Strong Kleene-Diense Logic: a variant of the infinite-valued Kleene-Diense Logic*

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[Abstract] Kleene first investigated a three-valued system which follows the evaluations of the Łukasiewicz infinite-valued logic ŁC with respect to negation, conjunction, and disjunction, and treats $\rightarrow$ as material-like implication in the sense that $A \rightarrow B$ is defined as $\sim A \vee B$ in its evaluation. Diense and Rescher extended it to many-valued logic and infinite-valued logic, respectively. This paper investigates a variant of the infinite-valued Kleene-Diense logic KD, which we shall call *strong Kleene-Diense logic* (sKD): sKD has the same evaluations as KD except that sKD takes a variant of Kleene-Diense implication. Following the idea of Dunn [2], we provide algebraic completeness for sKD together with its deduction theorem.

[keywords] sKD, KD, infinite-valued logic, algebraic semantics.

1. Introduction

Kleene [7] first introduced a three-valued system, which follows the evaluation (1) below with respect to implication and those of the Łukasiewicz infinite-valued logic ŁC with respect to negation, conjunction, and disjunction. Diense [1] extended it as many-valued system, and Rescher [11] as infinite-valued one. Let us consider this logic as infinite one, and call it *Kleene-Diense Logic KD*.[1] KD is interesting in the sense that

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1) By $S^{\supset}{}_{\aleph}$, Rescher expressed this logic. But we call it *KD* in honor of Kleene and Diense who first gave the idea of it as many-valued logic.

it treats $\rightarrow$ as material-like implication. Note that $A \rightarrow B$ is defined as $\sim A \vee B$ in its evaluation (see (1)). Thus KD can be thought of as a natural many-valued extension of the Classical Propositional Logic (CPL).

Let an evaluation be a function v: PA $\rightarrow$ [0, 1] (PA: set of propositional variables, [0, 1]: the rationals between 0 and 1). KD follows the evaluation

(1) $v(A \rightarrow B) = \max(1 - v(A), v(B))$.

As we mentioned above, this idea was in fact suggested by Rescher [11], However, as he stated in it, this logic does have no tautologies in case it has the sole designated value 1, the greatest element, and in case it has as designated all the elements except for the least element 0, the tautologies of CPL are those of KD, and vice versa. Thus, with respect to the second case KD is not interesting in the sense that it collapses into CPL just by taking axioms and rules for CPL as those for KD.

Now let us instead consider a variant of (1) as follows:

(2) $$v(A \rightarrow B) = \begin{cases} 1 & \text{if } v(A) \leq v(B); \\ \max(1 - v(A), v(B)) & \text{otherwise.} \end{cases}$$

And take as designated the greatest. We call an implication satisfying (2) *strong Kleene-Dienes implication (sKDI)*, and the KD with sKDI (in place of (1)) *strong Kleene-Diense Logic (sKD)*. Note that under the value-range [0,1] above, as designated sKD has the greatest 1.[2)]

Dunn [2] showed that the relevance logic **R**-mingle (**RM**) is pretabular in the sense that while it does not itself have a finite characteristic matrix, any normal extension of it does, and he gave algebraic completeness for **RM** and its normal extensions. He and Meyer [4] also showed the pretabularity of the Dummett's infinite-valued logic LC, obtained from the intuitionistic propositional logic **H** of Heyting by adding to **H** the A10 below as an axiom scheme, and they gave algebraic completeness for LC and its normal extensions.

In connection with **RM** and LC above, sKD is interesting in the sense that it may be regarded as the contractionless LC with respect to positive part of it, and moreover as relevant because with respect to negation $\sim$ (together with conjunction $\wedge$ and disjunction $\vee$) the principles the "absurdity" ("from any proposition of the form $A \wedge \sim A$ any proposition whatever can be deduced") and the "triviality" ("from any proposition whatever there can be deduced any proposition of the form $A \vee \sim A$"),

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2) KD can be regarded as a system, which has as non-logical constants vague sentences we can assign degrees of truth (and falsity) as evaluations to them, but still has as logical constant material-like (but not exactly material) implication in the sense that like material implication its implication is defined by negation and disjunction, i.e., $A \rightarrow B := \sim A \vee B$, (see (1)). As a neighbor of KD, we can think of sKD as a system still having material-like implication but *strengthened* in the sense that whenever degree of truth of the evaluation of consequent is greater than that of antecedent, we regard degree of truth of the evaluation of implication as *absolutely true* (see (2)).
While each definition of KD implication and sKD implication is very analogous to that of material implication of CPL, surprisingly each implication is very different from material implication in the sense that each implication of the former does not result in 'paradoxes of material implication'.

which give the 'paradoxes of (material) implication' and thus irrelevance between premises and conclusion, are not valid in it. Thus, sKD may be thought of in some sense either as a relevant contractionless LC or as a relevance system with a variant of the intuitionistic implication.

However, sKD is not satisfactory as a relevance logic because it has still "positive paradox". Note that in case a formula A is a theorem in sKD, B $\rightarrow$ A is also a theorem and so the latter gives an irrelevance between the antecedent B and the consequent A. Hence, we had better think of sKD as a *minimal* relevance system by regarding the condition that both the absurdity and the triviality do not hold in a system L as the *minimal condition* for relevance between premises (or antecedent) and conclusion (or consequent) in L.

(Note that sKD may be also (partially) paraconsistent in the (weak) sense that the above absurdity is not either valid in sKD (see Remark 4 in section 5.2).)

In this paper we first give algebraic completeness for sKD. Note that sKD omits the contraction (**W**). Thus, the "self-distribution", which principle is very important in the deduction for LC (as well as CPL and **H**), is not valid in sKD. Hence, it seems to us that sKD must have a deduction different from LC. Next, we give a deduction theorem for sKD.

We note that with the help of Dunn's (and Meyer's) algebraic completeness for **RM** (and LC), we give the completeness for sKD. Also, for convenience, we adapt ideas from their proofs in [2], [4].

2. Strong Kleene-Dienes algebras

To prove algebraic completeness for sKD, we should discover an algebra whose class will characterize it. We shall call that algebra a *strong Kleene-Dienes algebra*, which depends on a bounded de Morgan (b-DM) lattice and a variant of Henle algebra. For convenience, we shall also adopt the notation, interpretation, and results related with the algebras that are found in [3], [4], [9], [10], and assume familiarity with them.

We first define a *strong Kleene-Dienes (skd) algebra* to be a structure $(A, \top, \bot, \sim, \wedge, \vee, \rightarrow)$ where $(A, \top, \bot, \sim, \wedge, \vee)$ is a b-DM lattice, i.e., $(A, \wedge, \vee)$ is a distributive lattice with the greatest element $\top$ and the least $\bot$, and $\sim$ is a unary operation on A which is an involution:

(3) $\sim\sim a = a$;
(4) $\sim(a \vee b) = \sim a \wedge \sim b$,

$(A, \top, \bot, \rightarrow)$ is a variant of Henle algebra as follows: (in any partially ordered set $(A, \leq)$ with the greatest element $\top$ and the least one $\bot$)

$$(i_{skd})\ a \rightarrow b = \begin{cases} \top & \text{if and only if (iff) } a \leq b \\ \max(\top - a, b) & \text{otherwise,} \end{cases}$$

and $\sim$ is the precomplement in the sense that

(5) $\sim a = a \rightarrow \bot$.

We shall call the condition (i_{skd}) *strong Kleene-Dienes* *implication* and its algebra (A, $\top$, $\bot$, $\rightarrow$) a *strong Kleene-Dienes implication (skdi) algebra.* An *skd algebra* is a b-DM lattice satisfying (i_{skd}) and

(6) (a $\rightarrow$ b) $\vee$ (b $\rightarrow$ a) = $\top$,

called "prelinearity axiom" by Hájek [6]. An skd algebra is *linearly ordered* if the ordering of its algebra is linear, i.e., a $\leq$ b or b $\leq$ a (equivalently, a $\wedge$ b = a or a $\wedge$ b = b) for each pair a, b.

Since $\top$ is the dual of $\bot$, i.e., $\top$ = $\sim\bot$, join $\vee$ can be defined by using $\rightarrow$ and meet $\wedge$ (see df1 below), and $\sim$ by $\rightarrow$ and $\bot$ (see df2 below), an skd algebra (A, $\top$, $\bot$, $\sim$, $\wedge$, $\vee$, $\rightarrow$) may be abbreviated to (A, $\bot$, $\wedge$, $\rightarrow$).

3. Tables, axiom schemes, and rules for sKD

For convenience, we present only the tables for evaluation, the axiom schemes, and the rules of inference for sKD. We shall use the biconditional $\leftrightarrow$, where A $\leftrightarrow$ B = (A $\rightarrow$ B) $\wedge$ (B $\rightarrow$ A), and the falsity **f**. For the remainder we shall follow the customary notation and terminology. We use the axiom systems to provide a consequence relation.

An *evaluation* for sKD is a function v: PV $\rightarrow$ {$-\omega$, $\cdots$, $-n$, $-n + 1$, $\cdots$, 0, 1, $\cdots$, $n - 1$, n, $\cdots$, ω} that is extended to all well-formed formulas of L($\sim$, $\rightarrow$, $\wedge$, $\vee$, p_0, p_1, $\cdots$) by the following tables: (PV: set of propositional variables, {$-\omega$, $\cdots$, $-n$, $-n + 1$, $\cdots$, 0, 1, $\cdots$, $n - 1$, n, $\cdots$, $-\omega$}: set of integers with $-\omega$ and ω)

TABLES

T1. $v(\sim A) = \omega$ if $v(A) = -\omega$
$-\omega$ if $v(A) = \omega$
0 if $v(A) = 0$
$-n$ (or n) if $v(A) = n$ (or $-n$),

T2. $v(A \rightarrow B) = \omega$ if $v(A) \leq v(B)$;
$\max(v(\sim A), v(B))$ otherwise,

T3. $v(A \wedge B) = \min(v(A), v(B))$,

T4. $v(A \vee B) = \max(v(A), v(B))$.

AXIOM SCHEMES

A1. $A \rightarrow A$ (self-implication)
A2. $(A \rightarrow B) \rightarrow ((C \rightarrow A) \rightarrow (C \rightarrow B))$ (prefixing)
A3. $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$ (permutation)
A4. $A \rightarrow (B \rightarrow A)$ (positive paradox)
A5. $(A \wedge B) \rightarrow A$, $(A \wedge B) \rightarrow B$ ($\wedge$-elimination)
A6. $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$ ($\wedge$-introduction)
A7. $A \rightarrow (A \vee B)$, $B \rightarrow (A \vee B)$ ($\vee$-introduction)
A8. $((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$ ($\vee$-elimination)
A9. $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$ (distributive law)
A10. $(A \rightarrow B) \vee (B \rightarrow A)$ (chain)
A11. $\sim\sim A \leftrightarrow A$ (double negation)
A12. $(A \rightarrow \sim B) \rightarrow (B \rightarrow \sim A)$ (contraposition)
A13. $(\sim A \vee B) \rightarrow (A \rightarrow B)$
A14. $(A \rightarrow B) \vee ((A \rightarrow B) \rightarrow (\sim A \vee B))$
A15. $(A \rightarrow (A \rightarrow \sim A)) \rightarrow (A \rightarrow \sim A)$ (special contraction)

RULES

$A \rightarrow B, A \vdash B$ (modus ponens (MP))
$A, B \vdash A \wedge B$ (adjunction (AD))

DEFINITIONS
df1. $A \vee B := ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A)$
df2. $\sim A := A \rightarrow \mathbf{f}$.

Note that by df1 and df2, we may concern ourselves with $\rightarrow$, $\wedge$, and $\mathbf{f}$ as propositional connectives for sKD. Thus T1 and T4 are redundant. Note that $\wedge$ can not be defined as $A \wedge B := A \,\&\, (A \rightarrow B)$ and thus the axiom $(A \,\&\, (A \rightarrow B)) \rightarrow (B \,\&\, (B \rightarrow A))$ of BL (the basic logic for residuated fuzzy logics) in [5], [6] is not valid in it. However, we can obtain in place of it $(A \wedge B) \rightarrow (B \wedge A)$ as a theorem of sKD. Note also that "$\sim$", "$\wedge$", and "$\vee$" are used ambiguously as propositional connectives and as algebraic operators, but context should make their meaning clear.

4. sKD algebras and KDs algebras

To prove algebraic completeness for sKD, we need to consider algebras, more exactly matrices as algebras with designated element(s). The algebras which we shall consider will be ordered sextuples $\mathbf{M} = (A, \top, \sim, \wedge, \vee, \rightarrow)$, where A is a non-empty set of elements, $\top$ is the greatest element as designated, $\sim$ is unary operation on A interpreting the negation sign $\sim$, and $\wedge$, $\vee$, and $\rightarrow$ are binary operations on A interpreting the signs of conjunction ($\wedge$), disjunction ($\vee$), and implication ($\rightarrow$), respectively. In addition, we shall use the binary operation $\leftrightarrow$, where $a \leftrightarrow b := (a \rightarrow b) \wedge (b \rightarrow a)$ interpreting the sign of coimplication ($\leftrightarrow$), which we regard as a definitional abbreviation in the usual way.

Note that we can get $\perp$, the least element, as the dual of $\top$, and vice versa; and that $\sim$ can be defined by $\rightarrow$ and $\perp$ (see df2), and $\vee$ by $\rightarrow$ and $\wedge$ (see df1). Thus in considering (sKD) algebras, we need only concern ourselves with $\rightarrow$, $\wedge$, and $\perp$.

We employ the customary algebra notions, e.g., subalgebra, evaluation, validity, without special definition. We shall call an algebra in which all of the theorems of sKD are valid an *sKD algebra*. By a *normal* algebra, we mean one in which for all elements a and b,

(i) $a \leftrightarrow b = \top$ iff $a = b$;
(ii) a, $a \rightarrow b = \top$ only if $b = \top$; and
(iii) a, $b = \top$ only if $a \wedge b = \top$.

It is easy to verify, by inspection of the axiom schemes of sKD, that a normal sKD algebra is a de Morgan (DM) lattice with $\wedge$, $\vee$, and $\sim$, and that for elements a, b, $a \leq b$ iff $a \rightarrow b = \top$. Also, it follows from (ii) and (iii) for normality that the set of designated element D, $= \{\top\}$, is a filter. In the sequel, we shall regard normal sKD algebras as DM lattices in this way without special comment.

Certain normal sKD algebras will be especially important. We consider an algebra (among sKD algebras) which we shall call KD^s_ω, whose elements are the integers with $-\omega$ and ω, which are least and greatest elements, whose designated element is the ω, and whose operations $\sim$, $\rightarrow$, $\wedge$, and $\vee$ are defined as the above $\sim$, $\rightarrow$, $\wedge$, and $\vee$ tables, respectively; by KD^s_n, the subalgebra of KD^s_ω whose set is $\{-\omega, -n/2, -n/2 + 1, \cdots, -1, 1, \cdots, n/2 - 1, n/2, \omega\}$ if n is even and $\{-\omega, -n+1/2, -n+1/2 + 1, \cdots, -1, 0, 1, \cdots, n-1/2 - 1, n-1/2, \omega\}$ if n is odd, each with $n + 2$ elements.

We take KD^s_0 to consist of just $-\omega$ and ω. We exclude the degenerate one element algebra.

Generalizing, by a *KD^s algebra*, we shall mean any algebra whose elements form a chain with least and greatest elements, and whose operations are defined in an analogous way. All KD^s algebras are normal sKD algebras. (Note that it can be easily proved that A is a theorem of sKD only if A is valid in KD^s_ω, i.e., soundness.)

5. Algebraic completeness for sKD

5.1 Filters in sKD algebras

Let **M** be a normal KD^s algebra. A *filter* on **M** is a non-empty set $F \subseteq \mathbf{M}$ such that for each $x, y \in \mathbf{M}$,

(F1) $x \in F$ and $y \in F$ imply $x \wedge y \in F$,
(F2) $x \in F$ and $x \leq y$ (or $x \rightarrow y \in F$) imply $y \in F$.

F is a *prime filter* iff for each pair of elements x, y ($\in \mathbf{M}$) such that $x \vee y \in F$,

(PF) $x \in F$ or $y \in F$.

Where **M** is a normal KD^s algebra and F is a filter of **M** such that $D \subseteq F$, then if for elements a and b we define $a \equiv b$ iff $a \leftrightarrow b \in F$, then $\equiv$ is a congruence relation on **M** with respect to F, i.e., $\equiv$ is an equivalence relation, and if $a \equiv b$ and $c \equiv d$, then

(i) $a \in F$ iff $b \in F$;
(ii) $a \wedge c \equiv b \wedge d$; and
(iii) $a \rightarrow c \equiv b \rightarrow d$.

These properties of $\equiv$ are obtainable from the definition of normality together with easily accessible theorems of sKD. Note that in establishing the transitivity of $\equiv$ and also in establishing (i), it is useful to use (F2), which principle rests upon the fact that $A \rightarrow ((A \rightarrow B) \rightarrow B)$ is a theorem of sKD.

For a normal (skd) algebra **M** and such a filter F we can define the quotient algebra (modulo F) **M**/F as follows: the elements of **M**/F consist of the equivalence classes $|a|$ of all elements of b of **M** such that $b \equiv a$; the designated element(s) of **M**/F consist(s) of all $|a|$ such that $a\ (= \top) \in F$; and operations and special elements are defined representative-wise on the equivalence classes, so that $\sim|a| = |\sim a|$, $|a| \wedge |b| = |a \wedge b|$, $|a| \vee |b| = |a \vee b|$, $|a| \rightarrow |b| = |a \rightarrow b|$, $|\mathbf{t}| = \top$, and $|\mathbf{f}| = \bot$.[3)]

Note that we need not concern ourselves with $\sim|a| = |\sim a|$ and $|a| \vee |b| = |a \vee b|$ because $\sim$ and $\vee$ can be defined by $\rightarrow$, **f**, and $\wedge$. (Since $\mathbf{t} = \sim\mathbf{f}$ and $\sim$ and $\vee$ can be defined by $\rightarrow$, f, and $\wedge$, we need only concern ourselves with $\rightarrow$, **f**, and $\wedge$ with respect to equivalence classes.) Then, it is obvious that

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3) It can be ensured that this definition is correct due to (the definition of sKD algebras and) the provabilities as follows (we just need to check that $\leftrightarrow$ is a congruence with respect to $\wedge$ and $\rightarrow$: we check just one direction. Let $\vdash A \rightarrow B$. With respect to $\wedge$, by A5 and transitivity, $(A \wedge C) \rightarrow B$, and thus $(A \wedge C) \rightarrow (B \wedge C)$ by A5, A6, AD, and MP; with respect to $\rightarrow$, by transitivity, it is almost immediate that $(B \rightarrow C) \rightarrow (A \rightarrow C)$ and $(C \rightarrow A) \rightarrow (C \rightarrow B)$.

Proposition 1 If **M** is a normal sKD algebra and F is a filter of **M**, then **M**/F is a normal sKD algebra and is a homomorphic image of **M** under the natural homomorphism, h(a) = |a|.

Proposition 2 Let **M** and F be as in Proposition 1, and yet F be prime. Then **M**/F is a KD^s algebra.

Proof That **M**/F is a chain follows from the algebraic consideration of the axiom scheme A10 of sKD and the primeness of F. For the operations, which are defined as on a KD^s algebra, the axiom schemes A5 to A9, A11, and A12 ensure that $\wedge$, $\vee$, and $\sim$ satisfy DM lattice properties, i.e., each of them is as on a KD^s algebra. A14 together with the theorems of sKD (7) $(A \rightarrow B) \rightarrow ((A \rightarrow B) \leftrightarrow \mathbf{t})$, (8) $((A \rightarrow B) \rightarrow (\sim A \vee B)) \rightarrow ((A \rightarrow B) \leftrightarrow (\sim A \vee B))$ ensures that $\rightarrow$ is as on a KD^s algebra. Thus **M**/F is a KD^s algebra.

We check as an example $\rightarrow$. We first note that for each $a \in$ **M**, $a \in F$ iff $a = \top$ in F, i.e. $|a| = |\top|$, and that (∗) $a \rightarrow b \in F$ iff $|a| \leq |b|$. (cf. see the proof of Lemma 2.3.12 in [6]). Then, the axiom scheme A14 ensures that $\rightarrow$ is all right:

Since F is prime, either $a \rightarrow b \in F$ or $(a \rightarrow b) \rightarrow (\sim a \vee b) \in F$. First, let $a \rightarrow b\ (= \top) \in F$. Then $|a \rightarrow b|\ (= |\top|)$ by (∗), and thus $|a| \rightarrow |b| = |\top|$ by Proposition 1. Next, let $a \rightarrow b\ (\neq \top) \notin F$. Then by primeness, $(a \rightarrow b) \rightarrow (\sim a \vee b) \in F$. Then, by (∗) $|a \rightarrow b| \leq |\sim a \vee b|$. Since by algebraic consideration of A13 $(\sim a \vee b) \rightarrow (a \rightarrow b) = \top$ in F, and thus $|\sim a \vee b| \leq |a \rightarrow b|$. Hence $|a \rightarrow b| = |\sim a \vee b|$, as it should. □

Proposition 3 Let **M**, F be as in Proposition 1. Then if a is an element of **M** such that a $\neq \top$, there exists a homomorphism h of **M** onto a KD^s algebra such that h(a) $\neq \top$.

Proof By Propositions 1, 2, and the Stone Prime Filter Separation Theorem. □

Remark 1 As Dunn's (and Meyer's) consideration in [2], [4], our proofs of Propositions 1 to 3 may be regarded as generalizations of Stone's work.

Remark 2 By a construction used by Stone, from Proposition 3 it follows that every normal sKD algebra is isomorphic to a subdirect product of KD^s algebras. Since $KD^s{}_0$ is the only KD^s algebra that is a Boolean algebra (excluding the degenerate one element algebra), this may be also regarded as another generalization of embedding theorem of Stone's for Boolean algebras.

Proposition 4 Let $KD^s{}_0$, $KD^s{}_1$, $KD^s{}_2$, ⋯ be the sequence of KD^s algebras. If a sentence A is valid in $KD^s{}_i$, then A is valid in $KD^s{}_j$, for all $j \leq i$, such that if i is odd, then j is any non-negative integer ($\leq$ i), and that i is even only if j is even.

Proof Since each $KD^s{}_j$ is a subalgebra of $KD^s{}_i$, it is immediate. □

Note that when i is even, $KD^s{}_i$ will include a valid sentence A that is not valid in any odd-valued $KD^s{}_j$, $j \leq i$. This may be shown by considering (9) $(\sim(A \leftrightarrow \mathbf{f}) \wedge \sim(A \leftrightarrow \sim\mathbf{f})) \rightarrow ((A$

$\leftrightarrow$ ~A) $\leftrightarrow$ **f**) which is valid in every even-valued KD^s_i, but not in KD^s_1 (and thus not in any odd-valued KD^s_j). This implies that every valid sentence in KD^s_ω must be valid in KD^s_ω - 0 obtained from KD^s_ω by deleting 0 from the elements of KD^s_ω, i.e., KD^s_ω excluding 0, but there will be valid sentences in KD^s_ω - 0 that are not in KD^s_ω. Note that it implies that while each **RM** and LC is pretabular in the sense that any normal extension of it has a finite characteristic algebra (see [2], [4]), sKD is not because some normal extension of it may have an infinite characteristic algebra KD^s_ω - 0 (see section 5.2).

5.2 Completeness for sKD

To achieve the completeness for sKD, first, we define the Lindenbaum algebra of sKD. Our work is parallel to that of Dunn (and Meyer) in [2], [4]. We can construct a normal characteristic algebra for sKD (**A**(sKD)) as follows: for sentences P and Q, we define P $\equiv$ Q iff P $\leftrightarrow$ Q is a theorem of sKD; the elements of **A**(sKD) consist of the equivalence classes [P], where [P] is the set of all sentences of Q such that Q $\equiv$ P; operations are defined representative-wise on the equivalence classes, so that ~[P] = [~P], [P] $\wedge$ [Q] = [P $\wedge$ Q], [P] $\vee$ [Q] = [P $\vee$ Q], [P] $\rightarrow$ [Q] = [P $\rightarrow$ Q]; and [**t**] and [**f**] are greatest and least elements, respectively. (Note that the greatest element as designated consists of all [P] such that P is a theorem of sKD. Note also that we need only concern our selves with $\wedge$, $\rightarrow$, and **f** (see df1 and df2).)

This definition parallels the definition of a quotient algebra in section 5.1, and thus we can convince that this is a well-defined algebra because of the same theorems of sKD that justified the definition of the quotient algebra. Let us call this algebra the *normal Lindenbaum algebra* for sKD **A**(sKD), since evidently **A**(sKD) is a normal sKD algebra, and in fact is characteristic for sKD since any non-theorem A may be falsified under the *canonical* evaluation v_c which sends every sentence B to [B].

We shall call a propositional calculus X an *extension* of sKD iff X has the same sentences as sKD and every theorem of sKD is a theorem of X; an extension X *proper* iff X does not have exactly same theorems as sKD; and an extension X *normal* iff X is closed under rules of sKD. Where X is an extension of sKD, by an X-algebra we mean an algebra in which all of the theorems of X are valid. Where X is a normal extension of sKD, the Lindenbaum construction above can be modified by defining $P \equiv Q$ iff $P \leftrightarrow Q$ is a theorem of X, thereby producing the *normal Lindenbaum algebra* for X (**A**(X)).

By these definitions, we can give completeness for sKD. To do this, we mimic Theorems 6 to 10 in [2] and Theorems 5, 6 in [4]. Where X is a propositional calculus and **V** is a set of propositional variables, let X/**V** be that propositional calculus like X except that its sentences contain no propositional variables other than those in **V**. Then, it is obvious that

Proposition 5 Let X be a normal extension of sKD. Then **A**(X/**V**) is a normal X-algebra and is characteristic for X/**V**, since any non-theorem may be falsified under the canonical evaluation v_c that sends every sentence A to [A].

The hard part of the (weak) completeness result for sKD is showing that if a sentence A is not a theorem, then there is some KD^s algebra KD^s_n such that A is not valid in KD^s_n. This is contained in the following theorem, but generalized to arbitrary normal extensions of sKD, follows from Propositions 5 and 3.

Proposition 6 Let X be a normal extension of sKD. Then if a sentence A is not a theorem of X, then there exists some KD^s X-algebra KD^s such that A is not valid in KD^s.

Proof Let A be not a theorem of X. Then by Proposition 5, A is falsifiable in the normal X-algebra **A**(X/**V**) by the canonical evaluation v_c. But since $[A] \neq [\mathbf{t}]$, i.e., the greatest $\top$, in **A**(X/**V**), then by Proposition 3 there is a homomorphism h of **A**(X/**V**) onto a KD^s algebra KD^s such that KD^s is an X-algebra and $h([A]) \neq \top$ in KD^s. Then the composition h and v_c, $h \circ v_c(A) = h([A])$, is an evaluation which falsifies A in KD^s. Let this be the evaluation v such that $v(A) = h([A])$. Since KD^s is a KD^s algebra and $h([A]) \neq \top$ in KD^s, we may falsify A in KD^s by the evaluation $v(B) = h([B])$. Since every KD^s algebra is normal, it only remains to observe that KD^s is an X-algebra since it is a homomorphic image of **A**(X/**V**), which is an X-algebra. It is true by a general theorem of Łos's in [8]. □

Note that in Proposition 6 the KD^s algebra KD^s need not be finite. We can consider the case that KD^s is finite as follows: where there are only finitely many n propositional variables, let $\mathbf{V}^n$ be their set **V** and let $\mathbf{A}(X/\mathbf{V}^n)$ be that subalgebra of **A**(X/**V**) generated by elements, i.e., $[p_1], \cdots, [p_n]$, corresponding to propositional variables $p_1, \cdots, p_n$. Let us suppose that a sentence A, in which n propositional variables occur, is not a theorem of some normal extension X of sKD. Then, by Proposition 5, $\mathbf{A}(X/\mathbf{V}^n)$ is a normal X-algebra. Since $[A] \neq [\mathbf{t}]$ in $\mathbf{A}(X/\mathbf{V}^n)$, by almost the same argument as for Proposition 6, we may falsify A in some KD^s X-algebra KD^s that is a homomorphic image of $\mathbf{A}(X/\mathbf{V}^n)$ under some homomorphism h. Clearly, KD^s is generated by $h([p_1]), \cdots, h([p_n])$. But it is obvious that every finitely generated KD^s algebra KD^s is finite, and that if KD^s has n generators, then KD^s has at most 2n+2 elements. It is also obvious that every finite KD^s algebra containing at least two elements, the greatest and the least, is isomorphic to some KD^s_i. Hence, KD^s is isomorphic to some KD^s_i. So we have

Proposition 7 Let X be a normal extension of sKD. Then if A is a sentence containing n propositional variables and A is not a theorem of X, then there exists some KD^s X-algebra KD^s_i with at most 2n+2 elements such that A is not valid in KD^s_i.

When X is sKD itself, we may replace KD^s_i in Proposition 7 with some KD^s_j, $j \leq i$, because of Proposition 4 and the fact that every KD^s algebra is an sKD algebra. We thereby remove the final deficiency in Proposition 6, as promised after its proof, for the special case where X is sKD. Thus we get the (weak) completeness result:

Theorem 1 (weak completeness) If A is a sentence but n propositional variables, then A is a theorem of sKD iff A is valid in some KD^s_j, $j \leq 2n+2$.

Remark 3 sKD itself has no finite characteristic algebra, which can be easily shown by the proof similar to that of Sugihara in [12]. Thus, we can be ensured that sKD is not pretabular in the sense as in section 1, since some proper normal extension of sKD may have as an infinite characteristic algebra $KD^s_\omega - 0$.

Remark 4 sKD may be (regarded as) just partially paraconsistent because the implicative spread law $A \rightarrow (\sim A \rightarrow B)$ is valid in it, while the conjunctive spread law $(A \wedge \sim A) \rightarrow B$ is not. sKD may be also just weakly relevant in the sense that that $A \rightarrow B$ is a theorem does not imply that A and B share a sentential variable, which is the relevance principle of Anderson and Belnap, because of A4, while the "triviality" and the "absurdity" (see section 1) are not valid in it.

Note that, given the algebraic work, we can also easily prove that the following strong completeness theorem.

Theorem 2 (strong completeness) Let Γ be a set of sentences and A be a sentence. Then a necessary and sufficient condition for A to be deducible from Γ in sKD is that every evaluation in a KD^s algebra which gives every sentence of Γ a designated value also gives A a designated one.

Proof *(Sketch)* The necessity is obvious since KD^s algebras are normal. By contraposition, we prove its sufficiency. Suppose A is not deducible from Γ. Consider the normal Lindenbaum algebra for sKD **A**(sKD). The set F of all elements [B] such that B is deducible from Γ is a filter containing every designated element of **A**(sKD). Thus by applying Proposition 3, we can construct an evaluation into some KD^s algebra, as we did in the proof of Theorem 6, so that every sentence deducible from Γ receives a designated value and yet A does not. □

6. Deduction Theorem for sKD

Before giving a deduction for sKD, let us first consider the "iterated implication" $A \rightarrow^2 B$ abbreviating the number of implication, i.e., which is an abbreviation of $A \rightarrow (A \rightarrow B)$. We define the "elliptical implication" $\Rightarrow$ as standing for $\rightarrow$ or $\rightarrow^2$

Where Γ is a list of formulas of sKD (thought of as hypothesis), we define a *deduction* from Γ to be a sequence B_1, B_2, ⋯ , B_n, where for each B_i, $1 \leq i \leq n$, either (i) B_i is in Γ, or (ii) B_i is an axiom, or (iii) B_i follows from preceding members of the sequence by the rules of sKD. A formula A is called to be *deducible* from Γ, in symbols $\Gamma \vdash_{sKD} A$, just in case there is some deduction from Γ ending in A. Then, since sKD has MP and AD as its rules, it is obvious

Lemma 1 If $\Gamma \vdash_{sKD} A$ and $\Gamma \vdash_{sKD} A \rightarrow B$, then $\Gamma \vdash_{sKD} B$; and if $\Gamma \vdash_{sKD} A$ and $\Gamma \vdash_{sKD} B$, then $\Gamma \vdash_{sKD} A \wedge B$.

Then, we can obtain the elliptical deduction theorem (EDT) as follows:

Theorem 3 (EDT for sKD) If $\Gamma, A \vdash_{sKD} B$, then $\Gamma \vdash_{sKD} A \Rightarrow B$.

Proof Assume $\Gamma, A \vdash_{sKD} B$. Then there is a deduction B_1, B_2, ⋯ , B_m (B is B_m from Γ, A. We prove

(EDT) $\Gamma \vdash_{sKD} A \Rightarrow B_i$

for i = 1, 2, ⋯ , m. Taking i = m in EDT, we have Theorem 3.

Now we prove EDT by induction on i. Thus by induction hypothesis we assume that EDT holds for all values of i that are less than some fixed value of i and prove EDT for that fixed value of i.

Case 1. B_i satisfies (i). Then B_i is a member of Γ or A. Let Γ be a list of a finite sequence $A_1, A_2, \cdots, A_{n-1}$ and A be A_n. Then B_i is A_j for some j = 1, 2, ⋯ , n.

Subcase 1.1. Let j = 1, 2, ⋯ , n−1, i.e., $A_j \in \Gamma$. Then by (i), $\Gamma \vdash_{sKD} A_j$ and thus $\Gamma \vdash_{sKD} B_i$. By A4, $\Gamma \vdash_{sKD} B_i \rightarrow (A \rightarrow B_i)$. Hence, by Lemma 1, $\Gamma \vdash_{sKD} A \rightarrow B_i$, and thus $\Gamma \vdash_{sKD} A \Rightarrow B_i$.
Subcase 1.2. Let j = n, i.e. A_j be A. Then $A \rightarrow B_i$ is $A \rightarrow A$. Thus, $\Gamma \vdash_{sKD} A \rightarrow B_i$ by A1, and thus $\Gamma \vdash_{sKD} A \Rightarrow B_i$.

Case 2. B_i satisfies (ii). Then B_i is an axiom scheme of sKD. Thus, $\Gamma \vdash_{sKD} B_i$. By the proof similar to Subcase 1.1., $\Gamma \vdash_{sKD} A \Rightarrow B_i$.

Case 3. B_i satisfies (iii). Then there are j, k < i such that either (3.1) B_k is $B_j \rightarrow B_i$ or (3.2) B_i is $B_j \wedge B_k$.

Subcase 3.1. By the induction hypothesis

(**) $\Gamma \vdash_{sKD} A \rightarrow B_j$

and

$\Gamma \vdash_{sKD} A \rightarrow B_k$.

Then,

(***) $\Gamma \vdash_{sKD} A \rightarrow (B_j \rightarrow B_i)$.

By (***), A3, and Lemma 1,

(****) $\Gamma \vdash_{sKD} B_j \rightarrow (A \rightarrow B_i)$.

By (**), (****), and suffixing as a theorem, $\Gamma \vdash_{sKD} A \rightarrow (A \rightarrow B_i)$, and thus $\Gamma \vdash_{sKD} A \Rightarrow B_i$.

Subcase 3.2. By the induction hypothesis we can state (**) above. Then by A6 and Lemma 1,

(*****) $\Gamma \vdash_{sKD} A \rightarrow (B_j \land B_k)$.

Thus, since B_i is $B_j \land B_k$, $\Gamma \vdash_{sKD} A \rightarrow B_i$. Hence, $\Gamma \vdash_{sKD} A \Rightarrow B_i$. This completes the proof of this theorem. □

Since by A4 $(A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B))$, we can obtain from Theorem 3 the Deduction Theorem (DT) for sKD as follows.

Corollary 1 (DT for sKD) If T, A $\vdash_{sKD}$ B, then T $\vdash_{sKD} A \rightarrow (A \rightarrow B)$.

We note that we can easily show that the converse of Corollary 1, i.e., if T $\vdash_{sKD}$ A → (A → B), then T, A $\vdash_{sKD}$ B. Thus we can obtain that

Corollary 2 T, A $\vdash_{sKD}$ B iff T $\vdash_{sKD}$ A → (A → B).

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