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UNIFORM DECAY OF SOLUTIONS FOR VISCOELASTIC PROBLEM WITH NONLINEAR BOUNDARY DAMPING AND MEMORY TERM

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ABSTRACT. We consider the existence of solutions of viscoelastic degenerate problem of Kirchhoff type with nonlinear boundary damping and memory term. Moreover,we consider the uniform decay of the energy for the problem.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^2 boundary $\Gamma := \partial\Omega$ such that $\Gamma = \Gamma_0 \cup \Gamma_1$, $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$ and Γ_0, Γ_1 have positive measures. In this paper, we are concerned with the following problem:

$$
(1.1)\qquad \begin{aligned}
&K(x,t)u'' - (1 + \|\nabla u\|^2)\Delta u - \Delta u' + \int_0^t h(t-\tau)\Delta u(\tau)d\tau = 0 \\
&\qquad\qquad \text{on } Q = \Omega \times (0,\infty), \\
&u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \text{ on } x \in \Omega, \\
&u = 0 \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,\infty), \\
&(1 + \|\nabla u\|^2)\frac{\partial u}{\partial \nu} + \frac{\partial u'}{\partial \nu} \\
&- \int_0^t h(t-\tau)\frac{\partial u}{\partial \nu}(\tau)d\tau + u' + \alpha(t)(|u'|^\rho u' - |u|^\gamma u) = 0 \\
&\qquad\qquad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,\infty),
\end{aligned}
$$

where ν denotes the unit outer normal vector pointing towards Γ. From physical point of view, when $\Gamma_0 = \emptyset$ and $K = I$, equation (1.1) with

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$h=0$ has its roots in the model for the small amplitude vibrations of an elastic string in which the dependence of the tension on the deformation cannot be ignored [7], and equation (1.1) with $h \neq 0$ may be used to describe the dynamics of an extensible string with fading memory.

When $K = I$, there exists many literature about viscoelastic problems, that is, the one with integral memory term, acting in the domain. Among the numerous works in this direction, we can cite Rivera [4]. For the blow-up properties of the solutions of wave equation with nonlinear damping and source term acting in the domain, see Georgiev et al [3]. For the existence results for Kirchhoff type wave equation with $\Gamma_0 = \emptyset$ and $h = 0$, see Brito [1], Matsuyama [5] and Ikehata [6]. Author [8, 9] has studied the existence and uniform decay of solutions of Kirchhoff type wave equations with $h = 0$ and nonlinear boundary damping and nonlinear boundary source term.

On the other hand, Torrejon [10] has considered the global existence of quasilinear wave equation with memory;

$$\begin{aligned}
&u_{tt} - M(\|\nabla u\|^2)\Delta u - \int_0^t h(t-r)N(\|\nabla u(r)\|^2)\Delta u(r)dr = f \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{on } Q = \Omega \times [0,T], \\
&u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on} \quad x \in \Omega, \\
&u = 0 \quad \text{on} \quad \partial\Omega \times [0,T],
\end{aligned}$$

where h, M, N are real valued nondecreasing functions with $M(\cdot) \geq N(\cdot) \geq 0$. Also, Cavalcanti et al [2] have studied the following degenerate equation:

$$\begin{aligned}
&K_1(x,t)u'' + K_2(x,t)u' - \Delta u = 0 \quad \text{on} \quad Q = \Omega \times (0,\infty), \\
&u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \quad \text{on} \quad x \in \Omega, \\
&u = 0 \quad \text{on} \quad \Sigma_1 = \Gamma_1 \times (0,\infty), \\
&\frac{\partial u}{\partial \nu} + u' + \alpha(t)(|u'|^\rho u' - |u|^\gamma u) = 0 \quad \text{on} \quad \Sigma_0 = \Gamma_0 \times (0,\infty).
\end{aligned} \tag{1.2}$$

This kinds of problem is specially related to the study of transonic gas dynamics. Cavalcanti [2] have considered the existence of strong solutions and uniform decay of energy for the above problem (1.2), assuming that $K_1(x,t)$ can vanish on Q, in this case we call the equation is degenerate.

In this paper, we will investigate the existence of solutions of viscoelastic degenerate problem of Kirchhoff type (1.1) with nonlinear boundary damping and memory source term. Moreover, we will consider the uniform decay of the energy of the problem (1.1). To obtain the existence of solutions, we use Faedo-Galerkin's approximation, and also to show the uniform stabilization we use the perturbed energy method.

2. Assumptions and main result

Throughout this paper we define

$$V = \{u \in H^1(\Omega);\ u = 0 \quad \text{on} \quad \Gamma_1\},\ (u, v) = \int_\Omega u(x)v(x)dx,$$

$$(u, v)_\Gamma = \int_\Gamma u(x)v(x)d\Gamma,\ \|u\|_{p,\Gamma}^p = \int_\Gamma |u(x)|^p dx \text{ and } \|u\|_\infty = \|u\|_{L^\infty(\Omega)}.$$

For simplicity we denote $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{2,\Gamma}$ by $\|\cdot\|$ and $\|\cdot\|_\Gamma$, respectively. In order to obtain the existence for regular solutions, we need the following assumptions.

(A_1) Let us consider $u_0 \in V \cap H^2(\Omega)$ and $u_1 \in V$ verifying the compatibility conditions

$$u_0 = \Delta u_0 = \Delta u_1 = 0 \quad \text{on} \quad \Gamma_1,$$

$$(1 + \|\nabla u_0\|^2)\frac{\partial u_0}{\partial \nu} + \frac{\partial u_1}{\partial \nu} + u_1 + \alpha(0)(|u_1|^\rho u_1 - |u_0|^\gamma u_0) = 0 \quad \text{on} \quad \Gamma_0.$$

(A_2) Let K, α be functions in $W^{1,\infty}(0, \infty)$, K, $\alpha \geq 0$ such that

$$-m_0\alpha(t) \leq \alpha'(t) \leq -m_1\alpha(t),\ \ K(t) \geq 0,\ \ -K'(t) \geq \delta > 0\ \forall\, t \geq 0.$$

(A_3) Let the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonnegative and bounded C^2-function such that $l = 1 - \int_0^\infty h(r)dr > 0$ and for some ξ_i, $i = 1, 2, 3$

$$-\xi_1 h(t) \leq h'(t) \leq -\xi_2 h(t) \text{ and } 0 \leq h''(t) \leq \xi_3 h(t) \quad \forall\, t \geq 0.$$

We note that the assumption (A_1) is needed to show the boundedness of the norm of $u''(0)$, and the assumption (A_2) is used in degenerate problem. Now we state our main result.

THEOREM 2.1. *Under assumptions (A_1)-(A_3), we assume $\rho \geq \gamma$ and $0 < \gamma, \rho \leq \frac{1}{n-2}$ if $n \geq 3$, or $\gamma, \rho > 0$ if $n = 1, 2$. Then problem (1.1) has a unique strong solution $u : \Omega \to \mathbb{R}$ such that $u \in L^\infty(0, \infty; V \cap H^2(\Omega))$, $u' \in L^\infty(0, \infty; V)$, $\sqrt{K}u'' \in L^\infty(0, \infty; L^2(\Omega))$, $u'' \in L^2(0, \infty; L^2(\Omega))$. Moreover, if $\rho = \gamma$ and $m_1 > 2(\gamma+2)$, then there exist positive constants C_1 and C_2 such that $E(t) \leq C_1 E(0)\exp(-C_2\epsilon t) \quad t \geq 0$.*

3. Proof of Theorem 2.1

In this section we are going to show the existence of solution for problem (1.1). We represent by $\{w_j\}_{j\in N}$ *a basis in* $V\cap H^2(\Omega)$ *which is orthonormal in* $L^2(\Omega)$, *by* V_m *the finite dimensional subspace of* V *generated by the first* m *vectors. Next we define for each* $\epsilon>0$, $K_\epsilon(t)=K(t)+\epsilon$ *and* $u_{\epsilon m}(t)=\Sigma_{j=1}^{m}\gamma_{\epsilon jm}(t)w_j$, *where* $u_{\epsilon m}(t)$ *is the solution of the following problem:*

(3.1)
$$\begin{aligned}&(K_\epsilon(t)u''_{\epsilon m}(t),w)+(1+\|\nabla u_{\epsilon m}(t)\|^2)(\nabla u_{\epsilon m}(t),\nabla w)+(\nabla u'_{\epsilon m}(t),\nabla w)\\&\quad+(u'_{\epsilon m}(t),w)_{\Gamma_0}+\alpha(t)(|u'_{\epsilon m}(t)|^\rho u'_{\epsilon m}(t),w)_{\Gamma_0}\\&\quad-\alpha(t)(|u_{\epsilon m}(t)|^\gamma u_{\epsilon m}(t),w)_{\Gamma_0}\\&=\int_0^t h(t-r)(\nabla u_{\epsilon m}(r),\nabla w)dr\quad \textit{for all } w\in V_m,\end{aligned}$$

(3.2)
$$\begin{aligned}u_{\epsilon m}(0)=u_{0m}&=\sum_{j=1}^{m}(u_0,w_j)w_j\to u_0\quad \textit{strongly in}\quad V\cap H^2(\Omega),\\u'_{\epsilon m}(0)=u_{1m}&=\sum_{j=1}^{m}(u_1,w_j)w_j\to u_1\quad \textit{strongly in}\quad V,\end{aligned}$$

and $(1+\|\nabla u_{0m}\|^2)\frac{\partial u_{0m}}{\partial\nu}+\frac{\partial u_{1m}}{\partial\nu}+u_{1m}+\alpha(0)(|u_{1m}|^\rho u_{1m}-|u_{0m}|^\gamma u_{0m})=0$ *on* Γ_0. *Note that we can solve the system (3.1) by Picard's iteration method. In fact, the problem (3.1) has a unique solution on some interval* $[0,T_{\epsilon m})$ *depending on* m. *The extension of these solutions to the whole interval* $[0,T]$, *for all* $T>0$, *is a consequence of the first estimate which we are going to prove below.*

A priori estimate I

Multiplying (3.1) by $\gamma'_{\epsilon jm}(t)$, *summing over* j, *we obtain*

(3.3)
$$\begin{aligned}&\frac{d}{dt}[E_m(t)+\frac{\alpha(t)}{\gamma+2}\|u_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}]-\frac{1}{2}(K'(t),|u'_{\epsilon m}(t)|^2)+\|u'_{\epsilon m}(t)\|^2_{\Gamma_0}\\&\quad+\|\nabla u'_{\epsilon m}(t)\|^2+\alpha(t)\|u'_{\epsilon m}(t)\|^{\rho+2}_{\rho+2,\Gamma_0}+h(0)\|\nabla u_{\epsilon m}(t)\|^2\\&=2\alpha(t)(|u_{\epsilon m}(t)|^\gamma u_{\epsilon m}(t),u'_{\epsilon m}(t))_{\Gamma_0}+\frac{\alpha'(t)}{\gamma+2}\|u_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}\\&\quad+\frac{d}{dt}\int_0^t h(t-r)(\nabla u_{\epsilon m}(r),\nabla u_{\epsilon m}(t))dr\end{aligned}$$

$$-\int_0^t h'(t-r)(\nabla u_{\epsilon m}(r), \nabla u_{\epsilon m}(t))dr,$$

where $E_m(t) = \frac{1}{2}\|\sqrt{K_\epsilon(t)}u'_{\epsilon m}(t)\|^2 + \frac{1}{2}\|\nabla u_{\epsilon m}(t)\|^2 + \frac{1}{4}\|\nabla u_{\epsilon m}(t)\|^4$. *Note that Hölder's inequality and Young's inequality give us*

$$\begin{aligned}(3.4)\qquad & 2\alpha(t)(|u_{\epsilon m}(t)|^\gamma u_{\epsilon m}(t), u'_{\epsilon m}(t))_{\Gamma_0} \\ & \le 2\alpha(t)\|u_{\epsilon m}(t)\|^{\gamma+1}_{\gamma+2,\Gamma_0}\|u'_{\epsilon m}(t)\|_{\gamma+2,\Gamma_0} \\ & \le C_1(\eta)\alpha(t)\|u_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \eta\alpha(t)\|u'_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}.\end{aligned}$$

Since $\rho \ge \gamma$, $L^{\rho+2}(\Gamma_0) \hookrightarrow L^{\gamma+2}(\Gamma_0)$ *and therefore we can obtain*

$$(3.5)\qquad \eta\|u'_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \le C_2(\eta) + \eta\|u'_{\epsilon m}(t)\|^{\rho+2}_{\rho+2,\Gamma_0}.$$

Considering Schwarz's inequality and taking the assumption (A_3) *into account, we deduce*

$$\begin{aligned}(3.6)\qquad & \int_0^t h'(t-r)(\nabla u_{\epsilon m}(r), \nabla u_{\epsilon m}(t))dr \\ & \le \xi_1\|\nabla u_{\epsilon m}(t)\|\int_0^t h(t-r)\|\nabla u_{\epsilon m}(r)\|dr \\ & \le \frac{\xi_1^2}{2}\|\nabla u_{\epsilon m}(t)\|^2 + \frac{1}{2}\|h\|_{L^1(0,\infty)}\int_0^t h(t-r)\|\nabla u_{\epsilon m}(r)\|^2dr.\end{aligned}$$

Combining the above inequalities and integrating it over $(0,t)$, *we have*
(3.7)

$$\begin{aligned} & E_m(t) + \frac{\alpha(t)}{\gamma+2}\|u_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \frac{\delta}{2}\int_0^t \|u'_{\epsilon m}(s)\|^2ds \\ & + \int_0^t \|\nabla u'_{\epsilon m}(s)\|^2ds + (1-\eta)\int_0^t \alpha(s)\|u'_{\epsilon m}(s)\|^{\rho+2}_{\rho+2,\Gamma_0}ds \\ & + \int_0^t \|u'_{\epsilon m}(s)\|^2_{\Gamma_0}ds \\ \le\ & E_m(0) + \frac{\alpha(0)}{\gamma+2}\|u_{0m}\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\ & + \int_0^t \alpha(s)(C_2(\eta) + C_1(\eta)\|u_{\epsilon m}(s)\|^{\gamma+2}_{\gamma+2,\Gamma_0})ds \\ & + (\frac{\xi_1^2}{2} - h(0))\int_0^t \|\nabla u_{\epsilon m}(s)\|^2ds + \int_0^t h(t-r)(\nabla u_{\epsilon m}(r), \nabla u_{\epsilon m}(t))dr \\ & + \frac{1}{2}\|h\|_{L^1(0,\infty)}\int_0^t\int_0^s h(s-r)\|\nabla u_{\epsilon m}(r)\|^2drds.\end{aligned}$$

Taking Fubini's Theorem into account, we deduce

$$
\begin{aligned}
&\frac{1}{2}\|h\|_{L^1(0,\infty)}\int_0^t\int_0^s h(s-r)\|\nabla u_{\epsilon m}(r)\|^2 dr ds \\
&\le \frac{1}{2}\|h\|^2_{L^1(0,\infty)}\int_0^t \|\nabla u_{\epsilon m}(r)\|^2 dr.
\end{aligned} \tag{3.8}
$$

Also, Schwarz's inequality implies

$$
\begin{aligned}
&\int_0^t h(t-r)(\nabla u_{\epsilon m}(r),\nabla u_{\epsilon m}(t))dr \\
&\le \|\nabla u_{\epsilon m}(t)\|\int_0^t h(t-r)\|\nabla u_{\epsilon m}(r)\|dr \\
&\le \frac{1}{4}\|\nabla u_{\epsilon m}(t)\|^2 + \|h\|_{L^1(0,\infty)}\|h\|_{L^\infty(0,\infty)}\int_0^t \|\nabla u_{\epsilon m}(r)\|^2 dr.
\end{aligned} \tag{3.9}
$$

Combining the above inequalities (3.7)-(3.9), employing Gronwall's lemma we obtain the first estimate:

$$
\begin{aligned}
&\|\sqrt{K_\epsilon(t)}u'_{\epsilon m}(t)\|^2 + \|\nabla u_{\epsilon m}(t)\|^2 + \alpha(t)\|u_{\epsilon m}(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} \\
&+ \int_0^t (\|u'_{\epsilon m}(s)\|^2 + \|\nabla u'_{\epsilon m}(s)\|^2 + \|u'_{\epsilon m}(s)\|^2_{\Gamma_0} \\
&+ \alpha(s)\|u'_{\epsilon m}(s)\|^{\rho+2}_{\rho+2,\Gamma_0})ds \le L_1,
\end{aligned} \tag{3.10}
$$

where $L_1 > 0$ is a positive constant independent of m and $t > 0$.

A priori estimate II

First of all, we are estimating $u''_{\epsilon m}(0)$ in the L^2-norm. Considering $w = u''_{\epsilon m}(0)$ in (3.1), from the fact $(1+\|\nabla u_{0m}\|^2)\frac{\partial u_{0m}}{\partial \nu} + \frac{\partial u_{1m}}{\partial \nu} + u_{1m} + \alpha(0)(|u_{1m}|^\rho u_{1m} - |u_{0m}|^\gamma u_{0m}) = 0$ on Γ_0, we have

$$
\begin{aligned}
\|\sqrt{K_\epsilon(0)}u''_{\epsilon m}(0)\|^2 &= (1+\|\nabla u_{0m}\|^2)(\Delta u_{0m}, u''_{\epsilon m}(0)) + (\Delta u_{1m}, u''_{\epsilon m}(0)) \\
&\le [(1+\|\nabla u_{0m}\|^2)\|\Delta u_{0m}\| + \|\Delta u_{1m}\|]\|u''_{\epsilon m}(0)\|.
\end{aligned}
$$

Since $\sqrt{K_\epsilon(0)} \ge \epsilon > 0$,

$$
\|u''_{\epsilon m}(0)\| \le L_2, \tag{3.11}
$$

where $L_2 > 0$ is a positive constant independent of m and $t > 0$.

Now, getting the derivative of (3.1), multiplying the result by $\gamma''_{\epsilon jm}(t)$ and summing over j, we have

(3.12)

$$\begin{aligned}
&\frac{d}{dt}E_{1m}(t) + \frac{1}{2}(K'(t), |u''_{\epsilon m}(t)|^2) + \|\nabla u''_{\epsilon m}(t)\|^2 + \|u''_{\epsilon m}(t)\|^2_{\Gamma_0} \\
&+ \alpha'(t)(|u'_{\epsilon m}(t)|^\rho u'_{\epsilon m}(t), u''_{\epsilon m}(t))_{\Gamma_0} \\
&+ (\rho+1)\alpha(t)(|u'_{\epsilon m}(t)|^\rho, (u''_{\epsilon m}(t))^2)_{\Gamma_0} \\
=\ & \alpha'(t)(|u_{\epsilon m}(t)|^\gamma u_{\epsilon m}(t), u''_{\epsilon m}(t))_{\Gamma_0} \\
&+ (\gamma+1)\alpha(t)(|u_{\epsilon m}(t)|^\gamma u'_{\epsilon m}(t), u''_{\epsilon m}(t))_{\Gamma_0} \\
&+ 3(\nabla u_{\epsilon m}(t), \nabla u'_{\epsilon m}(t))\|\nabla u'_{\epsilon m}(t)\|^2 + h(0)\frac{d}{dt}(\nabla u_{\epsilon m}(t), \nabla u'_{\epsilon m}(t)) \\
&- h(0)\|\nabla u'_{\epsilon m}(t)\|^2 + \frac{d}{dt}\int_0^t h'(t-r)(\nabla u_{\epsilon m}(r), \nabla u'_{\epsilon m}(t))dr \\
&- h'(0)(\nabla u_{\epsilon m}(t), \nabla u'_{\epsilon m}(t)) - \int_0^t h''(t-r)(\nabla u_{\epsilon m}(r), \nabla u'_{\epsilon m}(t))dr,
\end{aligned}$$

here $E_{1m}(t) = \frac{1}{2}[\|\sqrt{K_\epsilon(t)}u''_{\epsilon m}(t)\|^2 + (1 + \|\nabla u_{\epsilon m}(t)\|^2)\|\nabla u'_{\epsilon m}(t)\|^2 + 2(\nabla u_{\epsilon m}(t), \nabla u'_{\epsilon m}(t))^2]$. *Now, Hölder's inequality and Young's inequality give us*

$$\begin{aligned}
&\alpha'(t)(|u_{\epsilon m}(t)|^\gamma u_{\epsilon m}(t), u''_{\epsilon m}(t))_{\Gamma_0} \\
&\le m_0\alpha(t)\|u_{\epsilon m}(t)\|^{\gamma+1}_{2\gamma+2,\Gamma_0}\|u''_{\epsilon m}(t)\|_{\Gamma_0} \\
&\le C_3(\eta)\|\alpha\|^2_{L^\infty(0,\infty)}\|\nabla u_{\epsilon m}(t)\|^{2(\gamma+1)} + \eta\|u''_{\epsilon m}(t)\|^2_{\Gamma_0} \\
&\le C_4(\eta)\|\alpha\|^2_{L^\infty(0,\infty)}L_1^\gamma\|\nabla u_{\epsilon m}(t)\|^2 + \eta\|u''_{\epsilon m}(t)\|^2_{\Gamma_0}
\end{aligned} \tag{3.13}$$

and

(3.14)

$$\begin{aligned}
\alpha'(t)(|u'_{\epsilon m}(t)|^\rho u'_{\epsilon m}(t), u''_{\epsilon m}(t))_{\Gamma_0} &\le m_0^2\alpha(t)\|u'_{\epsilon m}(t)\|^{\rho+2}_{\rho+2,\Gamma_0} \\
&+ \eta\alpha(t)(|u'_{\epsilon m}(t)|^\rho, (u''_{\epsilon m}(t))^2)_{\Gamma_0}.
\end{aligned}$$

Considering the same argument, we obtain

$$\begin{aligned}
&(\gamma+1)\alpha(t)(|u_{\epsilon m}(t)|^\gamma u'_{\epsilon m}(t), u''_{\epsilon m}(t))_{\Gamma_0} \\
&\le \alpha(t)(\gamma+1)\|u_{\epsilon m}(t)\|^\gamma_{2\gamma+2,\Gamma_0}\|u'_{\epsilon m}(t)\|_{2\gamma+2,\Gamma_0}\|u''_{\epsilon m}(t)\|_{\Gamma_0} \\
&\le C_5(\eta)\|\alpha\|^2_{L^\infty(0,\infty)}\|\nabla u_{\epsilon m}(t)\|^{2\gamma}\|\nabla u'_{\epsilon m}(t)\|^2 + \eta\|u''_{\epsilon m}(t)\|^2_{\Gamma_0} \\
&\le C_6(\eta)\|\alpha\|^2_{L^\infty(0,\infty)}L_1^\gamma\|\nabla u'_{\epsilon m}(t)\|^2 + \eta\|u''_{\epsilon m}(t)\|^2_{\Gamma_0}.
\end{aligned} \tag{3.15}$$

Combining the arguments (3.13)-(3.15) and assumption on K, integrating it over $(0,t)$, we have

$$
\begin{aligned}
& E_{1m}(t) + \int_0^t \|\nabla u''_{\epsilon m}(s)\|^2 ds \\
& + (1-2\eta)\int_0^t \|u''_{\epsilon m}(s)\|^2_{\Gamma_0} ds \\
& + (\rho+1-\eta)\alpha(t)(|u'_{\epsilon m}(t)|^\rho, (u''_{\epsilon m}(t))^2)_{\Gamma_0} \\
\le\ & E_{1m}(0) + 3\int_0^t \|\nabla u_{\epsilon m}(s)\|\|\nabla u'_{\epsilon m}(s)\|^3 ds \\
& + \frac{1}{2}\|K'\|_{L^\infty(0,\infty)}\int_0^t \|u''_{\epsilon m}(s)\|^2 ds \\
& + h(0)\|\nabla u_{\epsilon m}(t)\|\|\nabla u'_{\epsilon m}(t)\| \\
& + h(0)\|\nabla u_{0m}\|\|\nabla u_{1m}\| + m_0^2\int_0^t \alpha(s)\|u'_{\epsilon m}(s)\|^{\rho+2}_{\rho+2,\Gamma_0} ds \\
& + C_1(h)\int_0^t \|\nabla u_{\epsilon m}(r)\|^2 dr + C_2(h)\int_0^t \|\nabla u'_{\epsilon m}(s)\|^2 ds,
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
C_1(h) = & \frac{1}{2}[\|h\|^2_{L^1(0,\infty)} + \|h\|_{L^1(0,\infty)}\|h\|_{L^\infty(0,\infty)} + |h'(0)|] \\
& + \frac{1}{2}C_4(\eta)L_1^\gamma\|\alpha\|^2_{L^\infty(0,\infty)}
\end{aligned}
$$

and

$$
C_2(h) = \frac{\xi_1^2+\xi_3^2+|h'(0)|}{2} - h(0) + C_6(\eta)\|\alpha\|^2_{L^\infty(0,\infty)}L_1^\gamma.
$$

Considering the first estimate and employing Gronwall's inequality, for sufficiently small $\eta > 0$, we have

$$
\begin{aligned}
& \|\sqrt{K_\epsilon(t)}u''_{\epsilon m}(t)\|^2 + \|\nabla u'_{\epsilon m}(t)\|^2 \\
& + \int_0^t \Big(\|\nabla u''_{\epsilon m}(s)\|^2 + \|u''_{\epsilon m}(s)\|^2 + \|u''_{\epsilon m}(s)\|^2_{\Gamma_0}\Big) ds \le L_3,
\end{aligned}
\tag{3.17}
$$

where L_3 is a positive constant independent of $m \in N$ and $t \in [0,T]$.

By the estimates we can extract subsequence $(u_{\epsilon\mu})$ of $(u_{\epsilon m})$ such that

$$u_{\epsilon\mu} \rightarrow u_\epsilon \quad \text{weak star} \quad L^\infty(0,T;V), \tag{3.18}$$

$$u'_{\epsilon\mu} \rightarrow u'_\epsilon \quad \text{weak star} \quad L^\infty(0,T;V), \tag{3.19}$$

$$\sqrt{K_\epsilon}u''_{\epsilon\mu} \rightarrow \sqrt{K_\epsilon}u''_\epsilon \quad \text{weak star} \quad L^\infty(0,T;L^2(\Omega)), \tag{3.20}$$

$$u_{\epsilon\mu} \rightarrow u_\epsilon \quad \text{weak star} \quad L^\infty(0,T;L^{\gamma+2}(\Gamma_0)), \tag{3.21}$$

$$u'_{\epsilon\mu} \rightarrow u'_\epsilon \quad \text{weak star} \quad L^\infty(0,T;L^{\rho+2}(\Gamma_0)), \tag{3.22}$$

$$u'_{\epsilon\mu} \rightarrow u'_\epsilon \quad \text{weak} \quad L^2(0,T;L^2(\Gamma_0)), \tag{3.23}$$

$$u''_{\epsilon\mu} \rightarrow u''_\epsilon \quad \text{weak} \quad L^2(0,T;L^2(\Gamma_0)). \tag{3.24}$$

The convergence (3.18)-(3.20) and (3.23) are sufficient to pass to the limit in the linear terms of problem (3.1). Next we are going to consider the nonlinear ones.

Analysis of the nonlinear terms

From the first and second estimates we deduce

$$(u_{\epsilon m}) \quad \text{is bounded in} \quad L^2(0,T;H^{\frac{1}{2}}(\Gamma_0)), \tag{3.25}$$

$$(u'_{\epsilon m}) \quad \text{is bounded in} \quad L^2(0,T;H^{\frac{1}{2}}(\Gamma_0)), \tag{3.26}$$

$$(u''_{\epsilon m}) \quad \text{is bounded in} \quad L^2(0,T;L^2(\Gamma_0)). \tag{3.27}$$

From (3.25)-(3.27), taking into consideration that the imbedding $H^{\frac{1}{2}}(\Gamma) \hookrightarrow L^2(\Gamma)$ is continuous and compact and using Aubin compactness theorem, we can extract a subsequence $(u_{\epsilon\mu})$ of $(u_{\epsilon m})$ such that

$$u_{\epsilon\mu} \rightarrow u_\epsilon \quad \text{a.e. on} \quad \Sigma_0 \quad \text{and} \quad u'_{\epsilon\mu} \rightarrow u'_\epsilon \quad \text{a.e. on} \quad \Sigma_0 \tag{3.28}$$

and therefore

$$|u_{\epsilon\mu}|^\gamma u_{\epsilon\mu} \to |u_\epsilon|^\gamma u_\epsilon \quad \text{and} \quad |u'_{\epsilon\mu}|^\rho u'_{\epsilon\mu} \to |u'_\epsilon|^\rho u'_\epsilon \quad \text{a.e. on} \quad \Sigma_0. \tag{3.29}$$

On the other hand, from the first and second estimate we obtain

$$(|u_{\epsilon\mu}|^\gamma u_{\epsilon\mu}) \quad \text{is bounded in} \quad L^2(\Sigma_0), \tag{3.30}$$

$$(|u'_{\epsilon\mu}|^\rho u'_{\epsilon\mu}) \quad \text{is bounded in} \quad L^2(\Sigma_0). \tag{3.31}$$

Combining (3.29)-(3.31), we deduce that

$$\begin{aligned} &|u_{\epsilon\mu}|^\gamma u_{\epsilon\mu} \to |u_\epsilon|^\gamma u_\epsilon \quad \text{weakly in} \quad L^2(\Sigma_0), \\ &|u'_{\epsilon\mu}|^\rho u'_{\epsilon\mu} \to |u'_\epsilon|^\rho u'_\epsilon \quad \text{weakly in} \quad L^2(\Sigma_0). \end{aligned} \tag{3.32}$$

The last convergence is sufficient to pass to the limit in the nonlinear terms of problem (3.1). Using standard arguments, we can show from the above estimates that

(3.33)

$$K(t)u''(t) - (1 + \|\nabla u(t)\|^2)\Delta u(t) - \Delta u'(t) - \int_0^t h(t-r)\Delta u(r)dr = 0 \quad \text{in } L^2_{loc}(0,T;L^2(\Omega)),$$

and also making use of the generalized Green's formula we deduce that

$$\begin{aligned} &(1 + \|\nabla u(t)\|^2)\frac{\partial u}{\partial \nu}(t) + \frac{\partial u'}{\partial \nu}(t) - \int_0^t h(t-r)\frac{\partial u}{\partial \nu}(r)dr + u'(t) \\ &+ \alpha(t)(|u'(t)|^\rho u'(t) - |u(t)|^\gamma u(t)) = 0 \quad \text{in} \quad L^2_{loc}(0,T;L^2(\Gamma_0)). \end{aligned} \tag{3.34}$$

This completes the proof of the existence of solutions of problem (1.1).

Uniqueness

Let u and v be strong solutions to problem (1.1). Define $z = u - v$, then we deduce

$$\begin{aligned} &(K(t)z''(t), w) + (z'(t), w)_{\Gamma_0} + \alpha(t)(|u'(t)|^\rho u'(t) - |v'(t)|^\rho v'(t), w)_{\Gamma_0} \\ &+ (\nabla z'(t), \nabla w) + \alpha(t)(|u(t)|^\gamma u(t) - |v(t)|^\gamma v(t), w)_{\Gamma_0} \\ = &\int_0^t h(t-r)(\nabla z(r), \nabla w)dr \\ &+ (1 + \|\nabla v(t)\|^2)(\nabla v(t), \nabla w) - (1 + \|\nabla u(t)\|^2)(\nabla u(t), \nabla w). \end{aligned}$$

Substituting $w = z'(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{K(t)}z'(t)\|^2 - K'(t)\|z'(t)\|^2 + \|z'(t)\|^2_{\Gamma_0} + \|\nabla z'(t)\|^2$$
$$+ \alpha(t)(|u'(t)|^\rho u'(t) - |v'(t)|^\rho v'(t), z'(t))_{\Gamma_0}$$
$$= \alpha(t)(|v(t)|^\gamma v(t) - |u(t)|^\gamma u(t), z'(t))_{\Gamma_0} + \int_0^t h(t-r)(\nabla z(r), \nabla z'(t))dr$$
$$+ (1 + \|\nabla v(t)\|^2)(\nabla v(t), \nabla z'(t)) - (1 + \|\nabla u(t)\|^2)(\nabla u(t), \nabla z'(t)).$$

On the other hand, we note that

$$(1 + \|\nabla v(t)\|^2)(\nabla v(t), \nabla z'(t)) - (1 + \|\nabla u(t)\|^2)(\nabla u(t), \nabla z'(t))$$
$$= -\frac{1}{2}\frac{d}{dt}[(1 + \|\nabla u(t)\|^2)\|\nabla z(t)\|^2] + (\nabla u(t), \nabla u'(t))\|\nabla z(t)\|^2$$
$$+ [\|\nabla v(t)\|^2 - \|\nabla u(t)\|^2](\nabla v(t), \nabla z'(t))$$
$$\le -\frac{1}{2}\frac{d}{dt}[(1 + \|\nabla u(t)\|^2)\|\nabla z(t)\|^2] + C_1(\eta)\|\nabla z(t)\|^2 + \eta\|\nabla z'(t)\|^2.$$

Also

$$\alpha(t)(|v(t)|^\gamma v(t) - |u(t)|^\gamma u(t), z'(t))_{\Gamma_0}$$
$$\le C(\gamma)\int_{\Gamma_0}(|u(t)|^\gamma + |v(t)|^\gamma)|z(t)||z'(t)|d\Gamma$$
$$\le C_1(\gamma)[\|u(t)\|^\gamma_{\gamma+2,\Gamma_0} + \|v(t)\|^\gamma_{\gamma+2,\Gamma_0}]\|z(t)\|_{2\gamma+2,\Gamma_0}\|z'(t)\|_{\Gamma_0}$$
$$\le C_2(\gamma,\eta)\|\nabla z(t)\|^2 + \eta\|z'(t)\|^2_{\Gamma_0}$$

and

$$\int_0^t h(t-r)(\nabla z(r), \nabla z'(t))dr \le \|\nabla z'(t)\|\int_0^t h(t-r)\|\nabla z(r)\|dr$$
$$\le \|\nabla z'(t)\|\|h\|^{\frac{1}{2}}_{L^1(0,\infty)}\left(\int_0^t h(t-r)\|\nabla z(r)\|^2dr\right)^{\frac{1}{2}}$$
$$\le \eta\|\nabla z'(t)\|^2 + C_3(\eta)\|h\|_{L^1(0,\infty)}\int_0^t h(t-r)\|\nabla z(r)\|^2dr.$$

Thus we have

$$\frac{1}{2}\frac{d}{dt}[\|\sqrt{K(t)}z'(t)\|^2 + (1 + \|\nabla u(t)\|^2)\|\nabla z(t)\|^2] + (1-2\eta)\|\nabla z'(t)\|^2$$
$$+ \delta\|z'(t)\|^2 + (1-\eta)\|z'(t)\|^2_{\Gamma_0}$$
$$\le [C_1(\eta) + C_2(\gamma,\eta)]\|\nabla z(t)\|^2$$
$$+ C_3(\eta)\|h\|_{L^1(0,\infty)}\int_0^t h(t-r)\|\nabla z(r)\|^2dr.$$

Integrating it over $(0,t)$, *we have*

$$\begin{aligned}&\frac{1}{2}[\|\sqrt{K(t)}z'(t)\|^2+(1+\|\nabla u(t)\|^2)\|\nabla z(t)\|^2]\\&+(1-2\eta)\int_0^t\|\nabla z'(s)\|^2ds\\&+\delta\int_0^t\|z'(s)\|^2ds+(1-\eta)\int_0^t\|z'(s)\|^2_{\Gamma_0}ds\\\leq&\ [C_1(\eta)+C_2(\gamma,\eta)+C_3(\eta)\|h\|^2_{L^1(0,\infty)}]\int_0^t\|\nabla z(s)\|^2ds,\end{aligned}$$

where we have used

$$\int_0^t\int_0^s h(s-r)\|\nabla z(r)\|^2drds\leq\|h\|_{L^1(0,\infty)}\int_0^t\|\nabla z(s)\|^2ds.$$

Using Gronwall inequality, we have $\|z'(t)\|=\|\nabla z(t)\|=0$. *This concludes the proof of uniqueness for strong solutions.* □

4. Uniform decay

In this section we prove the decay estimates for the energy of (1.1). We define the energy $E(t)$ *of the problem (1.1) by*

$$E(t)=\frac{1}{2}\|\sqrt{K(t)}u'(t)\|^2+\frac{1}{2}\|\nabla u(t)\|^2+\frac{1}{4}\|\nabla u(t)\|^4. \tag{4.1}$$

Then the derivative of energy is given by

$$\begin{aligned}E'(t)=&\frac{1}{2}(K'(t),|u'(t)|^2)+\int_0^t h(t-r)(\nabla u(r),\nabla u'(t))dr-\|\nabla u'(t)\|^2\\&-\|u'(t)\|^2_{\Gamma_0}-\alpha(t)\|u'(t)\|^{\rho+2}_{\rho+2,\Gamma_0}+\alpha(t)(|u(t)|^\gamma u(t),u'(t))_{\Gamma_0}.\end{aligned} \tag{4.2}$$

We define $(h\,\Box\,u)$ *and the modified energy* $e(t)$ *by*

$$(h\,\Box\,u)(t):=\int_0^t h(t-r)\|u(t)-u(r)\|^2dr, \tag{4.3}$$

$$\begin{aligned}e(t)=&\frac{1}{2}\|\sqrt{K(t)}u'(t)\|^2+\frac{1}{4}\|\nabla u(t)\|^4+\frac{1}{2}(h\,\Box\,\nabla u)(t)\\&+\frac{1}{2}(1-\int_0^t h(r)dr)\|\nabla u(t)\|^2+\frac{1}{\gamma+2}\alpha(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}.\end{aligned} \tag{4.4}$$

Then we have

(4.5)

$$e'(t) = \frac{1}{2}(K'(t), |u'(t)|^2) - \|\nabla u'(t)\|^2 - \|u'(t)\|^2_{\Gamma_0} - \alpha(t)\|u'(t)\|^{\rho+2}_{\rho+2,\Gamma_0}$$
$$+ \frac{1}{\gamma+2}\alpha'(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + 2\alpha(t)(|u(t)|^\gamma u(t), u'(t))_{\Gamma_0}$$
$$- \frac{1}{2}h(t)\|\nabla u(t)\|^2 + \frac{1}{2}(h' \square \nabla u)(t).$$

Considering Young's inequality, we get

(4.6)

$$2\alpha(t)(|u(t)|^\gamma u(t), u'(t))_{\Gamma_0} \leq \eta\alpha(t)\|u'(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} + \eta^{-\frac{1}{\gamma+1}}\alpha(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}.$$

Thus for $\gamma = \rho$, the assumption (A_2) implies

(4.7)

$$e'(t) \leq -\frac{\delta}{2}\|u'(t)\|^2 - \|\nabla u'(t)\|^2 - \|u'(t)\|^2_{\Gamma_0} - \frac{\xi_2}{2}(h \square \nabla u)(t)$$
$$- \alpha(t)(1-\eta)\|u'(t)\|^{\rho+2}_{\rho+2,\Gamma_0} - \alpha(t)(\frac{m_1}{\gamma+2} - \eta^{-\frac{1}{\gamma+1}})\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}$$
$$- \frac{1}{2}h(t)\|\nabla u(t)\|^2.$$

Choosing $\eta = 2^{-(\gamma+1)}$ then $1 - \eta > \frac{1}{2}$ and

(4.8)

$$e'(t) \leq -\frac{\delta}{2}\|u'(t)\|^2 - \|\nabla u'(t)\|^2 - \|u'(t)\|^2_{\Gamma_0} - \frac{1}{2}\alpha(t)\|u'(t)\|^{\rho+2}_{\rho+2,\Gamma_0}$$
$$- \beta\alpha(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0} - \frac{1}{2}h(t)\|\nabla u(t)\|^2 - \frac{\xi_2}{2}(h \square \nabla u)(t),$$

where $\beta = \frac{m_1}{\gamma+2} - 2 > 0$. On the other hand we note that from assumption (A_3)

(4.9)

$$E(t) \leq \frac{1}{2}\|\sqrt{K(t)}u'(t)\|^2 + \frac{1}{2l}(1 - \int_0^t h(r)dr)\|\nabla u(t)\|^2 + \frac{1}{4}\|\nabla u(t)\|^4$$
$$\leq l^{-1}e(t)$$

and therefore it is enough to obtain the desired exponential decay for the modified energy $e(t)$ which will be done below.

For this purpose let λ be the positive number such that $\|v\|^2 \leq \lambda\|\nabla v\|^2$, $\forall\, v \in V$ and for every $\epsilon > 0$ let us define the perturbed modified energy by

$$e_\epsilon(t) = e(t) + \epsilon\psi(t), \quad \text{where} \quad \psi(t) = (K(t)u'(t), u(t)).$$

Applying Cauchy Schwarz's inequality, we easily obtain the following inequality.

PROPOSITION 4.1. *We have the inequality for any $\epsilon > 0$*

$$| e_\epsilon(t) - e(t)| \le \epsilon\lambda^{\frac{1}{2}}\|K\|_\infty^{\frac{1}{2}} e(t), \quad \forall\, t \ge 0.$$

Now, we can prove the decay estimates by the following Proposition.

PROPOSITION 4.2. *There exist $C_1 > 0$ and ϵ_1 such that for $\epsilon \in (0, \epsilon_1]$*

$$e'_\epsilon(t) \le -\epsilon C_1 e(t).$$

Proof. Using the equation (1.1), we have

(4.10)

$$\begin{aligned}\psi'(t) = & -e(t) + \frac{3}{2}\|\sqrt{K(t)}u'(t)\|^2 + (K'(t)u'(t), u(t)) - \frac{1}{2}\|\nabla u(t)\|^2 \\ & - \frac{3}{4}\|\nabla u(t)\|^4 - (\nabla u'(t), \nabla u(t)) \\ & + \frac{1}{2}(h \,\Box\, \nabla u)(t) - (u'(t), u(t))_{\Gamma_0} \\ & - \frac{1}{2}\int_0^t h(r)dr\|\nabla u(t)\|^2 - \alpha(t)(|u'(t)|^\rho u'(t), u(t))_{\Gamma_0} \\ & + \frac{\gamma+1}{\gamma+2}\alpha(t)\|u(t)\|_{\gamma+2,\Gamma_0}^{\gamma+2} + \int_0^t h(t-r)(\nabla u(r), \nabla u(t))dr.\end{aligned}$$

Now applying Schwarz's inequality and (4.3), we get

(4.11)

$$\int_0^t h(t-r)(\nabla u(r), \nabla u(t))dr \le \frac{1}{2}(h \,\Box\, \nabla u)(t) + \frac{3}{2}\|\nabla u(t)\|^2 \int_0^t h(r)dr,$$

(4.12)

$$(K'(t)u'(t), u(t)) \le \lambda\theta_1(\eta)\|K'\|_\infty\|u'(t)\|^2 + \eta\|\nabla u(t)\|^2,$$

(4.13)

$$|(|u'(t)|^\rho u'(t), u(t))_{\Gamma_0}| \le \theta_2(\eta)\|u'(t)\|_{\rho+2,\Gamma_0}^{\rho+2} + \eta\|u(t)\|_{\rho+2,\Gamma_0}^{\rho+2},$$

(4.14)

$$|(u'(t), u(t))_{\Gamma_0}| \le \eta\|\nabla u(t)\|^2 + \frac{\mu}{4\eta}\|u'(t)\|_{\Gamma_0}^2,$$

(4.15)

$$|(\nabla u'(t), \nabla u(t))| \le \eta\|\nabla u(t)\|^2 + \frac{1}{4\eta}\|\nabla u'(t)\|^2,$$

where $\mu > 0$ *such that* $\int_{\Gamma_0} |u|^2 d\Gamma \leq \mu \int_\Omega |\nabla u|^2 dx \quad \forall\, u \in V$. *Combining the results in (4.11)-(4.15), we have*

(4.16)

$$\begin{aligned}\psi'(t) \leq & -e(t) + (\frac{3}{2}\|K\|_\infty + \lambda\theta_1(\eta)\|K'\|_\infty)\|u'(t)\|^2 - \frac{3}{4}\|\nabla u(t)\|^4 \\ & - (\frac{1}{2} - 3\eta - \int_0^t h(r)dr)\|\nabla u(t)\|^2 + (h \,\Box\, \nabla u)(t) + \frac{\mu}{4\eta}\|u'(t)\|^2_{\Gamma_0} \\ & + \frac{1}{4\eta}\|\nabla u'(t)\|^2 + \theta_2(\eta)\alpha(t)\|u'(t)\|^{\rho+2}_{\rho+2,\Gamma_0} + \eta\alpha(t)\|u(t)\|^{\rho+2}_{\rho+2,\Gamma_0} \\ & + \frac{\gamma+1}{\gamma+2}\alpha(t)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}.\end{aligned}$$

Choosing small $\eta > 0$ *and* $\|h\|_{L^1(0,\infty)}$ *such that* $\frac{1}{2} - 3\eta - \int_0^t h(r)dr \geq 0$. *Thus from (4.8), (4.16) and the assumption* (A_2) *and considering* $\rho = \gamma$, *we get*

(4.17)

$$\begin{aligned}e'_\epsilon(t) =& \; e'(t) + \epsilon\psi'(t) \\ \leq & -\epsilon e(t) - (\frac{\delta}{2} - \epsilon[\frac{3}{2}\|K\|_\infty + \lambda\theta_1(\eta)\|K'\|_\infty])\|u'(t)\|^2 \\ & - \frac{3\epsilon}{4}\|\nabla u(t)\|^4 - (\frac{\xi_2}{2} - \epsilon)(h \,\Box\, \nabla u)(t) - (1 - \frac{\mu\epsilon}{4\eta})\|u'(t)\|^2_{\Gamma_0} \\ & - (1 - \frac{\epsilon}{4\eta})\|\nabla u'(t)\|^2 - \alpha(t)(\frac{1}{2} - \epsilon\theta_2(\eta))\|u'(t)\|^{\rho+2}_{\rho+2,\Gamma_0} \\ & - \alpha(t)(\beta - (\eta + \frac{\gamma+1}{\gamma+2})\epsilon)\|u(t)\|^{\gamma+2}_{\gamma+2,\Gamma_0}.\end{aligned}$$

Defining $\epsilon_1 = \min\{\frac{\xi_2}{2}, \frac{4\eta}{\mu}, 4\eta, \frac{1}{2\theta_2(\eta)}, \frac{\beta(\gamma+2)}{(\gamma+2)\eta+\gamma+1}, \frac{\delta}{3\|K\|_\infty + 2\lambda\theta_1(\eta)\|K'\|_\infty}\}$. *Then for each* ϵ *such that* $\epsilon \in (0, \epsilon_1]$, *we have*

$$e'_\epsilon(t) \leq -\epsilon C_1 e(t) \tag{4.18}$$

if $\|h\|_{L^1(0,\infty)}$ *is sufficiently small.* □

Continuing the proof of Theorem 2.1. Let $\epsilon_0 = \min\{\frac{1}{2\lambda^{\frac{1}{2}}\|K\|_\infty^{\frac{1}{2}}}, \epsilon_1\}$ and let us consider $\epsilon \in (0, \epsilon_0]$. As we have $\epsilon < \frac{1}{2\lambda^{\frac{1}{2}}\|K\|_\infty^{\frac{1}{2}}}$, we conclude from Proposition 4.1

(4.19)

$$\frac{1}{2}e(t) < (1 - \epsilon\lambda^{\frac{1}{2}}\|K\|_\infty^{\frac{1}{2}})e(t) < e_\epsilon(t) < (1 + \epsilon\lambda^{\frac{1}{2}}\|K\|_\infty^{\frac{1}{2}})e(t) < \frac{3}{2}e(t),$$

(4.18) and (4.19) implies

$$e'_\epsilon(t) \le -\frac{2}{3}C_1\epsilon e_\epsilon(t). \tag{4.20}$$

Thus

$$e(t) \le 3e(0)exp(-\frac{2}{3}C_1\epsilon t). \tag{4.21}$$

Hence from (4.9) and (4.21) we get

$$E(t) \le l^{-1}e(t) \le 3e(0)l^{-1}exp(-\frac{2}{3}C_1\epsilon t), \quad t \ge 0.$$

This conclude s the proof of Theorem 2.1. □

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