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Individual Strategies for Problem Solving

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Problem solving is an important aspect of learning mathematics and has been extensively researched into by mathematics educators. In this paper we analyze the difficulties students encounter in various steps involved in solving problems involving physical and geometrical applications of mathematical concepts. Our research shows that, generally students, in spite of possessing adequate theoretical knowledge, have difficulties in identifying the hidden data present in the problems which are crucial links to their successful resolutions. Our research also shows that students have difficulties in solving problems involving constructions and use of symmetry.

Keywords: problem solving, hidden data, constructions and use
ZDM Classification: D54
MSC2000 Classification: 97D50

1. INTRODUCTION

Problem solving is an integral part of learning and understanding mathematics at all levels. As early as the 17th century Descartes in his posthumously published *Rules of the Mind* presents a universal method for problem solving (See Polya 1962, Chapter 2). Several studies have analyzed the nature of difficulties experienced by students in problem solving attributing them to a variety of reasons. Bagni (2000) has shown that students are comfortable in solving problems only in the sectors explicitly considered and says that this is a remarkable obstacle in reaching good performances. In stating the importance of solving problems Schoenfeld (1994) observes "Mathematical thinking consists of a lot more than knowing facts, theorems, techniques, etc." I would characterize the mathematics a person understands by describing what that person *do* mathematically, rather than by an inventory of what the person "knows."

In his paper on "Teaching problem solving skills", Schoenfeld (1980) says that expert mathematicians, as a result of their problem solving experiences, use heuristic strategies to solve problems and that it is possible to teach some of these strategies to students. He conducted an experiment in which two groups of students were taught essentially the same course but with the difference that one group was taught these strategies. His finding is that the latter group performed much better than the former.

Booth & Thomas (1999) have studied the connection between spatial visualization and problem solving ability. Their research points out that students with higher spacial-visual ability performed significantly better in problem solving. Stylianou says that one should be skillful in recognizing and exploiting patterns like symmetry and become comfortable in manipulating visual images. She also adds that only when the visual notion of mathematical concepts are as strong as their analytic formulations one can perform transformations on them and use them as aids in problem solving. Yusof and David Tall explore the introduction of different problem solving strategies and they feel that only for students for whom mathematics "made sense" these strategies help to approach problem solving in a creative way. But those who stated that mathematics does not "make sense" treated problem solving techniques as a new sequence of routine procedures. Koeno Gravemeijer has suggested a dynamical strategy for solving word problems. His recommendation is that students should approach the application problems as situations to be "mathematized" and not as a situation to apply ready-made procedures. He says the main objective of the student is to make sense of the problem. If a student does not have ready-made procedures available, an organized approach may help the student to find informal solutions. He concludes by saying that approaching a new problem as a situation to be mathematized may have more mathematical value than any ready-made procedures learnt.

White & Mitchelmore (1996) in their research on conceptual understanding of introductory calculus state that most students develop an *abstract apart* concept of a variable hindering meaningful learning of calculus. They say that when students have a concept of a variable that is limited to algebraic symbols they have learnt to operate them without any regard to their conceptual meaning. They call such concepts abstract apart concepts. on contrasting with *abstract general* concepts they say that *abstract general* concepts requires the formation of links among a wide variety of superficially different contexts.

For general discussion on problem solving we refer the reader to the paper (*cf.* Wilson, Fernandez & Hadaway 1993).

Aim of this article

One of the heuristics frequently used by successful problem solvers is the notion of symmetry. It is an important tool consistently exploited by mathematicians, not only for solving problems, but also to guide them in their path to discoveries/inventions. The aims of this paper are two-fold:

(1) To explore whether students are able to use symmetry as a strategy in their approach to problem solving when it presents itself in the given problem situation.

(2) To understand the difficulties students experience in solving problems which require for their resolution the proper formulation of equations based on geometrical or physical constraints.

We address the difficulties of those students who, in spite of having understood the relevant mathematical facts and having acquired basic skills in solving routine problems, find it difficult to solve *application oriented* problems. We do not address the difficulties encountered in the understanding of the basic knowledge components required to solve the problems.

Transfer of knowledge

The theory of transfer has been studied extensively by educational psychologists. Transfer is the use of prior knowledge in new situations. In order to successfully complete a task, which requires application of knowledge gained as a result of previous experiences, it is imperative that transfer of knowledge take place. The greater the ability to transfer relevant knowledge components, the better are the chances of success in solving a given problem. A *positive* transfer takes place when the knowledge component that is transferred shares many structural components of the problem. However, one should not be distracted by superficial details of the problem situation as this might lead to *negative* transfer resulting in running into dead end. We can do no better than to quote the following passage taken from the book by Sara Medows.

"Learning and transfer of rules that are based on the important structural features of a task not on trivial surface characteristics will be easier if the learner encounters a range of examples with a common structure and different irrelevant characteristics. Initially varied examples may be confusing but if enough moderately varied examples are given the learner can increasingly infer the general principles which underlie them and transfer them to a wider range of examples." Sfard (1991) describes a three phase model for understanding mathematical concepts or 'objects.' The *interiorisation* phase occurs when a process is performed on familiar mathematical objects. When the process is condensed into manageable units it is called the *condensation* phase. The third phase is called the *reification* phase when the process becomes an object in its own right. The third phase is a transition from operational to structural mode.

Steps in problem solving

To facilitate our study, we identify six major steps involved in solving problems in the application of mathematical concepts to physical situations.

Before we elucidate the various steps involved in problem solving we explain the kind of problems to which these steps are particularly relevant. Typically the solution to these problems involves a combination of different concepts studied at various stages, creating an appropriate mathematical model of the problem before the techniques can be applied, imagination and visualization, and, language skills. With these aspects in mind we divide the different steps involved in solving such problems as follows:

Step 1. Translating the physical problem into a mathematical one. This involves creating an appropriate mathematical model.

Step 2. Recognizing the relevant concepts and theories needed to solve the problem. This means in-context problem solving as well as out-of-context problem solving. Here, by an in-context problem, we mean a problem which is placed in the same chapter dealing with the mathematical concept needed for its resolution (Transfer of relevant mathematical knowledge to the perceived problem situation).

Step 3. Recognizing the known and unknown quantities involved, distinguishing the variables, constants and dependent and independent variables.

Step 4. Bringing out the relevant *hidden* data, *i. e.*, data which are not given explicitly, which are crucial links in the steps leading to the solution. This involves recognizing the physical and geometrical constraints (This step calls for recognition of relevant structural characteristics ignoring the superficial details).

Step 5. Taking hints by working backwards.

To illustrate the above steps we consider the following problem taken from the mathematics textbook for senior secondary (Grade 12) students in India.

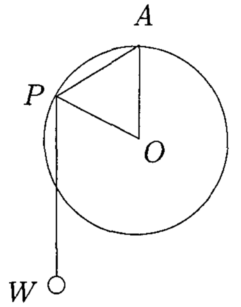
Problem. A ring of weight W which can slide freely on a smooth vertical circular hoop supported by a string attached to the highest point. If the string subtends an angle θ at the center find the tension in the string and reaction of the circle on the ring.

The following version is a reformulation of the above physical problem, which is the appropriate mathematical model, amenable for ready application of various concepts needed to solve the problem (Step 1).

Problem[revised]. There is a force of weight W acting vertically downwards at a point P on a circle. It is held in equilibrium by two other forces namely the force T (tension) along

PA where A is the highest point on the circle and a force R (reaction) along OP where $\angle AOP = \theta$.

In the above example the fact that the reaction is perpendicular to the circle is a hidden quantity which has to be identified (Step 4). The known and unknown quantities in this



problem are as follows: The length of the string, AP, the radii OA, OP, and the angle between them are known quantities. The weight of the ring, W, is also known. The tension in the string is the unknown quantity (Step 3).

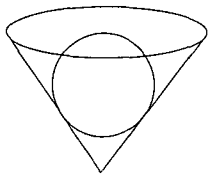
Since the three forces keep the system in equilibrium students should realize that the magnitude of the forces is proportional to sine of the angle between the other two forces. The ring in the problem has to be idealized to a point at which the mass is concentrated. It is constrained to move on the given circle. On account of its weight it slides to the point P so that PA equals the length of the string. The string experiences tension, a force along PA. This is recognizing relevant mathematical concepts needed to solve the problem (Step 2).

The following example illustrates the difficulty the students have in applying concepts from plane geometry to solving problems involving solid geometry. This example also throws light on the confusions in the minds of the students when solving problems involving application of maxima and minima. This is particularly the case with problems involving mensuration, calculus and geometry. The students had previously learned similarity of triangles, mensuration and trigonometry. The following problem was illustrated in a Grade 10 class of a secondary school in Chennai, India.

Problem. To determine the in-radius of a circle when the sides of the triangle are given.

After a week the students were tested on the following problem.

Problem[revised]. A tub in the shape of an inverted cone is filled with water. A spherical ball dropped into the tub just touches the top level of the cone. Find the volume of water displaced. Given that the radius of the cone is rm and height hm.



Only 3% of the students were able to solve the problem. The students did not seem to have problems in identifying the volume of water displaced as the volume of the sphere. But most of them were unable to compute the volume of the sphere as they could not compute its radius. They were not able to establish the link with the problem of finding the in-radius of a given triangle (Step 4). The rotation symmetry of the objects is the hidden clue to reduce the three dimensional representation to a two dimension one obtained by looking at the vertical cross section. This calls for a higher level of visual perception and imagery which the average student is yet to fully develop.

2. RESEARCH

Our research is based on teaching Grades 10–Grades 12, in senior secondary schools (junior college) in Chennai, afflicted to Central Board of Secondary Education (CBSE) of India. The mathematical curriculum is of a high standard and covers a wide range of topics. In Classes 11 and Classes 7 the students learn algebra, trigonometry, two dimensional and three dimensional analytical geometry, complex numbers, differential and integral calculus, differential equations, matrices and determinants, Boolean algebra, statics, dynamics and probability theory. The final examination taken at the end of Class 12 comprises only of problem solving.

Students studying in Class 10 are 15 years old and those who are studying in Class 12 are 17 years old. Teachers handling these classes have at least a master's degree in Mathematics and a bachelor's degree in education. The NCERT textbook which is mostly followed is written by a team of Professors working in the Mathematics departments of various colleges and Universities. The school in which the author is currently teaching is a school of high repute as the students admitted here have to go through stringent admission procedures. Generally the students are taught the different mathematical concepts in detail and problems of varying difficulty level are then worked out in the class. Many students, who seem to understand the concepts, are able to work out simple problems which are formulated in a way which are readily amenable for application of mathematical concepts learnt. However the same students have difficulty when given more involved ones. Our research shows that difficulties encountered by the students are not uniform. Students also respond to different strategies of approaches to problem solving and therefore we emphasize that one should aim at helping students develop their own strategies of problem solving.

Sample. The sample consisted of a class of 45 students from Grade 12 mathematics course and a class of 37 students in a Grade 10 mathematics course. The Grade 10 as well as the Grade 12 comprised of students of varying mathematical abilities.

Data Collection. The purpose of the present study was to investigate and identify the steps in which the students experience difficulties in solving problems. In order to accomplish this, data were collected by administering tests to the sample of students. These tests were of the following nature.

- Each set of problems given in the test varied slightly from each other.
- The problems required conceptual knowledge of more than one mathematical topic.
- The problems were either physical or geometrical application of mathematical concepts.
- All the problems had hidden data to be identified.

Question

Grade 12 students

The test given to the Grade 12 students involved applications to maxima and minima. The test, which was of forty minute duration, was given without prior announcement. The test consisted of the following three questions:

Question 1. Find the maximum area of a rectangle inscribed in an equilateral triangle with its side on the base of the triangle.

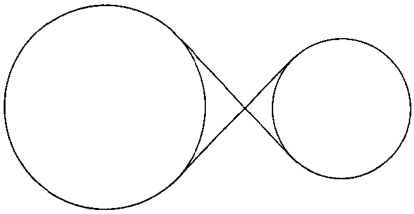
Question 2. Let $ABCD$ be a rhombus with angle $A = \pi/3$. Find the maximum area of a rectangle inscribed in the rhombus so that the sides of the rectangle are parallel to the diagonals of the rhombus.

Question 3. Find the maximum area of a rectangle inscribed in a triangle of angles $\pi/2, \pi/3$ and $\pi/6$.

Grade 10 students

The test given to the Grade 10 students comprised of the following three questions, which were slight variations of one another. The questions were given one after another. The students were given 20 minutes to work on question 1, 15 minutes each to work on questions 2 and 3.

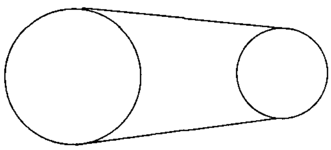
Question 1. Two circular wheels with radii 5cm and 3cm are to be wound by a belt around them. Find the length of the belt if it crosses over between the wheels at right angles.



Question 2. This question is similar to the one above except for the fact that the belt made an angle of 60 degrees when it crossed between the wheels.

Question 3. Two circular wheels with 5cm and 3cm are such that their centers are 10cm apart. A belt has to be wound around the wheels. Find the length of the belt if it does not cross between the wheels.

Figures were not given for any of the questions. The first question of the two tests can be



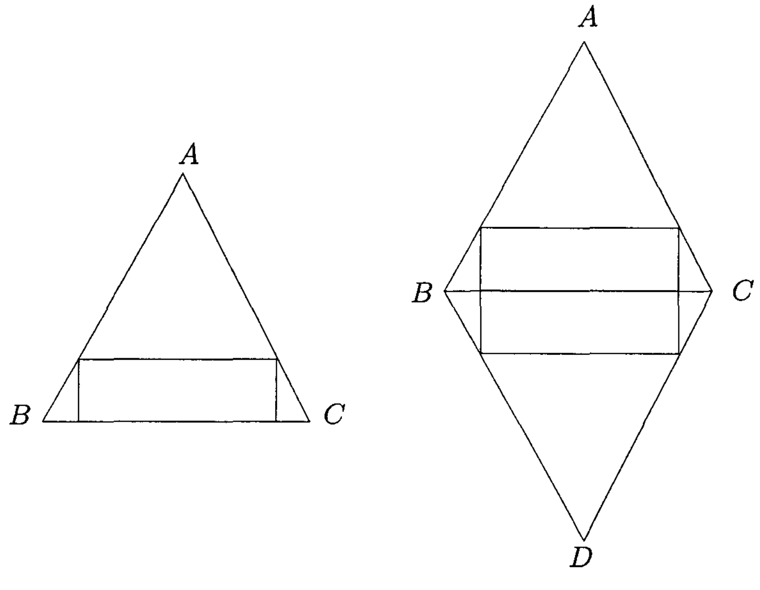
found in any standard textbook while the other questions are variations of these questions made by the author.

Data Analysis. The students' responses were graded and were analyzed thoroughly to understand the mental constructions made by the students while solving application problems. The errors in the students' responses were broadly fit into different steps as classified above.

Findings

Grade 12 Students

Out of 45 students who took the test 27% of the students answered all the questions. 38% did not answer any question correctly. 23% answered only the first question, 11% answered Questions 1 and 2. 4% answered questions 1 and 3. The statistics that 38% did not answer any question correctly reveals the fact that students have difficulty in Step-2 particularly when solving problems not pertaining to the topic currently being taught. This



study leads to the following interesting observation. Out of 17 students who answered the question 2 only 9 students deduced the solution from question 1 using symmetry as the diagonal BC of the rhombus bisects it into two triangles each congruent to triangle ABC. (The inscribed rectangle is bisected by BC each having base on the side of the equilateral triangles.) This was not apparent to the others as they seem to approach any given problem in an algorithmic fashion. These students had difficulty with Step-4, *i. e.*, in identifying and using hidden data. Although one student observed the symmetry in the problem he was not convinced that appealing to such arguments could constitute a mathematically sound proof.

We tabulate below the results of this test.

question no.	number of correct response	percentage of correct response
1 only	8	18%
2 only	0	0%
3 only	0	0%
1 and 2 only	5	11%
1 and 3 only	2	4%
all three	12	27%
none	17	39%

The following examples of students' responses and possible difficulties are representative of how the students approached each test question. We take up each question and give below some responses of the students.

Result of Quation 1.

Student 1. She obtains the correct expression for the area A and writes $A = \sqrt{3}b(a - b)$ but proceeds no further. She does not seem to realize that b is a variable and a is a constant in the problem.

Student 2. He writes the correct expression for the area: $A = 2(a - x)y$ and but fails to obtain the relation between x and y and so is unable to express A as a function of a single variable. Perhaps he does not realize he should differentiate with respect to x.

Student 3. This student obtains the relation $y = \sqrt{3}x$ and the expression $A = (a - 2x)y$. But he does not substitute for y and so he fails to express A is a function of a single variable. As a result the student is unable to complete t he problem.

We are led to the conclusion that failing to realize which are the constants and which are variables is a major stumbling block in solving this problem.

Result of Quation 2.

Student 4. He solved the first problem correctly. In the second problem, he draws the figure correctly but fails to notice the symmetry. He writes the correct expression for area in terms of the length of the shorter diagonal of the rhombus. But he fails to realize that it is the same as the length of the side of the rhombus.

Student 5. He solved the first problem correctly. In the second, he writes that the rectangle of maximum area that can be inscribed in a rhombus must be a square and computes its area. It is interesting to note that he had not applied the above idea in the earlier problem.

Result of Quation 3. The fact that many students are unable to solve the third problem correctly may be due to the fact that on seeing three different angles and hence three different sides they are confused as to how to relate them in order that they can express the area function as a function of a *single* variable.

Grade 10 Students

The following results show the level of performance of students and possible difficulties they encountered for each test question.

Result of Quation 1. Seven students got the right solution. Ten were able to draw the figure correctly but unable to solve the problem. The remaining students had trouble with the mathematical modeling of the problem. At the end of the test the figure was drawn on the board and an additional ten minutes were given for the students to complete the problem with the help of the diagram. Five more students were able to solve the problem. Ten of them did not take into consideration the length of the arcs which also formed a part of the belt. Two of them computed the length of the minor arcs instead of the major arcs. Many students drew two circles of the same radii, touching each other. One student drew the figure correctly but did not seem to realize that the tangents together with the radii drawn to the points of tangency form two squares.

Result of Quation 2. Now 90% of the students computed the length of the belt correctly. This is very likely due to the fact that since they had the visual in front of them, they had only to use some basic trigonometry to complete the problem.

Result of Quation 3. At the end of 15 minutes 2% of the students were able to find the length of the belt and none of the students completely solved the problem. The hint to extend the tangents to their point of intersection was given. Once the hint was given most of the students in the class successfully completed it. This clearly indicates the problem with identifying hidden data and the inability to carry out the construction by *working backwards*.

3. DISCUSSION AND CONCLUSION

In their approach to problem solving the students seem to develop *abstract apart* idea of a variable (*cf.* White & Mitchelmore 1996) which inhibits them from solving complex problems involving maxima and minima. In a formal calculus course the students acquire skills and get trained to perform operations in an algorithmic manner. But the need is to facilitate the students to develop their own strategies of problem solving in entirely novel situations where the earlier algorithmic approach would no longer be applicable readily.

The test conducted for Class 12 students on maxima and minima resulted in the following observations. Many students are unable to distinguish between variables and constants. They are unable to exploit symmetry present in the problem situation. This emphasizes the need to stress the importance of the role of symmetry in problem solving.

They also show difficulties in expressing the required function as a function of a *single* variable using the physical or geometrical constraints in the given problem, largely due to the fact that they are not able to identify the *hidden data.*

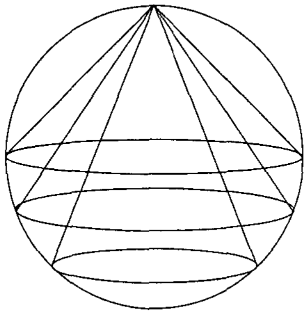
It is interesting to note that the results of a test conducted by White & Mitchelmore (1996) on a problem involving minimization (*cf.* White & Mitchelmore 1996, item 3 of Table 1) corroborates our findings that the students have difficulty in creating an appropriate mathematical model of the given problem and making crucial links between the known and unknown variables (our Steps 3 and Step 4). In their analysis of the students' response to a particular problem, White and Mitchelmore conclude that "there is a strong suggestion that defining and using new variables is qualitatively different from relating explicitly given variables in symbolic form." They further observe that "seeing how to combine them indicates the existence of a relationship at a higher level of abstraction."

The two different tests conducted for Class 10 students show that the students are unable to discern the familiar concepts of *planar* geometry when they manifest themselves in a given problem as a property of *three* dimensional objects. Their lack of ability in visualizing geometric objects adds to their difficulty in creating a suitable mathematical model.

We suggest pedagogical strategies which might help students acquire better problem solving skills with the help of the following example.

Example. To find the maximum volume of a cone inscribed in a sphere of radius r.

Plenty of diagrams suggesting different cones inscribed in a sphere can be drawn. The diagrams will help the students to understand that the radius of the sphere is a constant while the inscribed cones have varying radii and heights.



- Infinitely many cones can be inscribed in a sphere.
- Any point on the sphere can be the vertex and hence one can fix the north pole as the vertex using spherical symmetry. Once the vertex is fixed one cone can be drawn for each value of h between 0 and $2R$ where R is the radius of the sphere. This helps in understanding that $r = f(h)$. This then helps to express the volume of the cone as a function of r or h. Complete understanding of the problem and strengthening of the weak knowledge of functions are made possible by answering the following questions.
- If the radius is at least $R/8$ what is the maximum volume?
- If the height is at most $2R/5$ what is the change in the maximum volume.?
- Will an isosceles triangle of maximum area give rise to a cone of maximum volume? If not why not?

In the present context these kind of questions will help the student to understand that domain of the function is as important as the algebraic modeling in solving application problems of maxima and minima.

In order to improve the overall mathematical background and understanding of different concepts, the students should be encouraged to engage in self-inquiry. By 'self-inquiry' we mean that the students pose for themselves a number of questions such as:

- What are the constraints in the given problem?
- What are the known quantities and what is to be determined?
- Do the given data determine the given problem completely?
- Are there any hidden data which can serve as crucial links to the solution?

Perhaps this may help students to develop analytical and individual strategies for problem solving. This however has to be substantiated by further research.

4. ACKNOWLEDGEMENTS

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