langle S^-(y) \rangle\}$, and hence there exists a finite set $M \subset S^-(y) \subset A$ such that $x_0 \in \Gamma'_M$. Since $M \in \langle S^-(y) \rangle$ implies $y \in S(x)$ for all $x \in M$, we have $y \in F(x_0) \cap \bigcap\{S(x) \mid x \in M\}$. Therefore, $\emptyset \neq F(\Gamma'_M) \cap \bigcap\{S(x) \mid x \in M\} \subset F(\Gamma_M) \cap \bigcap\{S(x) \mid x \in M\}$; that is, $F(\Gamma_M) \not\subset G(M)$. This contradicts (2). This completes our proof. □

The following is a matching theorem:

THEOREM 3.4. *Let $(X, D; \Gamma)$ be a G-convex space, Y a topological space, $F \in \mathfrak{KO}(X, Y)$, and $A \in \langle D \rangle$. Let $S : A \multimap Y$ be a map such that*

(1) *S has closed values; and*
(2) *$S(A) = Y$.*

Then there exists a $B \in \langle A \rangle$ such that $F(\Gamma_B) \cap \bigcap\{S(b) \mid b \in B\} \neq \emptyset$.

Proof. Suppose that the conclusion does not hold. Define a map $G : A \multimap Y$ by $G(a) := Y \backslash S(a)$ for $a \in A$. Then each $G(a)$ is open and for each $B \in \langle A \rangle$,

$$F(\Gamma_B) \subset \bigcup_{b \in B} Y \backslash S(b) = \bigcup_{b \in B} G(b) = G(B).$$

Since $F \in \mathfrak{KO}(X, Y)$, the family $\{G(a)\}_{a \in A}$ has the finite intersection property. Therefore $\bigcap\{G(a) \mid a \in A\} \neq \emptyset$ and hence $S(A) \neq Y$, a contradiction. □

REMARK. A particular form of Theorem 3.4 is given by Balaj [1, Lemma 1].

We have another matching theorem:

THEOREM 3.4′. *Let* $(X, D; \Gamma)$ *be a G-convex space,* Y *a topological space,* $F \in \mathfrak{KC}(X, Y)$, *and* $A \in \langle D \rangle$. *Let* $T : A \multimap Y$ *be a map such that*

(1) T *has open values*; *and*
(2) $T(A) = Y$.

Then there exists $B \in \langle A \rangle$ *such that* $F(\Gamma_B) \cap \bigcap \{T(b) \,|\, b \in B\} \neq \emptyset$.

Proof. Suppose that the conclusion does not hold. Define a map $G : A \multimap Y$ by $G(a) := Y \backslash T(a)$ for $a \in A$. Then each $G(a)$ is closed and, for each $B \in \langle A \rangle$,

$$F(\Gamma_B) \subset \bigcup_{b \in B} Y \backslash T(b) = \bigcup_{b \in B} G(b) = G(B).$$

Since $F \in \mathfrak{KC}(X, Y)$, the family $\{G(a)\}_{a \in A}$ has the finite intersection property. Therefore, $\bigcap \{G(a) \,|\, a \in A\} \neq \emptyset$ and hence $T(A) \neq Y$, a contradiction. □

REMARK. Note that Theorem 3.4′ generalizes a result of Balaj [1, Lemma 7], which is a particular case of Theorem 3.4′ for $F \in \mathfrak{A}_c^\kappa(X, Y)$, $X = D = Y$ and $F = 1_X$.

The following is a new type of fixed point theorems:

THEOREM 3.5. *Let* $(X, D; \Gamma)$ *be a G-convex space,* Y *a topological space,* $F \in \mathfrak{KO}(X, Y)$, *and* $A \in \langle D \rangle$. *Let* $G : A \multimap Y$ *and* $T : Y \multimap Y$ *be two maps. Suppose that*

(1) $F(\Gamma_B) \subset G(B)$ *for each* $B \in \langle A \rangle$;
(2) *for each* $y \in Y$, $T(y) \supset G(x)$ *for some* $x \in A$; *and*
(3) *for each* $z \in G(A)$, $T^-(z)$ *is closed.*

Then T *has a fixed point.*

Proof. Define $S : A \multimap Y$ by

$$S(x) := \{y \in Y \,|\, G(x) \subset T(y)\} \text{ for } x \in A.$$

Then

$$S(x) = \{y \in Y \,|\, y \in T^-(z) \text{ for all } z \in G(x)\} = \bigcap_{z \in G(x)} T^-(z)$$

and hence each $S(x)$ is closed by (3). Moreover, for each $y \in Y$, there is an $x \in A$ such that $G(x) \subset T(y)$ by (2), and hence $y \in S(x)$. This shows $Y = S(A)$. Therefore, by Theorem 3.4, there exist a $B \in \langle A \rangle$ and a $y_0 \in Y$ such that $y_0 \in F(\Gamma_B) \subset G(B)$ and $y_0 \in S(b)$ for all $b \in B$. This implies $G(B) \subset T(y_0)$ and hence $y_0 \in F(\Gamma_B) \subset G(B) \subset T(y_0)$. □

REMARK. Note that Balaj [1, Theorem 2] is Theorem 3.5 for a T_1 regular space Y and $F \in \mathfrak{A}_c^\kappa(X, Y)$. From Theorem 3.5, we can also improve [1, Theorems 3, 4, 6 and Corollary 5].

Finally, in this section, we give a simple proof of the following generalization of a coincidence theorem in Balaj [1, Theorem 8]:

THEOREM 3.6. *Let $(X \supset D; \Gamma)$ be a G-convex space, Z a nonempty set, and $F, T : X \multimap Z$ two maps such that*

(1) *for each $y \in X$, the set $\{x \in X \mid F(x) \cap T(y) \neq \emptyset\}$ is Γ-convex;*
(2) *for each $z \in F(X)$, $T^-(z)$ is open; and*
(3) *$X = \bigcup_{x \in N} \{y \in X \mid F(x) \cap T(y) \neq \emptyset\}$ for some $N \in \langle D \rangle$.*

Then there exists $x_0 \in X$ such that $F(x_0) \cap T(x_0) \neq \emptyset$.

Proof. Define a map $G : X \multimap X$ by

$$G(y) := \{x \in X \mid F(x) \cap T(y) \neq \emptyset\} \text{ for } y \in X.$$

Then each $G(y)$ is Γ-convex. On the other hand,

$$\begin{aligned} G^-(x) &= \{y \in X \mid F(x) \cap T(y) \neq \emptyset\} \\ &= \{y \in X \mid y \in T^-(z) \text{ for some } z \in F(x)\} \\ &= \bigcup_{z \in F(x)} T^-(z) \end{aligned}$$

for $x \in X$. Then $G^-(x)$ is open as a union of open sets. By (3), $G^-(N) = X$ for some $N \in \langle D \rangle$. Therefore, by the Fan-Browder fixed point theorem for a G-convex spaces (Theorems 2.3 and 2.3′), G has a fixed point $x_0 \in X$; that is, $F(x_0) \cap T(x_0) \neq \emptyset$. □

REMARK. The other results in [1, Theorems 9 and 10] are simple consequences of Theorem 3.6.

We note that our new results may have a large number of particular cases because of the abstract nature of the generalized convex space theory, and the readers could easily find such cases.

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