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NORMALIZATION OF THE HAMILTONIAN AND THE ACTION SPECTRUM

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ABSTRACT. In this paper, we prove that the two well-known natural normalizations of Hamiltonian functions on the symplectic manifold (M, ω) canonically relate the action spectra of different normalized Hamiltonians on *arbitrary* symplectic manifolds (M, ω). The natural classes of normalized Hamiltonians consist of those whose mean value is zero for the closed manifold, and those which are compactly supported in Int$M$ for the open manifold. We also study the effect of the action spectrum under the π_1 of Hamiltonian diffeomorphism group. This forms a foundational basis for our study of spectral invariants of the Hamiltonian diffeomorphism in [8].

§1. Introduction

Unlike the classical action functional on the loop space of $\mathbb{R}^{2n}$ whose study of critical values plays such an important role in Rabinowitz's celebrated work [13] and its application to symplectic rigidity properties [2], [6] on $\mathbb{R}^{2n}$, the general action functional is not single valued on the (contractible) loop space $\Omega_0(M)$ of *non-exact* symplectic manifolds and has to be considered on its covering space.

To set-up our notations, we first recall construction of this covering space. Let (γ, w) be a pair of $\gamma \in \Omega_0(M)$ and w be a disc bounding γ, i.e., with $w|_{\partial D^2} = \gamma$. We say that (γ, w) is Γ-*equivalent* to (γ, w') if and only if

$$\omega([w' \# \overline{w}]) = 0 \quad \text{and} \quad c_1([w' \# \overline{w}]) = 0 \tag{1.1}$$

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where $\overline{w}$ is the map with opposite orientation on the domain and $w'\#\overline{w}$ is the obvious glued sphere. Here Γ stands for the group

$$\Gamma = \frac{\pi_2(M)}{\ker\,(\omega|_{\pi_2(M)}) \cap \ker\,(c_1|_{\pi_2(M)})}.$$

We denote by $[\gamma, w]$ the Γ-equivalence class of (γ, w) and by $p : \widetilde{\Omega}_0(M) \to \Omega_0(M)$ the canonical projection. We also call $\widetilde{\Omega}_0(M)$ the Γ-covering space of $\Omega_0(M)$. For each given such pair (γ, w), its action is defined by

$$\mathcal{A}_0(\gamma, w) = -\int w^*\omega. \tag{1.2}$$

Two Γ-equivalent pairs (γ, w) and (γ, w') have the same action and so the function $\mathcal{A}_0$ induces a well-defined function on $\widetilde{\Omega}_0(M)$. We call this the action functional on $\widetilde{\Omega}_0(M)$ which we also denote by the same letter $\mathcal{A}_0$. When a periodic Hamiltonian $H : (\mathbb{R}/\mathbb{Z}) \times M \to \mathbb{R}$ is given, we consider the perturbed action functional $\mathcal{A}_H : \widetilde{\Omega}(M) \to \mathbb{R}$ by

$$\mathcal{A}_H([\gamma, w]) = \mathcal{A}_0([\gamma, w]) - \int H(t, \gamma(t))dt. \tag{1.3}$$

and denote the set of critical values of $\mathcal{A}_H$ by Spec(H).

The mini-max theory of this action functional on the Γ-covering space has been implicitly used in the proof of Arnold's conjecture in the literature starting from Floer's original paper [4]. Recently the present author has developed this mini-max theory via the Floer homology further and applied it to the study of geometry of Hamiltonian diffeomorphism groups [11].

The less noticed and seemingly inessential nuisance in the study of Hamiltonian diffeomorphisms via the Floer theory of the action functional is that the action functional depends on the choice of Hamiltonians, not just on the final diffeomorphism. Even when we fix the path of Hamiltonian diffeomorphisms generating the given Hamiltonian diffeomorphism, there still remains an ambiguity or freedom of choosing a constant. The so called *action spectrum*, i.e., the set of critical values of the action functional depends only on the homotopy class of the Hamiltonian path *but upto overall shift of a constant.* The necessity of proper normalization first arose in Viterbo's construction [16], using the generating functions, of certain spectral invariants of Lagrangian embeddings in the contangent bundle T^*N parametrized by $H^*(N)$. Viterbo [16] proved that the difference of two such invariants is independent of the choice of generating functions but the invariants themselves are defined upto overall shift of a constant. To study change of such invariants under the Hamiltonian isotopy, one has to deal with this normalization problem and to eliminate this ambiguity. This is particulary so if one would like to study the effect on the invariants under the Hamiltonian loop.

There are two natural, both well-known, normalizations of Hamiltonians in symplectic geometry: the first one is the one obtained by restricting to the class $C_m^\infty(M)$ of Hamiltonians whose mean values are zero, i.e., those h such that

$$\int_M h d\mu = 0 \tag{1.4}$$

where $d\mu = \frac{\omega^n}{n!}$ is the Liouville measure. This normalization is natural for the case of *closed* M , i.e., compact M without boundary. The second one is natural for the case of *open* M. It could be non-compact or compact with non-empty boundary for the second case. In this case, the natural class of Hamiltonians (or Hamiltonian diffeomorphisms) are those which have compact support in the interior of M, which we denote by $C_c^\infty(M)$.

However it is not obvious what overall effect on the action spectrum will be when restricted to these subsets. For example, the following is an obvious question to ask. We denote $F \sim G$ if and only if $\phi_F^1 = \phi_G^1$ and $\{\phi_F^t\}_{0\leq t\leq 1}$ and $\{\phi_G^t\}_{0\leq t\leq 1}$ are homotopic with fixed ends.

QUESTION 1. Let $F, G : M \times [0,1] \to \mathbb{R}$ be two time-dependent Hamiltonians with $F \sim G$ and satisfying one of the above normalization conditions. Does this imply

$$\mathrm{Spec}(F) = \mathrm{Spec}(G) \tag{1.5}$$

as a subset of $\mathbb{R}$?

This question has been studied by Schwarz [14] and by Polterovich [12] on *closed* manifold with normalization (1.4) for the case of *symplectically aspherical* (M,ω) i.e., with $\omega|_{\pi_2(M)} = 0$ and $c_1|_{\pi_2(M)} = 0$. In this case, as far as the definition of the action functional is concerned, we do not need to go to the covering space $\widetilde\Omega_0(M)$ but to just work on $\Omega_0(M)$. One very interesting result proven by Schwarz [14] is that there is no monodromy effect on the spectrum in this case and so he answered affirmatively to the above question on $\Omega_0(M)$ without assuming $F \sim G$ as long as $\phi_F^1 = \phi_G^1$.

The first result of this paper answers the question affirmatively for both of the above two normalizations in general in the above setting on the covering space $\widetilde{\Omega}_0(M)$, which naturally appears in the Floer theory on non-exact symplectic manifolds.

THEOREM I. *Let ϕ be a given Hamiltonian diffeomorphism and let F, G be two normalized Hamiltonians such that $\phi_F^1 = \phi_G^1 = \phi$ and $F \sim G$. Then*

$$\mathrm{Spec}(F) = \mathrm{Spec}(G) \tag{1.6}$$

as a subset $\mathbb{R}$.

This enables us to safely analyze the behavior of homologically essential critical values of the action functional constructed via the Floer theory for a family of Hamiltonians functions which appear in the study of geometry of the group of Hamiltonian diffeomorphisms [14], [11]. For example, combined with the fact that $\mathrm{Spec}(H)$ is of measure zero subset (see [11]), the following is the prototype of results that are used frequently in the mini-max theory of action functional.

COROLLARY. *Suppose that $\{F^s\}_{s\in[0,1]}$ is a one parameter family of normalized Hamiltonians generating the same Hamiltonian diffeomorphism group ϕ. If $c : [0,1] \to \mathbb{R}$ is a continuous function with*

$$c(s) \in \mathrm{Spec}(F^s) \subset \mathbb{R} \tag{1.7}$$

for all $s \in [0,1]$, then c must be a constant function.

One natural construction of such continuous functions is the assignment of the homologically essential critical value $\rho(H; a)$ for each quantum cohomology class $a \in QH^*(M, \mathbb{Q})$ which are constructed in [11, 8] using the Floer theory (see [9], [10], [14] for the case without quantum effect): e.g., the assignment

$$s \mapsto \rho(F^s; a).$$

Theorem I will enable us to systematically study these invariants in [11].

One notable fact in both of the above normalizations is that the class of Hamiltonians for each case is invariant under the action by the symplectic diffeomorphism group. In the Lie algebra level, they form a Lie sub-ideal of $\mathfrak{g} = C^\infty(M)$ under the natural Poisson bracket. More precisely, let $\mathfrak{h} \subset \mathfrak{g}$ denote any of the two subsets above and $\{\cdot, \cdot\}$ be the Poisson bracket on $\mathfrak{g}$ associated to the symplectic form ω. Then we have

$$\{\mathfrak{h}, \mathfrak{g}\} \subset \mathfrak{h}. \tag{1.8}$$

It appears that this fact is a crucial link between the normalization of Hamiltonians and the action spectrum (see §3 for the relevant discussion). For example, it is not difficult to see that (1.8) will not hold true for the kind of normalization

$$\min H_t = 0 \tag{1.9}$$

on closed manifolds, but whether (1.6) holds or not in general is much less obvious to check. However once we have our normalization result, it is easy to construct an example of two Hamiltonians F, G normalized by (1.9) and for which $F \sim G$ but (1.6) fails to hold.

In §4, we analyze the action by $\pi_1(\mathcal{H}am(M,\omega), id)$ on the action spectrum where $\mathcal{H}am(M,\omega)$ is the group of Hamiltonian diffeomorphisms of (M,ω). Let h be an arbitrary loop in $\mathcal{H}am(M,\omega)$ based at the identity and $[h] \in \pi_1(\mathcal{H}am(M,\omega))$ be its fundamental class. To describe the result in §4, let us recall some basic facts. Let

$$\widetilde{h} : \widetilde{\Omega}_0(M) \to \widetilde{\Omega}_0(M)$$

be a lifting of the action $h : \Omega_0(M) \to \Omega_0(M)$ (see [15]). This lifting always exists but is not unique. According to Seidel [15], the set of lifts forms a group

$$\widetilde{G} \subset G \times \mathrm{Homeo}(\widetilde{\Omega}_0(M))$$

which is the set of pairs $(h, \widetilde{h})$ such that $\widetilde{h}$ is a lift of the h-action on $\Omega_0(M)$. And it has one-one correspondence with the set of normalized Hamiltonian fiber bundle with fiber (M,ω) over S^2 with a Γ-equivalence class of section [Lemma 2.13, 15]. Here $G = \Omega(\mathcal{H}am(M,\omega), id)$, the space of loops based at the identity. There exists an exact sequence

$$0 \to \Gamma \to \widetilde{G} \to G \to 0.$$

A standard calculation (Lemma 2.3 below) shows

$$\widetilde{h}^*(d\mathcal{A}_{H\#F}) = d\mathcal{A}_F$$

and so the difference $\widetilde{h}^*(\mathcal{A}_{H\#F}) - \mathcal{A}_F$ is constant on $\widetilde{\Omega}_0(M)$, which depends on F, H and the lifting $\widetilde{h}$ in general. The following theorem describes the dependence of this constant precisely.

THEOREM II. *Let $h \in \Omega(\mathcal{H}am(M,\omega))$ be a loop and let $(h,\widetilde{h})$ be any lift. Denote $[h,\widetilde{h}] \in \pi_0(\widetilde{G})$. Then we have*

$$\widetilde{h}^*(\mathcal{A}_{H\#F}) = \mathcal{A}_F + I_\omega([h,\widetilde{h}]) \tag{1.10}$$

for any $F \mapsto \phi$ and H generating the loop h, where $I_\omega([h,\widetilde{h}])$ is a constant depending only on $[h,\widetilde{h}] \in \pi_0(\widetilde{G})$ but independent of F. The shift is induced by a homomorphism

$$I_\omega : \pi_0(\widetilde{G}) \to \mathbb{R}$$

that satisfies

$$I_\omega([id,\gamma]) \in \Gamma_\omega. \tag{1.11}$$

for any lift γ of the identity.

We have a natural action of $\pi_0(\widetilde{G})$ on $\widetilde{\mathcal{H}am}(M,\omega) \times \widetilde{\Omega}_0(M)$

$$[h,\widetilde{h}] : (\widetilde{\phi},[z,w]) \mapsto ([h]\cdot\widetilde{\phi}, \widetilde{h}\cdot[z,w]). \tag{1.12}$$

Evaluating I_ω at $[id,0] \in \widetilde{\mathcal{H}am}(M,\omega)$, we obtain

$$I_\omega([id,0]) = \mathcal{A}_H(\widetilde{h}\cdot[p,\widehat{p}]) = -\int w_p^*\omega - \int_0^1 H(z_p(t),t)\,dt \tag{1.13}$$

where $[z_p,w_p] = \widetilde{h}\cdot[p,\widehat{p}]$ for any $p \in M$.

It follows from (1.11) that in the weakly exact case these values depend only on the loop h not on $\widetilde{h}$. Therefore we have a map

$$\overline{I}_\omega : \pi_1(\mathcal{H}am(M,\omega)) \to \mathbb{R}. \tag{1.14}$$

This is precisely the monodromy map considered by Schwarz in [14]. In the aspherical case, i.e, for those (M, ω) with $\omega|_{\pi_2(M)} = c_1|_{\pi_2(M)} = 0$, he proved that (1.14) is trivial, by relating the question to the geometry of Hamiltonian fibrations [14]. The following is an interesting question to ask.

QUESTION 2. Is the map (1.14) trivial for the general weakly exact case?

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§2. The loop space and the action functional

Let (M, ω) be any symplectic manifold, compact or not, and $\Omega_0(M)$ be the set of contractible loops and $\widetilde{\Omega}_0(M)$ be its the covering space mentioned before.

Let a periodic Hamiltonian $H : M \times (\mathbb{R}/\mathbb{Z}) \to \mathbb{R}$ be given and denote by $\mathrm{Per}(H)$ the set of periodic orbits of the Hamiltonian vector field X_H. We consider the action functional $\mathcal{A}_H : \widetilde{\Omega}(M) \to \mathbb{R}$.

DEFINITION 2.1. We define the *action spectrum* of H, denoted as $\mathrm{Spec}(H) \subset \mathbb{R}$, by

$$\mathrm{Spec}(H) := \{\mathcal{A}_H([z,w]) \in \mathbb{R} \mid [z,w] \in \widetilde{\Omega}_0(M), z \in \mathrm{Per}(H)\},$$

i.e., the set of critical values of $\mathcal{A}_H : \widetilde{\Omega}(M) \to \mathbb{R}$. For each given $z \in \mathrm{Per}(H)$, we denote

$$\mathrm{Spec}(H; z) = \{\mathcal{A}_H([z,w]) \in \mathbb{R} \mid [z,w] \in p^{-1}(z)\}$$

where $p : \widetilde{\Omega}_0(M) \to \Omega_0(M)$ is the covering map.

Note that $\mathrm{Spec}(H; z)$ is a principal homogeneous space modeled by the period group of (M, ω)

$$\Gamma_\omega := \{\omega(A) \mid A \in \pi_2(M)\} = \omega(\Gamma)$$

and

$$\mathrm{Spec}(H) = \cup_{z \in \mathrm{Per}(H)} \mathrm{Spec}(H; z).$$

Recall that Γ_ω is either a discrete or a countable dense subset of $\mathbb{R}$. The following was proven in [11].

LEMMA 2.2. Spec(H) *is a measure zero subset of* $\mathbb{R}$ *for any* H.

For given $\phi \in \mathcal{H}am(M,\omega)$, we denote by $F \mapsto \phi$ if $\phi_F^1 = \phi$, and denote

$$\mathcal{H}(\phi) = \{F \mid F \mapsto \phi\}.$$

We say that two Hamiltonians F and G are equivalent and denote $F \sim G$ if they are connected by one parameter family of Hamiltonians $\{F^s\}_{0\leq s\leq 1}$ such that $F^s \mapsto \phi$ for all $s \in [0,1]$. We denote by $[F]$ the equivalence class of F. Then the universal (étale) covering space $\widetilde{\mathcal{H}am}(M,\omega)$ of $\mathcal{H}am(M,\omega)$ is realized by the set of such equivalence classes. Note that the group $\Omega(\mathcal{H}am(M,\omega))$ of based loops (at the identity) naturally acts on the loop space $\Omega(M)$ by

$$(h\cdot\gamma)(t) = h(t)(\gamma(t))$$

where $h \in \Omega(\mathcal{H}am(M,\omega))$ and $\gamma \in \Omega(M)$. An interesting consequence of Arnold's conjecture is that this action maps $\Omega_0(M)$ to itself (see e.g., [Lemma 2.2, 15]). Seidel [Lemma 2.4, 15] proves that this action (by a based loop) can be lifted to the covering space $\widetilde{\Omega}_0(M)$.

Next if a Hamiltonian H generating the loop h is given, the assignment

$$z \mapsto h\cdot z \tag{2.1}$$

provides a natural one-one correspondence

$$h : \mathrm{Per}(F) \mapsto \mathrm{Per}(H\#F) \tag{2.2}$$

where $H\#F = H + F\circ(\phi_H^t)^{-1}$.

When the loop $h : S^1 \to \mathcal{H}am(M,\omega)$ with $h(0) = id$ is contractible, the above mentioned lifting to $\widetilde{\Omega}_0(M)$ of the action of h on $\Omega_0(M)$ can be described explicitly: Let $\widetilde{h} : D^2 \to \mathcal{H}am(M,\omega)$ be a contraction of the loop h to the identity, i.e., $\widetilde{h}|_{\partial D^2} = h$, $\widetilde{(0)} = id$. This contraction provides a natural lifting of the action of the loop h on $\Omega_0(M)$ to $\widetilde{\Omega}_0(M)$ which we define

$$\widetilde{h}\cdot[\gamma,w] = [h\cdot\gamma, \widetilde{h}\cdot w] \tag{2.3}$$

where $\widetilde{h}\cdot w$ is the natural map defined by

$$(\widetilde{h}\cdot w)(y) = \widetilde{h}(y)(w(y)) \tag{2.4}$$

for $y \in D^2$. We call this lifting the *canonical lifting* associated to the contraction $\widetilde{h}$. Then the assignment

$$[z, w] \mapsto \widetilde{h} \cdot [z, w]$$

provides a one-one correspondence

$$\widetilde{h} : \mathrm{Crit}(\mathcal{A}_F) \to \mathrm{Crit}(\mathcal{A}_{H\#F}). \tag{2.5}$$

Let $F, G \mapsto \phi$ and denote

$$f_t = \phi_F^t,\ g_t = \phi_G^t, \text{ and } h_t = f_t \circ g_t^{-1}. \tag{2.6}$$

Let $\{F^s\}_{0 \le s \le 1} \subset \mathcal{H}(\phi)$ with $F_1 = F$ and $F_0 = G$. We denote $f_t^s = \phi_{F^s}^t$. $\widetilde{h}$ provides a natural contraction of the loop h to the identity through

$$\widetilde{h} : s \mapsto f^s \circ g^{-1}; \quad f^s \circ g^{-1}(t) := f_t^s \circ g_t^{-1}.$$

where $\widetilde{h}$ can be considered as a map from D^2 because $h(0, s) \equiv id$. This contraction in turn provides a canonical lifting of the action of the loop h, which we again denote by $\widetilde{h}$, on $\Omega_0(M)$ to $\widetilde{\Omega}_0(M)$ as in (2.6). For non-contractible loops $h : S^1 \to \mathcal{H}am(M, \omega)$, the lifting is related to the notion of normalized Hamiltonian bundles (see §2 [15]).

LEMMA 2.3. *Let F, G be any Hamiltonian with $F, G \mapsto \phi$, and f_t, g_t and h_t be as above. Suppose that $H : M \times [0, 1] \to \mathbb{R}$ is the Hamiltonian generating the loop h. We also denote the corresponding action of h on $\Omega_0(M)$ by h. Let $\widetilde{h}$ be any lift of h to Homeo$(\widetilde{\Omega}_0(M))$. Then we have*

$$\widetilde{h}^*(d\mathcal{A}_F) = d\mathcal{A}_G \tag{2.7}$$

as a one-form on $\widetilde{\Omega}_0(M)$. In particular we have

$$\mathcal{A}_F \circ \widetilde{h} = \mathcal{A}_G + C(F, G, \widetilde{h})$$

where $C(F, G, \widetilde{h})$ is a constant depending only on F, G.

Proof. We recall that $d\mathcal{A}_0$ is the pull-back $p^*\alpha$ of the closed one form α on $\Omega_0(M)$ by the natural projection $p : \widetilde{\Omega}_0(M) \to \Omega_0(M)$ defined by

$$\alpha(\gamma)(\xi) = \int_0^1 \omega(\dot\gamma(t), \xi(t))\, dt$$

for any vector $\xi \in T_\gamma\Omega_0(M)$. Therefore we first compute

$$\begin{aligned} h^*\alpha(\gamma)(\xi) &= \alpha(h\cdot\gamma)(Th(\xi)) \\ &= \int_0^1 \omega(\frac{d}{dt}(h\cdot\gamma)(t), T_{\gamma(t)}h_t(\xi(t)))\, dt \\ &= \int_0^1 \omega(X_H(h_t(\gamma(t)) + T_{\gamma(t)}(\dot\gamma(t)), T_{\gamma(t)}h_t(\xi(t)))\, dt \\ &= \int_0^1 \omega(X_H(h_t(\gamma(t)), T_{\gamma(t)}h_t(\xi(t)))\, dt \\ &\quad + \int_0^1 \omega(T_{\gamma(t)}h_t(\dot\gamma(t)), T_{\gamma(t)}h_t(\xi(t)))\, dt. \end{aligned} \tag{2.8}$$

For the first term, we recall that the Hamiltonian H generating h_t is given by

$$\begin{aligned} H(x,t) &= F(x,t) + \overline{G}(f_t^{-1}(x), t) \\ &= F(x,t) - G(g_t\circ f_t^{-1}, t) = F(x,t) - G(h_t^{-1}(x), t) \end{aligned}$$

and so $H(h_t(x),t) = F(h_t(x),t) - G(x,t)$. Recall that

$$\overline{G}(x,t) = -G(\phi^t(G)(x), t)$$

is the Hamiltonian generating ϕ^{-1}. Therefore since h_t is symplectic, we have

$$\begin{aligned} &\omega(X_H(h_t(\gamma(t))), T_{\gamma(t)}h_t(\xi(t))) \\ &= \omega(h_t^*(X_{H_t})(\gamma(t)), \xi(t)) = \omega(X_{H_t(h_t)}(\gamma(t)), \xi(t)) \\ &= \omega(X_{(F_t(h_t)-G_t)}(\gamma(t)), \xi(t)) = (d(F_t\circ h_t) - dG_t)(\xi(t)). \end{aligned}$$

For the second term of (2.8), we have

$$\int_0^1 \omega(T_{\gamma(t)}h_t(\dot\gamma(t)), T_{\gamma(t)}h_t(\xi(t)))\, dt = \int_0^1 \omega(\dot\gamma(t), \xi(t))\, dt.$$

Combining all these, we derive

$$\begin{aligned}\widetilde{h}^* d\mathcal{A}_F(\gamma)(\xi) &= \widetilde{h}^*(p^*\alpha)(\xi) - \int_0^1 d(F\circ h_t)(\gamma(t))(\xi(t))\,dt \\ &= \int_0^1 (d(F_t\circ h_t) - dG_t)(\gamma(t))(\xi(t))\,dt \\ &\qquad + \int_0^1 \omega(\dot\gamma(t),\xi(t))\,dt \\ &\qquad - \int_0^1 d(F\circ h_t)(\gamma(t))(\xi(t))\,dt \\ &= \int_0^1 \omega(\dot\gamma(t),\xi(t))\,dt - \int_0^1 dG_t(\xi(t))\,dt = d\mathcal{A}_G(\gamma)(\xi).\end{aligned}$$

which proves (2.7). □

COROLLARY 2.4. *Suppose F, G are two Hamiltonians such that F, $G \mapsto \phi$. Then the action spectra of F and G coincide up to an overall shift by a constant $C = C(F, G, \widetilde{h})$.*

§3. Normalization of Hamiltonians and the action spectrum

Now we go back to the one-correspondence (2.5) and study its effect on the action spectra. Naturally one hopes that $\widetilde{h}$ preserves the action, i.e., satisfies

$$\mathcal{A}_{H\#F}(\widetilde{h}\cdot[z,w]) = \mathcal{A}_F([z,w]) \tag{3.1}$$

for any $[z,w] \in \mathrm{Crit}\mathcal{A}_F$ for any F and for any lifting $\widetilde{h}$ of h, *provided h is contractible.* The main theorem in this section is to prove that this is indeed the case, if we use a sub-class of Hamiltonians which are normalized accordingly depending on whether M is closed or open. On closed M, we consider

$$C^\infty_m(M) := \{f \in C^\infty(M) \mid \int_M f\,d\mu = 0\}$$

and on open M, we consider

$$C^\infty_c(M) := \{f \in C^\infty(M) \mid f \text{ has compact support in } \mathrm{Int}M\}.$$

For a given Hamiltonian diffeomorphism $\phi \in \mathcal{H}am(M,\omega)$, we denote by $\mathcal{H}_m(\phi)$ (resp. $\mathcal{H}_c(\phi)$) the set of $H : M \times [0,1] \to \mathbb{R}$ with $H \mapsto \phi$ such that $H_t \in C^\infty_m(M)$ (resp. $C^\infty_c(M)$).

THEOREM 3.1. *Let F, G be two normalized Hamiltonians in $\mathcal{H}_m(\phi)$ on closed M (resp. in $\mathcal{H}_c(\phi)$ on open M) with $F \sim G$. Then* (3.1) *holds and so the correspondence* (2.5) *is action-preserving which in turn implies*

$$\mathrm{Spec}(G) = \mathrm{Spec}(F) \tag{3.2}$$

as a subset $\mathbb{R}$.

Since the proof will be almost identical in both cases, we will focus on the closed case and consider the set $C^\infty_m(M)$. In the course of studying this case, we will also indicate the modification we need for the open case.

This theorem in particular proves that the spectrum $\mathrm{Spec}(G)$ is indeed an invariant of $\widetilde{\phi} \in \widetilde{\mathcal{H}am}(M, \omega)$.

DEFINITION 3.2. [Universal Action Functional] The *universal action functional*

$$\mathcal{A} : \widetilde{\mathcal{H}am}(M, \omega) \times \widetilde{\Omega}_0(M) \to \mathbb{R}$$

is defined by

$$\mathcal{A}(\widetilde{\phi}, [z, w]) := \mathcal{A}_F([z, w]) \tag{3.3}$$

for any normalized representative F with $\widetilde{\phi} = [\phi, F]$.

DEFINITION 3.3. [Action Spectrum Bundle] For

$$\widetilde{\phi} = [\phi, G] \in \widetilde{\mathcal{H}am}(M, \omega),$$

we define the *action spectrum* of $\widetilde{\phi}$ by

$$\mathrm{Spec}(\widetilde{\phi}) := \mathrm{Spec}(G)$$

for a (and so any) $G \in \mathcal{H}_0(M)$ with $\widetilde{\phi} = [\phi, G]$. We define the bundle of action spectrum of (M, ω) by

$$\mathfrak{Spec}(M, \omega) = \{(\widetilde{\phi}, \mathcal{A}(\widetilde{\phi}, [z, w])) \mid d\mathcal{A}_{\widetilde{\phi}}([z, w]) = 0\} \subset \widetilde{\mathcal{H}am}(M, \omega) \times \mathbb{R}$$

and denote by $\pi : \mathfrak{Spec}(M, \omega) \to \widetilde{\mathcal{H}am}(M, \omega)$ the natural projection.

It remains to prove Theorem 3.1. We first need some preparation following Polterovich's discussion in [Proposition 6.1.C, 12] but clarifying the role of (1.8) in the discussion. Let f_t^s be any two parameter family of Hamiltonian diffeomorphisms. Denote by $X_{s,t}$ and $Y_{s,t}$ be the Hamiltonian vector fields

$$\frac{\partial f_t^s}{\partial t} \circ (f_t^s)^{-1} = X_{s,t}$$
$$\frac{\partial f_t^s}{\partial s} \circ (f_t^s)^{-1} = Y_{s,t}.$$

Then the following is the key formula below (see [1] for its derivation):

$$\frac{\partial X_{s,t}}{\partial s} = \frac{\partial Y_{s,t}}{\partial t} + [X_{s,t}, Y_{s,t}]. \tag{3.4}$$

Now we represent the vector fields $X_{s,t}$ and $Y_{s,t}$ by the corresponding normalized Hamiltonians $F(s,t,\cdot)$ and $K(s,t,\cdot)$ which are uniquely determined by $X_{s,t}$ and $Y_{s,t}$ respectively. Since (3.4) does not constrain anything on the direction of (s,t), the Hamiltonian F and K will satisfy

$$\frac{\partial F}{\partial s}(s,t,x) = \frac{\partial K}{\partial t}(s,t,x) - \{F,K\}(s,t,x) + c(s,t) \tag{3.5}$$

for some function $c : [0,1]^2 \to \mathbb{R}$ which depends only on (s,t). Here $\{F,K\}$ is the Poisson bracket, i.e., $\{F,K\} = dF(X_K)$ which satisfies the relation

$$[X_F, X_K] = -X_{\{F,K\}}.$$

Recall that the Poisson bracket $\{F,K\}$ is automatically normalized by Liouville's theorem. Therefore integrating (3.5) over M proves $c \equiv 0$ if F, K are normalized . Hence F, K satisfy

$$\frac{\partial F}{\partial s} = \frac{\partial K}{\partial t} - \{F,K\}. \tag{3.6}$$

Note that the same argument applies to prove (3.6) for the subset $C_c^\infty(M)$ on open M, this time using the fact that

$$\text{supp}\{F,K\} \subset \text{supp}F \cap \text{supp}K$$

and considering the value at a point $p \in M \setminus (\text{Supp}F \cup \text{Supp}K)$ in (3.5). Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\mathcal{F} = \{F^s\}_{s\in[0,1]}$ be a path in $\mathcal{H}_m(\phi)$ (resp. $\mathcal{H}_c(\phi)$) such that $F^0 = G$ and $F^1 = F$. Denote $h_t^s = f_t^s \circ g_t^{-1}$ and $\widetilde{h}^s \cdot [z,w] = [h^s \cdot z, \widetilde{h}^s \cdot w]$ for any $z \in \mathrm{Per}(G)$ as in (2.5)-(2.6). We will prove that the function $\chi : [0,1] \to \mathbb{R}$ defined by

$$\chi(s) = A_{F^s}(\widetilde{h}^s \cdot [z,w])$$

is constant.

Let F, $K : M \times [0,1]^2 \to \mathbb{R}$ be the unique normalized Hamiltonians associated to $\{f_t^s\}$ as in Theorem 3.1. Since $f_1^s \equiv \phi$ and $f_0^s \equiv id$ for all $s \in [0,1]$, $K(s,1,x)$ and $K(s,0,x)$ must be constant for each s and in turn

$$K(s,1,\cdot) = K(s,0,\cdot) \equiv 0 \tag{3.7}$$

by the normalization condition. We now differentiate $\chi(s)$

$$\chi'(s) = d\mathcal{A}_{F^s}(\widetilde{h}^s \cdot [z,w])\Big(\frac{\partial}{\partial s}(\widetilde{h}^s \cdot [z,w])\Big) - \int_0^1 \frac{\partial F}{\partial s}(s,t,(h^s \cdot z)(t))\, dt.$$

The first term vanishes since $\widetilde{h}^s \cdot [z,w]$ is a critical point of $\mathcal{A}_{F^s}$. For the second term we note $z(t) = g_t(p)$ for a fixed point $p \in M$ of ϕ since z is a periodic solution of $\dot{x} = X_{G_t}(x)$. Therefore we have

$$(h^s \cdot z)(t) = h_t^s(z(t)) = (f_t^s \circ g_t^{-1}) \circ g_t(p) = f_t^s(p).$$

Hence we have

$$\int_0^1 \frac{\partial F}{\partial s}(s,t,(h^s \cdot z)(t))\, dt = \int_0^1 \frac{\partial F}{\partial s}(s,t,f_t^s(p))\, dt.$$

The vanishing of this integral is precisely [Lemma 12.4B, 12]. For the completeness' sake, we give its proof here. We derive from (3.6)

$$\begin{aligned}\frac{\partial F}{\partial s}(s,t,f_t^s(p)) &= \frac{\partial K}{\partial t}(s,t,f_t^s(p)) - \{F,K\}(s,t,f_t^s(p))\\ &= \frac{\partial K}{\partial t}(s,t,f_t^s(p))) + dK(X_F)(f_t^s(p))\\ &= \frac{\partial}{\partial t}(K(s,t,f_t^s(p))).\end{aligned}$$

Integrating the total derivative over $[0,1]$ and using (3.7), we derive

$$\int_0^1 \frac{\partial F}{\partial s}(s,t,(h^s \cdot z)(t))\, dt = K(s,1,f_1^s(p)) - K(s,0,f_0^s(p)) = 0.$$

This proves that χ must be constant. Therefore (3.1) immediately follows from $\chi(0) = 0$. (3.2) follows from the fact that both G and $H\#F$ generate g_t and are normalized, which in turn implies $G = H\#F$. □

§4. Action of $\pi_0(\widetilde{G})$ on the action spectrum

From now on, we will always assume that our Hamiltonians are normalized. In this section, we study effects of the action of *non-contractible* loops h on the action spectrum. Consider the action spectrum bundle

$$\pi : \mathfrak{Spec}(M, \omega) \to \widetilde{\mathcal{H}am}(M, \omega).$$

We recall $G = \Omega(\mathcal{H}am(M, \omega), id)$ is the based loop space of $\mathcal{H}am(M, \omega)$ and let $\widetilde{G}$ be the covering space as described in the introduction.

Let $h \in G$ and $[h] \in \pi_1(\mathcal{H}am(M, \omega))$ be its fundamental class and

$$\widetilde{h} : \widetilde{\Omega}_0(M) \to \widetilde{\Omega}_0(M) \tag{4.1}$$

be a lifting of the canonical action $h : \Omega_0(M) \to \Omega_0(M)$ defined by

$$(h \cdot z)(t) = h(t)(z(t)).$$

Let $H \mapsto h$ and $(h, \widetilde{h}) \in \widetilde{G}$ be a lift of h. For any $\widetilde{\phi} = [\phi, F]$, we consider the action of $(h, \widetilde{h})$ on $\widetilde{\Omega}_0(M)$,

$$[z, w] \mapsto \widetilde{h} \cdot [z, w].$$

We now study how this acts on $\mathrm{Spec}(\widetilde{\phi})$. Since $\widetilde{h}^*(d\mathcal{A}_{H\#F}) = d\mathcal{A}_F$ by Lemma 2.3, the difference

$$\mathcal{A}_{H\#F}(\widetilde{h} \cdot [z, w]) - \mathcal{A}_F([z, w])$$

is independent of $[z, w]$. We denote the common value by $I_\omega(h, \widetilde{h}; F)$.

LEMMA 4.1. *The value $I_\omega(h, \widetilde{h}; F)$ is independent of F but depends only on $(h, \widetilde{h})$. We define*

$$I_\omega(h, \widetilde{h}) := I_\omega(h, \widetilde{h}; F) \tag{4.2}$$

for any F (e.g., $F = 0$). Then the action (4.1) induces a canonical isometry between $\mathrm{Spec}(\widetilde{\phi})$ and $\mathrm{Spec}(\widetilde{h} \cdot \widetilde{\phi})$ which is just the restriction of translation by $I_\omega(h, \widetilde{h})$ on $\mathbb{R}$. Furthermore, $I_\omega([id, \gamma]) \equiv 0$ for any canonical lift γ of id.

Proof. A straightforward computation using the identity

$$H\#F(h\cdot z,t)-F(z,t)=H(h\cdot z,t),$$

shows

$$\begin{aligned}I_\omega(h,\widetilde{h};F)&=\mathcal{A}_{H\#F}(\widetilde{h}\cdot[z,w])-\mathcal{A}_F([z,w])\\&=-\int(\widetilde{h}\cdot w)^*\omega+\int H(h\cdot z)-\int w^*\omega\\&=\mathcal{A}_H(\widetilde{h}\cdot[z,w])-\mathcal{A}_0([z,w])=I_\omega(h,\widetilde{h})\end{aligned}$$

This proves the first statement. Then the second immediately follows from

$$I_\omega(h,\widetilde{h};F)=\mathcal{A}_{H\#F}(\widetilde{h}\cdot[z,w])-\mathcal{A}_F([z,w]).$$

Finally $I_\omega(id,\gamma)=0$ for any canonical lift γ of id is just a restatement of Theorem 3.1. □

PROPOSITION 4.2. *Let $\widetilde{\phi}\in\widetilde{\mathcal{H}am}(M,\omega)$. $I_\omega(h,\widetilde{h};\widetilde{\phi})$ depends only on $[h,\widetilde{h}]\in\pi_0(\widetilde{G})$, i.e., if there is a continuous path $\{(h^s,\widetilde{h}^s)\}_{s\in[0,1]}$ in $\widetilde{G}$, then*

$$I_\omega(h^0,\widetilde{h}^0)=I_\omega(h^1,\widetilde{h}^1). \tag{4.3}$$

The induced map

$$I_\omega:\widetilde{\mathcal{H}am}(M,\omega)\to\mathbb{R}$$

defines a group homomorphism and satisfies

$$I_\omega([h,\widetilde{h}])-I_\omega([h,\widetilde{h}'])\in\Gamma_\omega \tag{4.4}$$

where $(h,\widetilde{h})$ and $(h,\widetilde{h}')$ are the lifts of the same h.

Proof. The same proof as that of Theorem 3.1 proves (4.3). For the homomorphism property, it is enough to consider $I_\omega(h,\widetilde{h};F)$. In this case, we have

$$\begin{aligned}&I_\omega(h_2\cdot h_1,\widetilde{h_2\cdot h_1});F)\\&=\mathcal{A}_{H_2\#H_1\#F}(\widetilde{h_2\cdot h_1}\cdot[z,w])-\mathcal{A}_F([z,w])\\&=(\mathcal{A}_{H_2\#H_1\#F}(\widetilde{h_2\cdot h_1}\cdot[z,w])-\mathcal{A}_{H_1\#F}(\widetilde{h}_1\cdot[z,w]))\\&\quad+(\mathcal{A}_{H_1\#F}(\widetilde{h}_1\cdot[z,w])-\mathcal{A}_F([z,w]))\end{aligned} \tag{4.5}$$

for any $(h_2, \widetilde{h}_2) \cdot (h_1, \widetilde{h}_1) = (h_2 \cdot h_1, \widetilde{h_2 \cdot h_1})$. The second term of (4.5) gives rise to $I_\omega([h_1, \widetilde{h_1}]; \widetilde{\phi})$. On the other hand, we have

$$\begin{aligned} &\mathcal{A}_{H_2 \# H_1 \# F}(\widetilde{h_2 \cdot h_1} \cdot [z,w]) - \mathcal{A}_{H_1 \# F}(\widetilde{h}_1 \cdot [z,w]) \\ &= \mathcal{A}_{H_2 \# (H_1 \# F)}(\widetilde{h}_2 \cdot (\widetilde{h}_1 \cdot [z,w])) - \mathcal{A}_{H_1 \# F}(\widetilde{h}_1 \cdot [z,w])) = I_\omega(h_2, \widetilde{h}_2) \end{aligned}$$

where the last identity comes from definition of $I_\omega(h, \widetilde{h}; F)$ and its independence of F (Lemma 4.1).

For the proof of (4.4), we just note that

$$[h, \widetilde{h}'] = [id, \gamma] \cdot [h, \widetilde{h}]$$

where $\gamma = \widetilde{h}' \circ (\widetilde{h})^{-1}$. Then it follows from the homomorphism property of I_ω that

$$I_\omega([h, \widetilde{h}']) - I_\omega([h, \widetilde{h}]) = I_\omega([id, \gamma]).$$

On the other hand, we have

$$I_\omega([id, \gamma]) = I_\omega([id, \gamma]; 0) = \mathcal{A}_0(\gamma \cdot [p, \widehat{p}]) = -\omega([\gamma \cdot \widehat{p}]) \in \Gamma_\omega$$

where $\gamma \cdot \widehat{p}$ is a disc with the constant boundary map $\widehat{p}$ and so defines a map from the sphere. This finishes the proof of (4.4). □

From the construction, $I_\omega([h, \widetilde{h}])$ is the amount of the shift in $\mathbb{R}$ for the isometry

$$\widetilde{h} : \mathrm{Spec}(\widetilde{\phi}) \to \mathrm{Spec}([h] \cdot \widetilde{\phi});\ [z,w] \mapsto \widetilde{h} \cdot [z,w] \tag{4.6}$$

for any $\widetilde{\phi} \in \widetilde{\mathcal{H}am}(M, \omega)$. What we have established by now is the following generalization of Theorem 3.1 for non-contractile loops.

THEOREM 4.3. *Let $h \in \Omega(\mathcal{H}am(M,\omega))$ be a loop with $[h, \widetilde{h}] \in \pi_0(\widetilde{G})$. Then the amount of shift is independent of the lift $\widetilde{h}$ but only on its homotopy class $[h, \widetilde{h}] \in \pi_0(\widetilde{G})$. In other words, we have*

$$\widetilde{h}^*(\mathcal{A}_{H \# F}) = \mathcal{A}_F + I_\omega([h, \widetilde{h}]) \tag{4.7}$$

for any $F \mapsto \phi$, H generating the loop h and for any lift $(h, \widetilde{h})$.

Now let us consider the values $I_\omega([id, \gamma])$. In the weakly exact case where $\Gamma_\omega = 0$, (4.4) implies that I_ω depends only on $[h] \in \pi_1(\mathcal{H}am(M, \omega))$, but not on the lift $\widetilde{h} \in \mathrm{Homeo}(\widetilde{\Omega}_0(M))$. Therefore it defines a map

$$\overline{I}_\omega : \pi_1(\mathcal{H}am(M, \omega)) \to \mathbb{R}$$

defined by

$$\overline{I}_\omega(a) := I_\omega([h,\widetilde{h}]) = \mathcal{A}_H(\widetilde{h}\cdot[p,\widehat{p}]) = -\int w_p^*\omega - \int_0^1 H(z_p(t),t)\,dt$$

for $a = [h]$ and $(h,\widetilde{h})$ is any lifting of h, and $[z_p, w_p] = \widetilde{h}\cdot[p,\widehat{p}]$. This is precisely the monodromy map considered by Schwarz [14] on the weakly exact (M,ω). He proved that this is trivial for the symplectically aspherical case, i.e., under the additional assumption $c_1 = 0$. It would be interesting to see if this holds true for arbitrary weakly exact (M,ω).

REMARK 4.4. It turns out that the homomorphism

$$I_\omega : \pi_0(\widetilde{G}) \to \mathbb{R}$$

has the interpretation in terms of the Hamiltonian fibration: Each Hamiltonian loop h associates a Hamiltonian fibration $\pi : P \to S^2$ with h as the gluing map. Then each lift $\widetilde{h}$ corresponds to a section $s : S^2 \to P$ (see [15]). We consider a symplectic connection of P and its associated *coupling form* ω_P. According to Seidel [15], the pair (P,s) is called a normalized Hamiltonian bundle. We normalize the area of S^2 to be 1. Then we have the identity

$$I_\omega([h,\widetilde{h}]) = \int_{S^2} s^*\omega_P$$

(see [14]). In terms of this picture, the homomorphism property can be understood as the gluing formula for the integral $\int s^*\omega_P$ of the normalized Hamiltonian bundles (P,s) (see [7], [15] for the discussion of the gluing).

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