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PDE-PRESERVING PROPERTIES

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ABSTRACT. A continuous linear operator T, on the space of entire functions in d variables, is PDE-preserving for a given set $\mathbb{P} \subseteq \mathbb{C}[\xi_1, ..., \xi_d]$ of polynomials if it maps every kernel-set $\ker P(D)$, $P \in \mathbb{P}$, invariantly. It is clear that the set $\mathscr{O}(\mathbb{P})$ of PDE-preserving operators for $\mathbb{P}$ forms an algebra under composition. We study and link properties and structures on the operator side $\mathscr{O}(\mathbb{P})$ versus the corresponding family $\mathbb{P}$ of polynomials. For our purposes, we introduce notions such as the PDE-*preserving hull* and *basic sets* for a given set $\mathbb{P}$ which, roughly, is the largest, respectively a minimal, collection of polynomials that generate all the PDE-preserving operators for $\mathbb{P}$. We also describe PDE-preserving operators via a *kernel* theorem. We apply Hilbert's Nullstellensatz.

1. Introduction and Preliminaries

We let d be a fixed arbitrary natural number and denote by $\mathscr{H}$ the space of entire functions in d variables endowed with the compact-open topology. Thus $\mathscr{H}$ is a reflexive Fréchet space and a generating family of semi-norms is obtained by $\|f\|_n \equiv \sup_{|z|\leq n} |f(z)|$, $n \in \mathbb{N}$. Given $r > 0$, Exp_r denotes the Banach space of functions $\varphi \in \mathscr{H}$ such that $|\!|\!|\varphi|\!|\!|_r \equiv \sup_\xi |\varphi(\xi)|e^{-r|\xi|} < \infty$, equipped with the norm $|\!|\!|\cdot|\!|\!|_r$ thus defined. The space of exponential type functions, Exp, is the union $\cup_{r>0}\mathrm{Exp}_r$ provided with the corresponding inductive locally convex topology. We recall that the Fourier-Borel transform $\mathcal{F}$, $\mathscr{H}' \ni \lambda \mapsto \mathcal{F}\lambda(\xi) \equiv \lambda(e_\xi)$, is a topological isomorphism between $\mathscr{H}'$ and Exp when $\mathscr{H}'$ is equipped with the strong topology. Here, and below, $e_a \equiv e^{\langle \cdot, a\rangle} \in \mathrm{Exp} \subseteq \mathscr{H}$,

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$a \in \mathbb{C}^d$, where $\langle z, \xi \rangle \equiv \sum z_i \xi_i$. The *Martineau-duality* between $\mathscr{H}$ and Exp is defined by $\langle f, \varphi \rangle \equiv \mathcal{F}^{-1}\varphi(f)$.

The complex algebra of continuous linear operators on $\mathscr{H}$ is denoted by $\mathscr{L} = \mathscr{L}(\mathscr{H})$. A *convolution operator* is defined as a continuous linear operator that commutes with all translations $\tau_a : f \mapsto f(z+a)$, $a \in \mathbb{C}^d$. For a proof of the following well-known we refer to [3]:

PROPOSITION 1. *The set of convolution operators on $\mathscr{H}$, $\mathscr{C} = \mathscr{C}(\mathscr{H})$, is a commutative subalgebra of $\mathscr{L}$ and is formed by the operators $\varphi(D)$, $\varphi \in$ Exp, where $\varphi(D)f(z) \equiv \langle f, \varphi e_z \rangle = \lambda(\tau_z f)$ and $\mathcal{F}\lambda = \varphi$. The function $\varphi \in$ Exp ($\lambda \in \mathscr{H}'$) is unique for $\varphi(D)$.*

(Thus there are one-to-one correspondences between all the spaces $\mathscr{H}'$, Exp and $\mathscr{C}$.) We let $\mathscr{P}$ denote the subalgebra $\mathbb{C}[\xi_1, ..., \xi_d]$ of Exp formed by the polynomials and remark that if $P \in \mathscr{P} \subseteq$ Exp, then $P(D)$ is the differential operator obtained by replacing each variable ξ_j by $\partial/\partial z_j$ in P. Note also that, for any $\varphi(D) \in \mathscr{C}$, $\varphi(D)e_a = \varphi(a)e_a$ and the transpose of $\varphi(D)$ is given by "multiplication by φ", and is simply denoted by φ. We recall *Malgrange's Theorem* [7]: Every convolution operator $\varphi(D)$, $\varphi \neq 0$, is surjective.

DEFINITION 1. A continuous linear operator $T : \mathscr{H} \to \mathscr{H}$ is PDE-*preserving* for a set $\mathbb{P} \subseteq$ Exp if $T \ker \varphi(D) \subseteq \ker \varphi(D)$ (i.e. $\ker \varphi(D)$ is invariant under T) for all $\varphi \in \mathbb{P}$. The set of PDE-preserving operators for $\mathbb{P}$ is denoted by $\mathscr{O}(\mathbb{P})$.

By Malgrange's Theorem, $\operatorname{Im} \varphi = \ker \varphi(D)^\perp$ and in view of this it is not difficult to prove that $T \in \mathscr{O}(\mathbb{P})$ iff ${}^tT$ is IDEAL-*preserving* for $\mathbb{P}$ in the sense that ${}^tT \in \mathscr{L}(\text{Exp})$ and every principal ideal $\operatorname{Im} \varphi = \text{Exp} \cdot \varphi$, $\varphi \in \mathbb{P}$, forms an invariant set under ${}^tT$. Consequently, $T \mapsto {}^tT$ defines a one-to-one correspondence between $\mathscr{O}(\mathbb{P})$ and the set of IDEAL-preserving operators for $\mathbb{P}$. Further, for any set $\mathbb{P}$, $\mathscr{O}(\mathbb{P})$ forms a subalgebra of $\mathscr{L}$ and hence $\mathscr{H}$ is an $\mathscr{O}(\mathbb{P})$-module in a natural way. In turn, $\mathscr{C}$ forms a (commutative) subalgebra of $\mathscr{O}(\mathbb{P})$.

The objective in this note is to study how properties of the set $\mathbb{P}$, of algebraic nature, can be translated to the algebra and ring $\mathscr{O}(\mathbb{P})$ and vice versa. In particular we shall apply Hilbert's Nullstellensatz. (We recall the close connection between the zero-set $Z(P) \equiv \{a : P(a) = 0\}$ of, say, a polynomial P and the corresponding kernel-set $\ker P(D)$.)

For our purposes we divide the study into two parts, exposed in Sections 2 and 3 respectively, and in the following way: In Section 2 we investigate PDE-preserving properties on the operator side. In the main theorem, Theorem 2, we describe PDE-preserving operators. In fact, every element of $\mathscr{L}$ has a unique *symbol* (*kernel*) (Theorem 1) and we describe the symbols for PDE-preserving operators. In Section 3, we study polynomial sets $\mathbb{P} \subseteq \mathscr{P}$ in terms of PDE-preserving properties. The reason for restricting the study to polynomial sets is that we want to make use of the nice algebraic properties of $\mathscr{P}$ (UFD, Noetherian etc.) and theorems like Hilbert's Nullstellensatz. Moreover, in this way we link operator theory and algebraic geometry. In particular, given a set $\mathbb{P} \subseteq \mathscr{P}$, a natural question to ask is for what polynomials Q do we have that every operator in $\mathscr{O}(\mathbb{P})$ is also PDE-preserving for Q. And, in the other direction, can we find a minimal collection $\mathbb{B} \subseteq \mathscr{P}$ that generate $\mathbb{P}$ (or $\mathscr{O}(\mathbb{P})$) in the sense that $\mathscr{O}(\mathbb{P}) = \mathscr{O}(\mathbb{B})$. Based on these questions we introduce notions such as the PDE-*preserving hull*, $\hat{\mathbb{P}}$, and *basic sets* for a given set $\mathbb{P}$, which is the largest, respectively a minimal, collection of polynomials that generate $\mathbb{P}$, see Definitions 2 and 5 respectively. As a measure of how reducible a set $\mathbb{P} \in \mathscr{P}$ is in the context, we define (Definition 6) the PDE-*preserving dimension* of $\mathbb{P}$. The main theorem in Section 3, Theorem 5, gives a necessary and sufficient condition for a polynomial to be in $\hat{\mathbb{P}}$.

In the last section, Section 4, we propose some possible lines of further investigations. Our discussion raises several open problems – some are formulated more explicitely and others.

The previous study on PDE-preserving operators, has been concentrated on describing PDE-preserving operators, also for other spaces than $\mathscr{H}$ [8, 10, 9]. In particular, the following characterization results are known: We let $\mathbb{H}$ denote the set of homogeneous polynomials in $\mathscr{P}$ and H_n denotes the projector $f = \sum_{m \geq 0} f_m \mapsto f_n$ in $\mathscr{H}$ onto the set of n-homogeneous polynomials, $\mathscr{P}_n$, where $\sum_{m \geq 0} f_m$ is the power-series expansion of $f \in \mathscr{H}$ about the origin. Now:

Proposition 2. (See [9, 10].)

(1) $\mathscr{O}(\mathscr{P}) = \mathscr{C}$,

(2) $\mathscr{O}(\mathbb{H})$ *is formed by all operators of the form*

$$\Phi(D)f \equiv \sum_{n \geq 0} H_n \varphi_n(D) f,$$

where the sequence $\Phi = (\varphi_n)$ *in* Exp *satisfies* $\|\varphi_n\|_m \leq CM^n$ *for some* $C, M, m \geq 0$ *and is unique.*

(Note that if $\varphi \in$ Exp, *then* $\varphi(D) = \Phi(D) \in \mathscr{C}$ *where* $\Phi = (\varphi, \varphi, \ldots)$.*)*

As we have pointed out, the study under consideration links algebra (algebraic geometry) and operator theory. We believe that the best applications can be found based on this link. We give an example of one application from the invariant subspace theory, to the reader that wants more details on this topic, we refer to [3]:

EXAMPLE 1 (Hypercyclicity). We shall prove (Theorem 2) that if $T \in \mathscr{O}(\varphi)$, $\varphi \neq 0$, then T "almost" commutes with $\varphi(D)$ in the sense that $\varphi(D)T = T^{(\varphi)}\varphi(D)$ for some operator $T^{(\varphi)} \in \mathscr{L}$, which is unique (by Malgrange's Theorem) and is called the derivative of T with respect to φ. Now, if $T \in \mathscr{L}$ is hypercyclic and f a corresponding hypercyclic vector, i.e. the orbit $\{f, Tf, T^2f, ...\}$ is dense in $\mathscr{H}$, it follows that $T^{(\varphi)}$ also is hypercyclic and $\varphi(D)f$ is a hypercyclic vector. Thus, by studying PDE-preserving properties, and corresponding derivatives, of hypercyclic operators, we may obtain new such operators. In particular, a well-known theorem of Godefroy & Shapiro states: Every $\varphi(D) \in \mathscr{C}$, $\varphi \notin \mathbb{C}$, is hypercyclic [3]. In view of this result, it is of interest to find hypercyclic operators outside $\mathscr{C}$, see also [1]. In fact, in [1] it is proved that in the case of one variable $T_{\lambda:b}$, where $T_{\lambda:b}f(z) \equiv f'(\lambda z+b)$, forms a hypercyclic operator if; (i) $\lambda^n = 1$ for some $n \geq 1$ and $b \in \mathbb{C}$ is arbitrary, and if (ii) $|\lambda| \geq 1$ and $b = 0$. We note that $T_{\lambda:b} = \Phi(D) \in \mathscr{O}(\mathbb{H})$ where $\Phi = (\varphi_n(\xi) = \xi e^{\xi b}\lambda^n)$, so $T_{\lambda:b} \notin \mathscr{C}$ if $\lambda \neq 1$. Thus, with $\lambda, b \in \mathbb{C}$ as in (i) or (ii), $T_{\lambda:b}^{(P)}$ also forms a hypercyclic operator for any non-zero $P \in \mathbb{H}$. With $P(\xi) = \xi^m$, i.e. $P(D) = D^m$, we deduce that $T_{\lambda:b}^{(P)} = \lambda^m T_{\lambda:b}$, hence $\lambda^m T_{\lambda:b}$ is a hypercyclic operator and $f^{(m)}$ a corresponding hypercyclic vector for any such vector f for $T_{\lambda:b}$. (Note also that $T_{\lambda:b} \in \mathscr{O}(\xi^n - \alpha)$ for any $\alpha \in \mathbb{C}$ and positive $n \in \mathbb{N}$ if $\lambda^n = 1$. But $T_{\lambda:b}^{(\xi^n-\alpha)} = T_{\lambda:b}$, so this does not provide us with any new hypercyclic operators, we conclude instead that $f^{(m)} - \alpha f$ is hypercyclic for $T_{\lambda:b}$ if f is.)

To the readers convenience, we conclude this introduction by invoking a list of basic notations that we have or shall introduce:

$\mathscr{H}$, Exp	$\{$Entire functions$\}$, $\{$Exponential type functions$\}$ – d variables;
$\mathscr{P}$, $\mathscr{P}_n$, $\mathbb{H}$	$\{$Polynomials$\}$, $\{n$-homogeneous $P \in \mathscr{P}\}$, $\cup_n \mathscr{P}_n$ – d variables;
$\mathscr{L}$, $\mathscr{C}$	$\{$continuous linear operators$\}$, $\{$convolution operators$\}$(on $\mathscr{H}$).

For $\mathbb{P} \subseteq \mathscr{P}$:

$\mathscr{O}(\mathbb{P})$	The set of PDE-preserving operators for $\mathbb{P}$ (Definition 1, p. 574);
$\hat{\mathbb{P}}$	The PDE-preserving hull of $\mathbb{P}$ (Definition 2, p. 581);
$\mathbb{P}_\bullet$, $\mathbb{P}_\circ$	The set of non-units, respective the non-zero elements, in $\mathbb{P}$;
$\mathbb{P}_S$	$\{P/S : P \in \mathbb{P}\}$ where $S \in \mathscr{P}$ is a common divisor for $\mathbb{P}$;
$[\mathbb{P}]$	$\{[P] : P \in \mathbb{P}\}$ where $[P] \equiv \{Q\text{: } P, Q \text{ associates in } \mathscr{P}\}$ (p. 581);
$\|\mathbb{P}\|$	The number of elements, i.e. equivalence classes, in $[\mathbb{P}]$;
$(\mathbb{P})$, $\langle\mathbb{P}\rangle$	The ideal generated by $\mathbb{P}$ in $\mathscr{P}$ respective in Exp;
$V(\mathbb{P})$	The algebraic set in $\mathbb{C}^d$ defined by $\mathbb{P}$;
$\dim_{\mathcal{D}} \mathbb{P}$	The PDE-preserving dimension of $\mathbb{P}$ (Definition 6, p. 591).

2. Properties and characterization of PDE-preserving operators

Our first objective is to establish necessary and sufficient conditions for an operator T to be PDE-preserving for a given $\varphi \in \text{Exp}$ (Theorem 2). Since $\mathscr{O}(\mathbb{P}) = \cap_{\varphi\in\mathbb{P}}\mathscr{O}(\varphi)$, this will give information about $\mathscr{O}(\mathbb{P})$ for an arbitrary set $\mathbb{P} \subseteq \text{Exp}$.

LEMMA 1. *A set in* Exp *is bounded iff it is contained and bounded in some* Exp_n.

Proof. Using that $\mathcal{F}$ is a topological isomorphism, it suffices to prove that any bounded set $\Lambda \subseteq \mathscr{H}'$ is mapped into some Exp_n and is bounded there. Thus we must prove that there are constants $M, n > 0$ such that $|\lambda(e_\xi)| \leq Me^{n|\xi|}$ for all $\lambda \in \Lambda$ and $\xi \in \mathbb{C}^d$. Since $\mathscr{H}$ is barrelled, Λ is equicontinuous, hence there is a neighborhood of the origin $U = U_{n:\varepsilon} \equiv \{f : \|f\|_n \leq \varepsilon\}$ in $\mathscr{H}$ such that $\Lambda \subseteq U^{\bigcirc}$. Now, $\|e_\xi\|_n \leq e^{n|\xi|} \equiv M(\xi)$ and hence $\varepsilon e_\xi/M(\xi) \in U$ for all ξ and our claim follows. □

We denote by $\mathfrak{S}$ the set of entire mappings $P = P(z, \xi)$, in $2d$ variables $(z, \xi) \in \mathbb{C}^d \times \mathbb{C}^d$, with the following property: For every $n \geq 0$ there are $m = m_n, M = M_n \geq 0$ such that $\|P(\cdot, \xi)\|_n \leq Me^{m|\xi|}$ (i.e., by Lemma 1, $\{P(z, \cdot) : |z| \leq n\}$ forms a bounded set in Exp for all n).

THEOREM 1 (Kernel-Theorem). $T \mapsto P(z, \xi) \equiv e^{-\langle z,\xi\rangle}Te_\xi(z)$ *defines a bijection between* $\mathscr{L}$ *and* $\mathfrak{S}$. *P is called the symbol for T, we write* $T = P(\cdot, D)$ *and have that* $Tf(z) = \langle f, P(z, \cdot)e_z\rangle$. *The symbol* $R \in \mathfrak{S}$ *for* $P(\cdot, D)Q(\cdot, D)$ *is given by* $R(z, \xi) = e^{-\langle z,\xi\rangle}\langle Q(\cdot, \xi)e_\xi, P(z, \cdot)e_z\rangle$.

Proof. We prove that $P(z,\xi) \equiv e^{-\langle z,\xi\rangle}Te_\xi(z) \in \mathfrak{S}$ for any given $T \in \mathscr{L}$. Clearly, P is entire in z. From $Te_\xi(z) = {}^tTe_z(\xi)$ it follows that P is entire in ξ and bounded as required. Indeed, $\mathbb{C}^d \ni z \mapsto e_z \in \mathrm{Exp}$ and ${}^tT : \mathrm{Exp} \to \mathrm{Exp}$ are continuous and hence Lemma 1 gives that $P \in \mathfrak{S}$. Next, let $P \in \mathfrak{S}$ and define T by $Tf(z) \equiv \langle f, P(z,\cdot)e_z\rangle$. It is easily checked that $T \in \mathscr{L}$ and $e^{-\langle z,\xi\rangle}Te_\xi(z) = P(z,\xi)$. Thus, the map $\mathscr{L} \ni T \mapsto e^{-\langle z,\xi\rangle}Te_\xi(z) \in \mathfrak{S}$ is onto and by the fact that $\{e_\xi\colon \xi \in \mathbb{C}^d\}$ forms a total set in $\mathscr{H}$, it is one-to-one. The last statement is elementary. $\square$

REMARK 1. Note that the transpose of $T = P(\cdot,D)$ is given by ${}^tT\varphi(\xi) = \langle P(\cdot,\xi)e_\xi, \varphi\rangle$ and, by reflexivity, we obtain the entire algebra $\mathscr{L}(\mathrm{Exp})$ of continuous linear operators on Exp in this way. Thus, Theorem 1 is also a kernel-theorem for $\mathscr{L}(\mathrm{Exp})$. The symbol for any $\varphi(D) \in \mathscr{C}$ is of course φ.

The following division-theorem is crucial:

LEMMA 2. *Let $0 \neq \varphi \in \mathrm{Exp}$, $P \in \mathfrak{S}$ and assume $P(z,\xi) = \varphi(\xi)Q(z,\xi)$, where $Q(z,\cdot) \in \mathrm{Exp}$ for all $z \in \mathbb{C}^d$. Then $Q \in \mathfrak{S}$.*

Proof. $\varphi(D)$ is surjective and hence the transpose, $\varphi : \psi \mapsto \varphi\psi$, is an injective strict morphism on Exp for the weak topology $\sigma(\mathrm{Exp},\mathscr{H})$ [4, Prop. 3.13.3]. Thus, the inverse, $\varphi^{-1} : \mathrm{Im}\,\varphi \to \mathrm{Exp}$ is weakly continuous. Consequently, $z \mapsto \varphi^{-1}P(z,\cdot) = Q(z,\cdot)$ is continuous for the weak topology on Exp. Thus $\{Q(z,\cdot) : |z| \leq n\}$ forms a bounded set in Exp for any given n and Lemma 1 gives that Q is bounded as required. It remains to prove that Q is entire in z. For fixed z and j, $P_{(s)}(z,\cdot) \equiv (P(z+se_j,\cdot) - P(z,\cdot))/s \in \mathrm{Im}\,\varphi$ and $P_{(s)}(z,\cdot) \to \partial_j P(z,\cdot) \in \mathrm{Exp}$ in Exp as $s \to 0$ in $\mathbb{C}$. (Here e_j, $j = 1,\dots,d$, denote the standard unit basis vectors in $\mathbb{C}^d$ and $\partial_j \equiv \partial/\partial z_j$.) Since $\mathrm{Im}\,\varphi$ is closed, we deduce from this that $\partial_j P(z,\cdot) = \varphi Q_j(z,\cdot)$ for some $Q_j(z,\cdot) \in \mathrm{Exp}$. Thus, if $Q_{(s)}$ denotes the analogue of $P_{(s)}$, $Q_{(s)}(z,\cdot) - Q_j(z,\cdot) = \varphi^{-1}(P_{(s)}(z,\cdot) - \partial_j P(z,\cdot))$ which proves that $\partial_j Q$ exists and by Hartog's Theorem, Q is entire in z. $\square$

THEOREM 2 (Characterization-Theorem). *Let $\varphi \in \mathrm{Exp}$ and $T = P(\cdot,D) \in \mathscr{L}$. Then the following are equivalent:*

(1) *T is PDE-preserving for φ,*
(2) *$\varphi(D)T = S\varphi(D)$ for some $S \in \mathscr{L}$,*
(3) *$\varphi|\varphi(\xi+D)P(\cdot,\xi)(z)$ in $\mathfrak{S}$, i.e., $\varphi(\xi+D)P(\cdot,\xi)(z) = \varphi(\xi)Q(z,\xi)$ for some $Q \in \mathfrak{S}$. (Here $\varphi(\xi+D) \equiv (\tau_\xi\varphi)(D) \in \mathscr{C}$.)*

If $\varphi \neq 0$, then the operator S is unique and is called the derivative of $T \in \mathscr{O}(\varphi)$ with respect to φ and is denoted by $T^{(\varphi)}$. (We justify this terminology in Example 2.)

Proof. The uniqueness of S when $\varphi \neq 0$, follows by the surjectivity of $\varphi(D)$. Further, if $\varphi = 0$, all the equivalencies hold true so assume $\varphi \neq 0$. Now, $\varphi(D)Te_\xi(z) = \varphi(D)P(\cdot,\xi)e_\xi(z) = e^{\langle z,\xi\rangle}\varphi(\xi + D)P(\cdot,\xi)(z)$. Thus 3 is equivalent to that φ divides the symbol for $\varphi(D)T$. Hence 2 and 3 are equivalent and since 2 trivially implies 1, it suffices to prove that if 1 holds, then $\varphi | R$ in $\mathfrak{S}$ where $R(z,\xi) \equiv \varphi(D)Te_\xi(z)$ $(\in \mathfrak{S})$. For fixed $z \in \mathbb{C}^d$ let $\lambda_z(f) \equiv \varphi(D)Tf(z)$. Then $\lambda_z \in \mathscr{H}'$ and $\mathcal{F}\lambda_z(\xi) = R(z,\xi)$. We prove that $\mathcal{F}\lambda_z \in \operatorname{Im}\varphi = \ker\varphi(D)^\perp$. But for any $f \in \ker\varphi(D)$,

$$\langle f, \mathcal{F}\lambda_z\rangle = \lambda_z(f) = \varphi(D)Tf(z) = 0$$

since $T \in \mathscr{O}(\varphi)$. Thus, for every $z \in \mathbb{C}^d$ there is a unique $Q(z,\cdot) \in \operatorname{Exp}$ such that $R(z,\xi) = \varphi(\xi)Q(z,\xi)$, $\xi \in \mathbb{C}^d$. Lemma 2 completes the proof. □

EXAMPLE 2. If $T = \Phi(D) = \sum H_n\varphi_n(D) \in \mathscr{O}(\mathbb{H})$, see Proposition 2, and $P \in \mathbb{H} \setminus \{0\}$ is m-homogeneous, then $T^{(P)} = \Phi^{(m)}(D) \in \mathscr{O}(\mathbb{H})$, where $\Phi^{(m)} \equiv (\varphi_{n+m})$ [10, Theorem 7]. Thus, $T^{(P)}$ only depends on m, not on $P \in \mathscr{P}_m$. (This shift behaviour of the derivative, in this special case, is what we think justifies our terminology "the derivative" in Theorem 2. Especially, $T \equiv \sum_0^r H_n \in \mathscr{O}(\mathbb{H})$ is the *Taylor projector* of order r, that maps $f \in \mathscr{H}$ onto its Taylor polynomial of order r at the origin, and $T^{(P)}$ is thus the Taylor projector of order $r - m$ if $P \in \mathscr{P}_m$ and $m \leq r$.)

$\mathbb{H}$ is multiplicative closed in the sense that $\mathbb{H}\cdot\mathbb{H} \subseteq \mathbb{H}$. From Example 2, $\mathscr{O}(\mathbb{H})$ is invariant under $T \mapsto T^{(P)}$, $P \in \mathbb{H} \setminus 0$, and we shall see that the corresponding holds for any multiplicative closed set $\mathbb{P}$:

THEOREM 3.

(i) *Let $\varphi, \psi \in \operatorname{Exp}$ where $\varphi \neq 0$ and assume $T \in \mathscr{O}(\varphi)$. Then $T \in \mathscr{O}(\varphi\psi)$ iff $T^{(\varphi)} \in \mathscr{O}(\psi)$, and if this holds and $\psi \neq 0$, then $T^{(\varphi)(\psi)} = T^{(\varphi\psi)}$.*

(ii) *If $\mathbb{P} \subseteq \operatorname{Exp}$ is multiplicative closed, then $T^{(\varphi)} \in \mathscr{O}(\mathbb{P})$ for all $T \in \mathscr{O}(\mathbb{P})$ and $0 \neq \varphi \in \mathbb{P}$.*

(iii) *Let $\mathbb{P} \subseteq \operatorname{Exp}$ be $\mathscr{P}$-absorbing in the sense that $\mathscr{P}\cdot\mathbb{P} \subseteq \mathbb{P}$. Then $T \in \mathscr{O}(\mathbb{P})$ iff T is of the form $T = \varphi(D) + S$ where $\varphi(D) \in \mathscr{C}$ and $\operatorname{Im} S \subseteq \cap_{\psi\in\mathbb{P}} \ker\psi(D)$. Thus, if $T \in \mathscr{O}(\mathbb{P})$, then $T^{(\psi)} = \varphi(D) \in \mathscr{C}$, $0 \neq \psi \in \mathbb{P}$, for some $\varphi \in \operatorname{Exp}$.*

Proof. We prove (i) and we may assume $\psi \neq 0$. Assume $T \in \mathscr{O}(\varphi) \cap \mathscr{O}(\varphi\psi)$, thus $\varphi(D)T = T^{(\varphi)}\varphi(D)$ and $[\varphi\psi](D)T = T^{(\varphi\psi)}[\varphi\psi](D)$. Since $[\varphi\psi](D) = \psi(D)\varphi(D)$, we obtain $\psi(D)T^{(\varphi)}\varphi(D) = T^{(\varphi\psi)}\psi(D)\varphi(D)$ and thus, by the surjectivity of $\varphi(D)$, $\psi(D)T^{(\varphi)} = T^{(\varphi\psi)}\psi(D)$. Hence, $T^{(\varphi)} \in \mathscr{O}(\psi)$ with derivative $T^{(\varphi\psi)}$. The converse part is elementary and left to the reader.

(ii) follows from (i) and we prove (iii). The sufficiency is elementary since $\psi(D)(\varphi(D)+S) = \varphi(D)\psi(D) = \varphi(D)\psi(D)$ for all $\psi \in \mathbb{P}$ if $\operatorname{Im} S \subseteq \cap_{\mathbb{P}} \ker \psi(D)$ and $\varphi(D) \in \mathscr{C}$. So assume $T \in \mathscr{O}(\mathbb{P})$ and assume first that $\mathbb{P} = \mathscr{P} \cdot \psi$, $\psi \neq 0$. By (i), and in view of Proposition 2, $T \in \mathscr{O}(\mathbb{P})$ if and only if $T^{(\psi)} \in \mathscr{C}$. Thus $\psi(D)T = T^{(\psi)}\psi(D) = \psi(D)\varphi(D)$, where $\varphi(D) \equiv T^{(\psi)} \in \mathscr{C}$, and hence $T = \varphi(D) + S$ where $\operatorname{Im} S \subseteq \ker \psi(D) = \cap_{\mathbb{P}} \ker \phi(D)$. Next let $\mathbb{P}$ be an arbitrary $\mathscr{P}$-absorbing set, thus $\mathbb{P} = \cup_{\mathbb{P}} \mathscr{P}\phi$, and let $0 \neq \phi, \psi \in \mathbb{P}$. Then, from what we just have proved, $T = T^{(\phi)} + S$ and $T = T^{(\psi)} + L$, where $T^{(\phi)}, T^{(\psi)} \in \mathscr{C}$ and $\operatorname{Im} S \subseteq \ker \phi(D), \operatorname{Im} L \subseteq \ker \psi(D)$. We prove that $T^{(\phi)} = T^{(\psi)}$ and hence that $S = L$. But $\varepsilon(D) \equiv T^{(\phi)} - T^{(\psi)} \in \mathscr{C}$ and $\operatorname{Im} \varepsilon(D) \subseteq \ker \phi(D) + \ker \psi(D) \subseteq \ker[\phi\psi](D)$. This is impossible if $\varepsilon \neq 0$ in view of Malgrange's Theorem. Accordingly, $T^{(\phi)} = T^{(\psi)} \equiv \varphi(D)$ and $T = \varphi(D) + S$ where $\operatorname{Im} S \subseteq \ker \phi(D) \cap \ker \psi(D)$. By the arbitrary choice of ϕ and ψ, $\operatorname{Im} S \subseteq \cap_{\mathbb{P}} \ker \psi(D)$. $\square$

In particular, (iii) implies that the derivative $T^{(\psi)}$ does not depend on $\psi \in \mathbb{P}$ if $\mathbb{P}$ is $\mathscr{P}$-absorbing. Further, for any set $\mathbb{P} \subseteq \operatorname{Exp}$, the set of operators $S \in \mathscr{L}$ such that $\operatorname{Im} S \subseteq \cap_{\mathbb{P}} \ker \psi(D)$ forms a subspace $\mathscr{I} = \mathscr{I}(\mathbb{P})$ of $\mathscr{L}$ (in fact, a right ideal in $\mathscr{L}$), thus $\mathscr{L}/\mathscr{I}$ is well-defined. So if $\mathscr{C}/\mathscr{I}$ denotes the image of $\mathscr{C}$ under the canonical map $\mathscr{L} \to \mathscr{L}/\mathscr{I}$, then (iii) can be formulated: $T \in \mathscr{O}(\mathbb{P})$ iff $T + \mathscr{I} \in \mathscr{C}/\mathscr{I}$ (we write $T \in \mathscr{C}$ mod $\mathscr{I}$). The most important example of $\mathscr{P}$-absorbing sets $\mathbb{P}$, is when $\mathbb{P}$ forms an ideal $\mathbb{I}$ in Exp or in $\mathscr{P}$. We note also that if $\mathbb{I}$ is an ideal in Exp, then $\cap_{\mathbb{I}} \ker \psi(D) = (\cup_{\mathbb{I}} \operatorname{Im} \psi)^{\perp} = \mathbb{I}^{\perp}$, thus:

COROLLARY 1. *Let* $\mathbb{I}$ *be an ideal in* Exp *or in* $\mathscr{P}$. *Then* $T \in \mathscr{O}(\mathbb{I})$ *iff* $T \in \mathscr{C}$ mod $\mathscr{I}(\mathbb{I})$. *In particular, if* $\mathbb{I}$ *is an ideal in* Exp, *then* $S \in \mathscr{I}(\mathbb{I})$ *iff* $\operatorname{Im} S \subseteq \mathbb{I}^{\perp}$, *i.e.* $\operatorname{Im}\ {}^tS \supseteq \mathbb{I}$.

3. PDE-preserving properties of a set $\mathbb{P}$

In this section we concentrate our study on when $\mathbb{P} \subseteq \mathscr{P}$, which we motivated in the introduction.

3.1. PDE-preserving hull

DEFINITION 2. Let $\mathbb{P}$ be a given set of polynomials. The PDE-*preserving hull* of $\mathbb{P}$, is the largest set $\hat{\mathbb{P}}$ in $\mathscr{P}$ such that $\mathscr{O}(\mathbb{P}) = \mathscr{O}(\hat{\mathbb{P}})$, i.e., $\hat{\mathbb{P}} \equiv \{P \in \mathscr{P} : \mathscr{O}(\mathbb{P}) \subseteq \mathscr{O}(P)\}$. (For larger expressions $\{\cdot\}$, we use the notation $\{\cdot\}^{\wedge}$ for the hull.) A set $\mathbb{P}' \subseteq \mathscr{P}$ is said to *generate* $\mathbb{P}$ (or $\mathscr{O}(\mathbb{P})$) when $\hat{\mathbb{P}'} = \hat{\mathbb{P}}$, or equivalently, when $\mathscr{O}(\mathbb{P}) = \mathscr{O}(\mathbb{P}')$.

It is convenient to note the following:

$$\mathbb{P} \subseteq \hat{\mathbb{P}}, \qquad \hat{\hat{\mathbb{P}}} = \hat{\mathbb{P}}, \qquad \mathbb{P}' \subseteq \mathbb{P} \Longrightarrow \hat{\mathbb{P}'} \subseteq \hat{\mathbb{P}}.$$

Thus $\hat{\mathbb{P}}$ is PDE-*convex* in the sense that it coincides with its hull.

EXAMPLE 3. We claim that $\mathbb{H}$ is PDE-convex. Indeed, we must show that to any $Q \in \mathscr{P} \setminus \mathbb{H}$ there is a $T = \Phi(D) \in \mathscr{O}(\mathbb{H})$ (see Proposition 2) and $f \in \ker Q(D)$ such that $Tf \notin \ker Q(D)$. But by choosing f of the form $f = e_a$, where $Q(a) = 0$, it is an easy exercise to construct such an operator T, i.e., sequence Φ.

The primary objective in this section is to describe the hull $\hat{\mathbb{P}}$ of a set $\mathbb{P}$. The main result, Theorem 5, requires some ground work.

Division defines a partial order relation in $\mathscr{P}$ if we identify polynomials that are associates in $\mathscr{P}$. More precisely, let $[\mathscr{P}]$ denote the set of equivalence classes in $\mathscr{P}$ under the equivalence relation; $P \sim Q$ if P and Q are associates, i.e. Q is a non-zero scalar multiple of P. If $[P]$ denotes the equivalence class containing P, we obtain a partial order relation $\preceq$ in $[\mathscr{P}]$ by; $[P] \preceq [Q]$ when $P|Q$ in $\mathscr{P}$. Note that $[0]$ $(= \{0\})$ and $[1]$ $(= \{\text{units in } \mathscr{P}\})$ is a largest, respectively a smallest, element in $[\mathscr{P}]$. Since $\mathscr{P}$ is a UFD, $\gcd \mathbb{P} \equiv \inf[\mathbb{P}]$ (greatest common divisor) and $\operatorname{lcm} \mathbb{P} \equiv \sup[\mathbb{P}]$ (least common multiple) exist for any set $\mathbb{P} \subseteq \mathscr{P}$ where $[\mathbb{P}] \equiv \{[P] : P \in \mathbb{P}\}$. If $[S] = \gcd \mathbb{P}$ we write simply $S = \gcd \mathbb{P}$ and analogously for the lcm. Let us also define $\gcd \mathbb{P} \equiv 0$ and $\operatorname{lcm} \mathbb{P} \equiv 1$ when $\mathbb{P} = \emptyset$. Note that $\operatorname{lcm} \mathbb{P} \neq 0$ if and only if $[\mathbb{P}]$ is a finite set and $0 \notin \mathbb{P}$. By $|M|$ we denote the number of elements in a set M and $\|\mathbb{P}\| \equiv |[\mathbb{P}]|$ if $\mathbb{P} \subseteq \mathscr{P}$. In view of our purposes, it is convenient to introduce the notations $\mathbb{P}_\circ \equiv \mathbb{P} \setminus [0] = \mathbb{P} \setminus \{0\}$ and $\mathbb{P}_\bullet \equiv \mathbb{P} \setminus [1]$.

REMARK 2. If $P \in \hat{\mathbb{P}}$, then $[P] \subseteq \hat{\mathbb{P}}$ and $[\mathbb{P}'] = [\mathbb{P}]$ implies $\hat{\mathbb{P}'} = \hat{\mathbb{P}}$. Further, for any set $\mathbb{P}$, $0, 1 \in \hat{\mathbb{P}}$ and consequently $\hat{\mathbb{P}} = \hat{\mathbb{P}_\circ} = \hat{\mathbb{P}_\bullet}$.

LEMMA 3. *If $P, Q \in \mathscr{P}$, then* $\ker Q(D) \subseteq \ker P(D)$ *if and only if* $Q|P$ *in* $\mathscr{P}$.

Proof. The necessary part is obvious so we assume $\ker Q(D) \subseteq \ker P(D)$ and prove that $Q|P$. By Malgrange's Theorem, $\operatorname{Im}\varphi = \ker \varphi(D)^{\perp}$ for any $\varphi \in \mathrm{Exp}$ and hence, $\ker Q(D) \subseteq \ker P(D)$ is equivalent to $\operatorname{Im} Q \supseteq \operatorname{Im} P$. Thus $Q|P$ in Exp. But since a rational function is an entire function iff it is a polynomial (consequence of Liouville's Theorem and Lemma [11, 1.8.1]), the lemma follows. $\square$

PROPOSITION 3. *Let $\mathbb{P} \subseteq \mathscr{P}$ and assume $Q \in \hat{\mathbb{P}}$. Then* $\gcd \mathbb{P}_{\bullet}$ $|Q$ *if Q is not a unit and $Q|$* $\operatorname{lcm} \mathbb{P}_{\circ}$ *if $Q \neq 0$. In particular, $\|\hat{\mathbb{P}}\| < \infty$ iff $\|\mathbb{P}\| < \infty$.*

Proof. Put $S \equiv \gcd \mathbb{P}_{\bullet}$ and assume Q is not a unit. In view of Lemma 3 we must prove that $\ker S(D) \subseteq \ker Q(D)$. Assume not. Then there is an $f_0 \in \ker S(D) \setminus \ker Q(D)$. (Note that $\ker S(D) \subseteq \cap_{\mathbb{P}_{\bullet}} \ker P(D)$.) Choose $\varphi_0 \notin \ker Q(D)^{\perp} = \operatorname{Im} Q$ (possible since $Q \notin [1]$) and define $Tf \equiv \langle f, \varphi_0 \rangle f_0$. Then $\operatorname{Im} T \subseteq \cap_{\mathbb{P}_{\bullet}} \ker P(D)$ so $T \in \mathscr{O}(\mathbb{P})$. But if $f \in \ker Q(D)$ and $\langle f, \varphi_0 \rangle \neq 0$, $Tf \notin \ker Q(D)$ and consequently $T \notin \mathscr{O}(Q)$. Thus $Q \notin \hat{\mathbb{P}}$ which is a contradiction.

Next we assume $0 \neq Q \in \hat{\mathbb{P}}$ and prove that $Q|L \equiv \operatorname{lcm} \mathbb{P}_{\circ}$, i.e., $\ker Q(D) \subseteq \ker L(D)$ (Lemma 3). Assume not. Then there is an $f_0 \in \ker Q(D) \setminus \ker L(D)$. (Note that $\cup_{\mathbb{P}_{\circ}} \ker P(D) \subseteq \ker L(D)$). By Hahn-Banach Theorem, there is a $\varphi_0 \in \mathrm{Exp} \simeq \mathscr{H}'$ such that $\langle f_0, \varphi_0 \rangle = 1$ and $\varphi_0 \in \ker L(D)^{\perp} = \operatorname{Im} L$. In particular, $\varphi_0 \in \cap_{\mathbb{P}_{\circ}} (\ker P(D)^{\perp})$. Thus with $Tf \equiv \langle f, \varphi_0 \rangle e_a$, where $Q(a) \neq 0$, $T \in \mathscr{O}(\mathbb{P})$. But $Tf_0 = e_a \notin \ker Q(D)$ so $T \notin \mathscr{O}(Q)$ which contradicts that $Q \in \hat{\mathbb{P}}$.

Finally, if $\|\mathbb{P}\| < \infty$, then $L \equiv \operatorname{lcm} \mathbb{P}_{\circ} \neq 0$ and from what we just have proved, $\|\hat{\mathbb{P}}\| \leq |\{[P] : P|L \text{ or } P = 0\}| < \infty$. $\square$

DEFINITION 3. A set $\mathbb{P} \subseteq \mathscr{P}$ is said to be *D-relatively prime* when $\cap_{P \in \mathbb{P}} \ker P(D) = \{0\}$. $\mathbb{P}$ is pre-*D-relatively prime* when $\mathbb{P}_S \equiv \{P/S : P \in \mathbb{P}\}$, where $S \equiv \gcd \mathbb{P}$, forms a D-relatively prime set.

In particular, a single-element set $\{P\}$ is D-relatively prime iff P is a unit in $\mathscr{P}$. Clearly, every D-relatively prime set $\mathbb{P}$ is relatively prime. By Proposition 4, that follows, the converse holds if (and in fact only if) $d = 1$, accordingly – every set in $\mathscr{P}$ is pre-D-relatively prime when $d = 1$.

We introduce some notation. We shall let $\langle \mathbb{P} \rangle$ and $(\mathbb{P})$ denote the ideals in Exp and $\mathscr{P}$ respectively generated by a set $\mathbb{P} \subseteq \mathscr{P} \subseteq \mathrm{Exp}$. $Z(P)$ denotes the zero-set $\{a \in \mathbb{C}^d : P(a) = 0\}$ for $P \in \mathscr{P}$ and by $V(\mathbb{P})$ we denote the algebraic set $\cap_{\mathbb{P}} Z(P)$ defined by $\mathbb{P} \subseteq \mathscr{P}$, and now:

PROPOSITION 4. *Let $\mathbb{P} \subseteq \mathscr{P}$, then the following are equivalent:*

(i) $\mathbb{P}$ *is D-relatively prime,*
(ii) $\overline{\langle \mathbb{P} \rangle} = \mathrm{Exp}$,
(iii) $V(\mathbb{P}) = \emptyset$,
(iv) $(\mathbb{P}) = \mathscr{P}$,
(v) $P_1 Q_1 + \ldots + P_n Q_n = 1$ *for some* $Q_i \in \mathscr{P}$, $P_i \in \mathbb{P}$ (*a so called* Bezout identity).

If $d = 1$ *then* (i-v) *are all equivalent to* $\mathbb{P}$ *being relatively prime.*

Proof. That $\mathbb{P}$ being D-relatively prime is equivalent to (ii) follows by the following observation $\overline{\langle \mathbb{P} \rangle} = (\cup_{\mathbb{P}} \operatorname{Im} P)^{\perp\perp} = (\cap_{\mathbb{P}} \ker P(D))^{\perp}$, since $\operatorname{Im} P^{\perp} = \ker P(D)$. The equivalence between (iv) and (v) is of course obvious, and that (iii) and (iv) are equivalent is a consequence of Hilbert's (weak) Nullstellensatz, see [2, p. 20]. Next, if $a \in V(\mathbb{P})$ then $e_a \in \cap_{\mathbb{P}} \ker P(D)$. Consequently, (i) implies (iii). Since every D-relatively prime set is relatively prime, it remains only to prove that the converse holds true when $d = 1$. But a set $\mathbb{P}$ of one-variable polynomials is relatively prime iff the elements of $\mathbb{P}$ have no common zeros, i.e. when (iii), and thus (i), holds true. $\square$

EXAMPLE 4. For any polynomial P, the set $\{P, P+1\}$ is D-relatively prime. $\{\xi_1, \xi_2\}$ forms a relatively but not a D-relatively prime set. Finally, $\{\xi_1, \xi_1 + 1, \xi_1\xi_2\}$ is an example of a D-relatively prime set which is not pairwise relatively prime.

It follows from Proposition 4 that every D-relatively prime set $\mathbb{P}$ contains a finite subset which is also D-relatively prime. Another consequence of the proposition is that $\overline{\langle \mathbb{P} \rangle} = \mathrm{Exp}$ actually implies $\langle \mathbb{P} \rangle = \mathrm{Exp}$ and thus – there is no proper dense ideal in Exp that is generated by some set of polynomials.

PROPOSITION 5. *Let $S \in \mathscr{P}$ and $\mathbb{P} \subseteq \mathscr{P}$. Then the following are equivalent:*

(1) $\ker S(D) = \cap_{\mathbb{P}} \ker P(D)$,
(2) $S | P$ *for all* $P \in \mathbb{P}$ *and the set* $\mathbb{P}_S \equiv \{P/S : P \in \mathbb{P}\}$ *is D-relatively prime,*
(3) $S = \gcd \mathbb{P}$ *and* $\mathbb{P}$ *is* pre-*D-relatively prime.*

Proof. Clearly, 3 implies 2. We assume 1 and prove that 3 holds. Since $\ker S(D) \subseteq \ker P(D)$ for all $P \in \mathbb{P}$, S is a common divisor for the set $\mathbb{P}$ by Lemma 3. If $S = 0$, then $\mathbb{P} = \{0\}$ and thus $\mathbb{P}_S = \{1 = 0/0\}$ is D-relatively prime and $S = \gcd \mathbb{P}$. Next, assume $S \neq 0$. Let

$f \in \cap_{\mathbb{P}} \ker P_S(D)$, $P_S \equiv P/S$, and choose g with $S(D)g = f$. Then $P(D)g = P_S(D)f = 0$ for all $P \in \mathbb{P}$. Hence $g \in \ker S(D)$, i.e. $f = S(D)g = 0$. Thus $\mathbb{P}_S$ is D-relatively prime and we must prove that $S = \gcd \mathbb{P}$. Assume $S'|P$ for all $P \in \mathbb{P}$ and $S' = SR$. We must prove that R is a unit. Assume not. Then $R(D)f = 0$ for some $f \neq 0$. But for any $P_S = P/S \in \mathbb{P}_S$, $P_S(D)f = P_{S'}(D)R(D)f = 0$ and hence, since $\mathbb{P}_S$ is D-relatively prime, $f = 0$ which is a contradiction, thus 3 holds.

Assume 2. Since $S|P$ for all $P \in \mathbb{P}$, we deduce from Lemma 3 that $\ker S(D) \subseteq \cap_{\mathbb{P}} \ker P(D)$. Next, let $f \in \cap_{\mathbb{P}} \ker P(D)$. Then if $P_S \in \mathbb{P}_S$, $P_S(D)(S(D)f) = P(D)f = 0$. Hence $S(D)f = 0$. This proves that 2 implies 1 and hence the proposition. □

The attentive reader may have already noted that Proposition 5 extends parts of Proposition 4 and, for the sake of consistency, we remark that it is easily checked that a set $\mathbb{P}$ is pre-D-relatively prime iff $\langle\mathbb{P}\rangle$ forms a principal ideal $(\langle\gcd \mathbb{P}\rangle)$ in Exp or equivalently, $(\mathbb{P})$ is a principal ideal $((\gcd \mathbb{P}))$ in $\mathscr{P}$.

If $P = P_i Q_i$, $i = 1, ..., n$, for some polynomials P_i, Q_i, then, by the definition of the lcm, $\mathrm{lcm}\{P_i\}_i | P$ but we need the following more general:

Lemma 4. *Let $\mathbb{P} = \{P_1, ..., P_n\} \subseteq \mathscr{P}$ be a finite set and assume $\varphi = P_1\varphi_1 = ... = P_n\varphi_n \in \mathrm{Exp}$ for some $\varphi_i \in \mathrm{Exp}$. Then $L|\varphi$ in* Exp*, where $L \equiv \mathrm{lcm}\, \mathbb{P}$ $(\in \mathscr{P})$.*

Proof. We may assume $0 \notin \mathbb{P}$. Assume first that $n = 2$, $\mathbb{P} = \{P, Q\}$, and put $S \equiv \gcd \mathbb{P}$. Then $P' \equiv P/S$ and $Q' \equiv Q/S$ are relatively prime and $L = PQ' = P'Q = PQ/S$. Thus we only have to prove that if $P(\xi)u(\xi) = Q(\xi)v(\xi)$, i.e. $P'(\xi)u(\xi) = Q'(\xi)v(\xi)$, for some $u, v \in$ Exp, then $Q'|u$ (or equivalently $P'|v$) in Exp. Assume first that P' is irreducible. Then P' is not a factor of Q' and hence there is a point $a \in Z(P')$ such that $Q'(a) \neq 0$. (Indeed, by Hilbert's Nullstellensatz, an irreducible polynomial R' is a factor of $R \in \mathscr{P}$ iff $Z(R') \subseteq Z(R)$.) Thus $w \equiv u/Q'$ is holomorphic in a neighborhood of a and $P'w = v$ is entire. Since $P'(a) = 0$ and P' is irreducible, w is an entire function by [11, Lemma 1.7.3]. Now, $u = Q'w$ and since $u \in$ Exp, $w \in$ Exp by [11, Lemma 1.8.1] (see also [6, p. 183]). Next, if $P' = P'_1 ... P'_n$ where $P'_i \in \mathscr{P}$ are irreducible, we repeat these arguments for P'_1, and obtain $P'_2 ... P'_n u = Q'w_1$, and then for P'_2 etc. and obtain finally that $u = Q'w$ for some $w \in$ Exp.

Now, using that $\mathrm{lcm}\{P_1, ..., P_{n+1}\} = \mathrm{lcm}\{L_n, P_{n+1}\}$, where $L_n \equiv \mathrm{lcm}\{P_1, ..., P_n\}$, the lemma follows by induction. □

LEMMA 5. *If* $\mathbb{P} = \{P_1, ..., P_n\} \subseteq \mathscr{P}$ *and* $L \equiv \operatorname{lcm} \mathbb{P}$, *then* $\ker L(D) = \overline{\sum_i \ker P_i(D)}$. *(In particular,* $\ker[P_1(D)...P_n(D)] = \overline{\sum_i \ker P_i(D)}$ *if* $\mathbb{P}$ *is pairwise relatively prime.)*

Proof. In view of Lemma 4, the ideal $\cap_i \operatorname{Im} P_i$ in Exp is generated by L and hence

$$\begin{aligned}&\overline{\ker P_1(D) + ... + \ker P_n(D)}\\ &= (\cup_i \ker P_i(D))^{\perp\perp} = (\cap_i \operatorname{Im} P_i)^{\perp}\\ &= \operatorname{Im} L^{\perp} = \ker L(D).\end{aligned}$$ □

THEOREM 4. *If* $\mathbb{P} \subseteq \mathscr{P}$, *then:*

(1) $S \equiv \gcd \mathbb{P} \in \hat{\mathbb{P}}$ *iff* $\mathbb{P}$ *is* pre-D-*relatively prime or* S *is a unit.*
(2) $L \equiv \operatorname{lcm} \mathbb{P} \in \hat{\mathbb{P}}$.

Proof. The necessary part in 1 follows by Proposition 5 and we prove that if $S \equiv \gcd \mathbb{P}$ is a non-unit element of $\hat{\mathbb{P}}$, then $\mathbb{P}_S$ is D-relatively prime, i.e., $\ker S(D) = \cap_{\mathbb{P}} \ker P(D)$. Assume not. Then there is an $f_0 \in \cap_{\mathbb{P}} \ker P(D) \setminus \ker S(D)$. Choose $\varphi_0 \notin \ker S(D)^{\perp} = \operatorname{Im} S$ (possible since $S \notin [1]$) and put $Tf \equiv \langle f, \varphi_0 \rangle f_0$. Then $T \in \mathscr{O}(\mathbb{P}) \setminus \mathscr{O}(S)$ which contradicts that $S \in \hat{\mathbb{P}}$.

Next we prove 2. We may assume that $L \neq 0$ and thus that $\mathbb{P}$ is finite, $\mathbb{P} = \{P_1, ..., P_n\}$, and $P_i \neq 0$. By Lemma 5, $\ker L(D) = \overline{\sum_i \ker P_i(D)}$ and hence,

$$T \ker L(D) \subseteq T \overline{\sum_i \ker P_i(D)} \subseteq \overline{\ker L(D)} = \ker L(D)$$

for any $T \in \mathscr{O}(\mathbb{P})$. □

EXAMPLE 5. Let $\mathbb{P} = \{P, Q\}$ be a pair of non-constant polynomials and put $S \equiv \gcd \mathbb{P}$ and $L \equiv \operatorname{lcm} \mathbb{P}$. Then $L = P'Q'S$, where $P' = P/S$ and $Q' = Q/S$, and Theorem 4 gives that $[S, L] \subseteq \hat{\mathbb{P}}$ if $\mathbb{P}_S = \{P', Q'\}$ is D-relatively prime and $[L] \subseteq \hat{\mathbb{P}}$ if not. In fact, we shall see (apply Theorem 7 and Corollary 4 below) that these are the only possible additional non-constants in $\hat{\mathbb{P}}$. Summing up; if P', Q' are D-relatively prime, then $\hat{\mathbb{P}}$ is given by $[0, 1, S, P, Q, L]$ (S may here be 1), and if P' and Q' not are D-relatively prime, $\hat{\mathbb{P}} = [0, 1, P, Q, L]$.

COROLLARY 2. *Let* $\mathbb{P} = \{P_1, ..., P_n\}$ *be a finite set of non-zero polynomials. Then the product* $P_1...P_n \in \hat{\mathbb{P}}$ *iff the polynomials* P_i *are pairwise relatively prime.*

Proof. If the elements P_i are pairwise relatively prime, then $L \equiv \operatorname{lcm}\mathbb{P} = P_1...P_n$ and hence $P_1...P_n \in \hat{\mathbb{P}}$ by Theorem 4 (note that $L \neq 0$). Conversely, assume $P \equiv P_1...P_n \in \hat{\mathbb{P}}$. Then, by Proposition 3, $P|L$ and since $L|P$, P and L are associates. This implies that P_i, $i = 1, ..., n$, are pairwise relatively prime. □

LEMMA 6. *Every finitely generated ideal in* Exp *is closed. In particular, $\langle\mathbb{P}\rangle$ is closed if $\|\mathbb{P}\| < \infty$ where $\mathbb{P} \subseteq \mathscr{P}$.*

Proof. Assume the ideal $\mathbb{I}$ in Exp is generated by the elements $\varphi_1, ..., \varphi_n \in \text{Exp}$. Consider the operator $\Phi : \mathscr{H} \to \mathscr{H}^n$ defined by $\Phi f \equiv (\varphi_1(D)f, ..., \varphi_n(D)f)$. Then $\mathbb{I}$ is the range of the transpose ${}^t\Phi : \text{Exp}^n \to \text{Exp}$ when the product spaces $\mathscr{H}^n$ and Exp^n are put into duality in the natural way. Thus we have to prove that $\operatorname{Im}{}^t\Phi$ is closed. Now, Exp^n and Exp are the duals of the reflexive Fréchet spaces $\mathscr{H}^n$ (product topology) and $\mathscr{H}$ respectively and thus, by Banach's Theorem, or see [4, Prop. 3.17.17], $\operatorname{Im}{}^t\Phi$ is closed iff $\operatorname{Im}\Phi$ is closed. But every $\varphi_i(D)$ is a closed range operator in view of Malgrange's Theorem and hence $\operatorname{Im}\Phi$ is closed. □

We are now ready to describe the hull $\hat{\mathbb{P}}$ when $\|\mathbb{P}\|$ is finite. Since any constant polynomial belongs to $\hat{\mathbb{P}}$, it suffices to describe the non-constants in $\hat{\mathbb{P}}$.

THEOREM 5. *Let $\mathbb{P} \subseteq \mathscr{P}$ and assume $\|\mathbb{P}\| < \infty$. Let $Q = I_0^{r_0}...I_n^{r_n}$ (non-constant) be the factorisation of $Q \in \mathscr{P}$ into irreducible factors $I_i \in \mathscr{P}$ and put $\mathbb{P}_i \equiv \mathbb{P} \cap (I_i^{r_i}) = \{P \in \mathbb{P} : I_i^{r_i}|P\}$, $i \leq n$. Then the following are equivalent:*

(i) $Q \in \hat{\mathbb{P}}$,
(ii) $Q \in \langle\mathbb{P}_i\rangle$ *for all* i,
(iii) $\langle Q\rangle = \cap_i\langle\mathbb{P}_i\rangle$.

In particular, if $Q \in \hat{\mathbb{P}}$, then for every irreducible factor I_i there is a $P \in \mathbb{P}$ such that I_i is a factor of P of the same multiplicity r_i as for Q and moreover, $Q = \operatorname{lcm}\{\gcd\mathbb{P}_i\}_i$.

Proof. We prove that (i) implies (ii) so assume $Q \in \hat{\mathbb{P}}$. Assume that $Q \in \langle\mathbb{P}_i\rangle$ does not hold true for some i, say, $i = 0$. Since $\langle\mathbb{P}_i\rangle$ are ideals and $\langle\mathbb{P}_i\rangle = \overline{\langle\mathbb{P}_i\rangle}$ (Lemma 6), $Q \in \langle\mathbb{P}_i\rangle$ is equivalent to $\operatorname{Im} Q \subseteq \overline{\langle\mathbb{P}_i\rangle} = (\cap_{\mathbb{P}_i}\ker P(D))^\perp$. Thus, our assumption means that there is an $f_0 \in \cap_{\mathbb{P}_0}\ker P(D)$ outside $\ker Q(D)$. Next, assume first that $\mathbb{P}\setminus\mathbb{P}_0 \neq \emptyset$ and put $L_0 \equiv \operatorname{lcm}(\mathbb{P}\setminus\mathbb{P}_0)$. Then $L_0 \neq 0$ and we claim that there is a $\varphi_0 \in (\ker L_0(D))^\perp \setminus \ker Q(D)^\perp$. Indeed, if not, then $Q|L_0$ which is not possible by the definition of $\mathbb{P}_0$ (I_0 is a factor of L_0 of order $< r_0$). Now, with $Tf \equiv \langle f, \varphi_0 \rangle f_0$, $T \in \mathscr{O}(\mathbb{P}) \setminus \mathscr{O}(Q)$ which contradicts that $Q \in \hat{\mathbb{P}}$. If $\mathbb{P} \setminus \mathbb{P}_0$ is empty, we choose $\varphi_0 \notin \ker Q(D)^{\perp}$ and obtain a contradiction in the same way. Thus $\operatorname{Im} Q \subseteq \overline{\langle \mathbb{P}_i \rangle} = \langle \mathbb{P}_i \rangle$ for all i so (ii) holds true.

Conversely, assume $Q \in \langle \mathbb{P}_i \rangle = \overline{\langle \mathbb{P}_i \rangle}$, i.e. $\operatorname{Im} Q \subseteq (\cap_{\mathbb{P}_i} \ker P(D))^{\perp}$, for all i. Then

$$(1) \qquad \ker Q(D) \supseteq \cap_{\mathbb{P}_i} \ker P(D), \qquad i \leq n.$$

Let $T \in \mathscr{O}(\mathbb{P})$ be arbitrary. We must prove that $T \ker Q(D) \subseteq \ker Q(D)$. For any i we have that $\ker I_i^{r_i}(D) \subseteq \ker P(D)$ for all $P \in \mathbb{P}_i$. Hence,

$$T \ker I_i^{r_i}(D) \subseteq \cap_{\mathbb{P}_i} \ker P(D) \subseteq \ker Q(D), \qquad i \leq n.$$

By Lemma 5, $\ker Q(D) = \overline{\sum_i \ker I_i^{r_i}(D)}$ and we obtain

$$(2) \qquad T \ker Q(D) \subseteq T \overline{\sum_i \ker I_i^{r_i}(D)} \subseteq \overline{\ker Q(D)} = \ker Q(D).$$

Next, it is trivial that (iii) implies (ii) and we prove the converse. But every element φ of $\cap_i \langle \mathbb{P}_i \rangle$ is a multiple of all the elements $I_i^{r_i}$ and we deduce from Lemma 4 that $Q = \operatorname{lcm}\{I_i^{r_i}\}_i$ divides φ in Exp, i.e. $\varphi \in \langle Q \rangle$ and thus $\cap_i \langle \mathbb{P}_i \rangle \subseteq \langle Q \rangle$. Now, if (ii) holds true we clearly must have equality so (ii) implies (iii) and thus, (i-iii) are all equivalent.

The last statement now follows. Since assume that, for some i, every element of $\mathbb{P}_i$ contains the factor $I_i^{r_i+1}$. Then so does every polynomial in $\langle \mathbb{P}_i \rangle$ which contradicts that $Q \in \hat{\mathbb{P}}$ in view of (ii). Further, $Q \in \langle \mathbb{P}_i \rangle$ implies $I_i^{r_i} | \gcd \mathbb{P}_i | Q$. Thus, $\operatorname{lcm}\{\gcd \mathbb{P}_i\}_i | Q$ but also $Q = \operatorname{lcm}\{I_i^{r_i}\}_i | \operatorname{lcm}\{\gcd \mathbb{P}_i\}_i$, hence $Q = \operatorname{lcm}\{\gcd \mathbb{P}_i\}_i$. □

Remark 3. We remark the following reformulation. The primary decomposition of the ideal (Q) is $\mathbb{I}_0 \cap \ldots \cap \mathbb{I}_n$ where $\mathbb{I}_i \equiv (I_i^{r_i})$, and $\mathbb{P}_i = \mathbb{P} \cap \mathbb{I}_i$. Thus, $Q \in \hat{\mathbb{P}}$ iff $\mathbb{P}$ meets every primary component $\mathbb{I}_i$ of (Q) and in such a way that $\langle Q \rangle \in \cap_i \langle \mathbb{P} \cap \mathbb{I}_i \rangle$. Moreover, note that for *any* set $\mathbb{P} \subseteq \mathscr{P}$, if $Q \in \langle \mathbb{P} \rangle$ then Q must vanish on $V(\mathbb{P})$. Thus, by Hilbert's Nullstellensatz, $Q \in \langle \mathbb{P} \rangle$ implies that the polynomial Q belongs to the radical $\operatorname{rad}(\mathbb{P})$, i.e., $Q^m \in (\mathbb{P})$ for some $m \geq 1$. (See also Section 4.)

We obtain that the following properties must be shared by any two sets that generate the same set.

Corollary 3. *Assume* $\hat{\mathbb{P}} = \hat{\mathbb{B}}$, *where* $\|\mathbb{P}\| < \infty$ (*and thus* $\|\mathbb{B}\| < \infty$), *then*:

(1) $V(\mathbb{P}_\bullet) = V(\mathbb{B}_\bullet)$,
(2) $\operatorname{lcm} \mathbb{P}_\circ = \operatorname{lcm} \mathbb{B}_\circ$,

(3) $\gcd \mathbb{P}_\bullet = \gcd \mathbb{B}_\bullet$,
(4) $\mathbb{P}_\bullet$ *is pre-D-relatively prime iff* $\mathbb{B}_\bullet$ *is.*

Proof. 1 is elementary in view of (ii) in Theorem 5 and by noting that it suffices to prove 1 with $\mathbb{B} = \hat{\mathbb{P}}$. 2 follows by Proposition 3 and Theorem 4. 3 is a consequence of that S is common divisor of $\mathbb{P}_\bullet$ iff it is a common divisor for the non-units in $\hat{\mathbb{P}}$, which follows by (ii) in Theorem 5. Finally, 3 and Theorem 4 implis 4 (note that 4 reduces to 1 when $\gcd \mathbb{P}_\bullet = \gcd \mathbb{B}_\bullet = 1$). $\square$

In the following corollary we describe how to extend the hull.

COROLLARY 4. *Let* $\mathbb{P} \subseteq \mathscr{P}$ *and assume* $\|\mathbb{P}\| < \infty$. (i) *Assume* $L \in \mathscr{P}$ *is a multiple of all the elements in* $\mathbb{P}_\circ$, *i.e.* $\operatorname{lcm} \mathbb{P}_\circ | L$, *then* $\widehat{\{\mathbb{P} \cup \{L\}\}} = \hat{\mathbb{P}} \cup [L]$. (ii) *If* $S | \gcd \mathbb{P}_\bullet$, *then* $\widehat{\{\mathbb{P} \cup \{S\}\}} = \hat{\mathbb{P}} \cup [S] = S\widehat{\mathbb{P}_{\bullet S}}$.

Proof. (i) We may assume $L \neq 0$ and thus $\operatorname{lcm} \mathbb{P}_\circ \neq 0$. Clearly, it suffices to prove that $\widehat{\{\mathbb{P} \cup \{L\}\}} \subseteq \hat{\mathbb{P}} \cup [L]$. So let $Q = I_0^{r_0} \ldots I_n^{r_n}$ be a (non-constant) polynomial in the hull of $\mathbb{P} \cup \{L\}$. By Theorem 5, $Q \in \langle(\mathbb{P} \cup \{L\})_i\rangle$ for all i. Now, if $\mathbb{P}_i = \emptyset$ for some $i = i_0$, we must have that $(\mathbb{P} \cup \{L\})_{i_0} = \{L\}$ and hence $Q \in \operatorname{Im} L$. By Proposition 3, $Q | \operatorname{lcm}(\mathbb{P} \cup \{L\})_\circ = L$, hence Q and L are associates. On the other hand, if $\mathbb{P}_i \neq \emptyset$ for all i, then, since L is a multiple of all the elements in $\mathbb{P}_i$, $\langle(\mathbb{P} \cup \{L\})_i\rangle = \langle \mathbb{P}_i \rangle$ for all i. Thus $Q \in \cap_i \langle \mathbb{P}_i \rangle$ and hence $Q \in \hat{\mathbb{P}}$ by Theorem 5.

(ii) Next we prove that $\hat{\mathbb{P}} \cup [S]$ equals $S\widehat{\mathbb{P}_{\bullet S}}$. Let $Q \in \hat{\mathbb{P}} \cup [S]$. Then $S|Q$ and we must prove that $Q' \equiv Q/S \in \widehat{\mathbb{P}_{\bullet S}}$. We may assume Q' is not a constant and thus $Q \in \hat{\mathbb{P}}$. So let I be an irreducible factor of Q' of multiplicity $r' \geq 1$. Then I is a factor of Q of multiplicity $r \geq r'$. Thus, in view of Theorem 5, $Q = \sum_i \varphi_i P_i$ for some $\varphi_i \in \operatorname{Exp}$ and where $I^r | P_i = P_i' S \in \mathbb{P}_\bullet$ for all i. We deduce that $P_i' \in \mathbb{P}_{\bullet S}$ and $I^{r'} | P_i'$ so $Q' \in \widehat{\mathbb{P}_{\bullet S}}$. Conversely, let $Q = SQ' \in S\widehat{\mathbb{P}_{\bullet S}}$, where $Q' \in \widehat{\mathbb{P}_{\bullet S}}$. We may assume $Q' \notin [1]$, i.e. $Q \notin [S]$, and must thus prove that $Q \in \hat{\mathbb{P}}$. Let I be an irreducible factor of Q of multiplicity $r \geq 1$. Then I is a factor of Q' of multiplicity $r' \leq r$. If $r' > 0$ we obtain with arguments as above that $Q \in \langle\{P_i\}_i\rangle$ where $P_i \in \mathbb{P}$ and $I^r | P_i$. On the other hand, if $r' = 0$, then I is a factor of S of multiplicity r. Now choose an arbitrary irreducible factor J of Q'. Then $Q' \in \langle\{P_i'\}_i\rangle$ where $P_i' \in \mathbb{P}_{\bullet S}$ and $J | P_i'$. Thus, $P_i \equiv SP_i' \in \mathbb{P}$, $I^r | P_i$ and $Q \in \langle\{P_i\}_i\rangle$. Hence, by virtue of Theorem 5, $Q \in \hat{\mathbb{P}}$ and we have proved that $\hat{\mathbb{P}} \cup [S] = S\widehat{\mathbb{P}_{\bullet S}}$. Finally, with $\mathbb{P}' \equiv \mathbb{P} \cup \{S\}$, the hull of $\mathbb{P}'_{\bullet S}$ equals $\widehat{\mathbb{P}_{\bullet S}}$, so from what we just have proved, $\hat{\mathbb{P}}' = \hat{\mathbb{P}}' \cup [S] = S\widehat{\mathbb{P}'_{\bullet S}} = S\widehat{\mathbb{P}_{\bullet S}}$. □

The rest of this subsection is devoted to give a global description of the hull for some special cases.

PROPOSITION 6. *Assume $\|\mathbb{P}\| < \infty$ and that $\mathbb{P}$ satisfies the following property: For every irreducible polynomial I and $n \geq 1$ such that I is a factor of some $P \in \mathbb{P}$ of multiplicity n, the set $\mathbb{P} \cap (I^n) = \{P \in \mathbb{P} : I^n | P\}$ is* pre-*D-relatively prime. Then $\hat{\mathbb{P}}$ is formed by constants and the elements of the form* $\mathrm{lcm}\{\gcd \mathbb{P}_i\}_i$, *where $\mathbb{P}_i$ are* pre-*D-relatively prime subsets of $\mathbb{P}$.*

Proof. By Theorem 4, any element of the described form belongs to $\hat{\mathbb{P}}$. Thus, we must prove that every non-constant $Q \in \hat{\mathbb{P}}$ is of the form $\mathrm{lcm}\{\gcd \mathbb{P}_i\}_i$, where every $\mathbb{P}_i$ is a pre-D-relatively prime set in $\mathbb{P}$. But our assumption implies that every set $\mathbb{P}_i$ in Theorem 5 is pre-D-relatively prime and the last part of the same theorem completes the proof. □

Since every set in $\mathscr{P}$ is pre-D-relatively prime when $d = 1$, we obtain:

COROLLARY 5. *Assume $\|\mathbb{P}\| < \infty$ and that $d = 1$. Then $\hat{\mathbb{P}}$ is formed by constants and the elements of the form* $\mathrm{lcm}\{\gcd \mathbb{P}_i\}_i$, $\mathbb{P}_i \subseteq \mathbb{P}$ *(i.e. the elements $\vee_i \wedge_j P_{ij}$, $P_{ij} \in \mathbb{P}$, where $P \vee Q \equiv \mathrm{lcm}\{P, Q\}$ and $P \wedge Q \equiv$* $\gcd\{P, Q\}$*).*

We repeat, Corollary 5 is a consequence of the fact that every subset of $\mathscr{P}$ forms a pre-D-relatively prime set. The problem that occurs when $d \geq 2$, is that the irreducible factors in the "UFD-factorisation" may not be pairwise D-relatively prime. This suggests:

DEFINITION 4. A set $\mathbb{P} \subseteq \mathscr{P}$ is said to admit a *D-prime factorisation* if there is a set $\mathbb{J}$, of pairwise D-relatively prime polynomials, such that for every non-constant $P \in \mathbb{P}$, $P \in [J_0^{r_0} \dots J_n^{r_n}]$ for some $J_i \in \mathbb{J}$ and $r_i \geq 1$.

(Thus when $d = 1$ the ordinary factorisation into irreducible factors works in the sense that we may choose $\mathbb{J}$ as the set of irreducible polynomials.) Clearly, if $\mathbb{P}$ admits a D-prime factorisation, so does every subset of $\mathbb{P}$, and we can now extend Corollary 5:

COROLLARY 5 (continued). *Assume $\mathbb{P}$ admits a D-prime factorisation (d is arbitrary and $\|\mathbb{P}\| < \infty$). Then every subset of $\mathbb{P}$ is* pre-*D-relatively prime and thus, $\hat{\mathbb{P}}$ is formed by constants and the elements of the form* $\mathrm{lcm}\{\gcd \mathbb{P}_i\}_i$, $\mathbb{P}_i \subseteq \mathbb{P}$.

Proof. In view of Proposition 6, we only have to prove that every set $\mathbb{B}$ that admits a D-prime factorisation forms a pre-D-relatively prime set. By noting that $\mathbb{B}_S$, $S \equiv \gcd \mathbb{B}$, admits a D-prime factorisation if $\mathbb{B}$ does, it suffices to prove that every relatively prime set $\mathbb{B}$ that admits a D-prime factorisation, forms a D-relatively prime set. Assume, for simplicity, that $\mathbb{B} = \{P, Q\}$. Our assumption means that we can write $P \equiv P_1 ... P_n$ and $Q \equiv Q_1 ... Q_m$ where P_i, Q_j are D-relatively prime for all i, j. In view of Proposition 4, $V(P_i) \cap V(Q_j) = \emptyset$ and we must prove that $V(P) \cap V(Q)$ is empty. But $V(P) = \cup_i V(P_i)$, and analogously for Q, hence this follows by D'Morgan's rule. Analogous arguments hold for a general $\mathbb{B}$. $\square$

PROPOSITION 7. *Assume $\|\mathbb{P}\| < \infty$, then if $\mathbb{P}$ admits a D-prime factorisation, so does $\hat{\mathbb{P}}$. Thus, for any subset $\mathbb{B} \subseteq \hat{\mathbb{P}}$, $\hat{\mathbb{B}}$ is formed by constants and the elements of the form* $\mathrm{lcm}\{\gcd \mathbb{B}_i\}_i$, $\mathbb{B}_i \subseteq \mathbb{B}$.

Proof. By Theorem 5, every non-constant element of $\hat{\mathbb{P}}$ is of the form $\mathrm{lcm}\{\gcd \mathbb{P}_i\}_i$, $\mathbb{P}_i \subseteq \mathbb{P}$, and hence any "representing set" $\mathbb{J}$ for $\mathbb{P}$ also works for $\hat{\mathbb{P}}$. $\square$

3.2. Basic sets

Given a set $\mathbb{P} \subseteq \mathscr{P}$, it is then a natural question to ask if we can find a "small" set $\mathbb{B}$ that generates $\mathbb{P}$. This suggests:

DEFINITION 5. A set $\mathbb{B} \subseteq \mathscr{P}$ is said to be *basic* if $\mathbb{B}' \subsetneq \mathbb{B}$ implies $\hat{\mathbb{B}}' \subsetneq \hat{\mathbb{B}}$ (i.e. $\mathscr{O}(\mathbb{B}) \subsetneq \mathscr{O}(\mathbb{B}')$). We say that a basic set $\mathbb{B}$ is a basic set for $\mathbb{P}$, or $\mathscr{O}(\mathbb{P})$, if $\mathbb{B}$ generates $\mathbb{P}$.

It is convenient to note that a set $\mathbb{B}$ is basic, is equivalent to any of the following:

1. For all $P \in \mathbb{B}$, $P \notin \{\mathbb{B} \setminus \{P\}\}^{\frown}$.
2. For all $P \in \mathbb{B}$, $\mathscr{O}(\mathbb{B}) \subsetneq \mathscr{O}(\mathbb{B} \setminus \{P\})$.
3. For all $P \in \mathbb{B}$, $\{\mathbb{B} \setminus \{P\}\}^{\frown} \subsetneq \hat{\mathbb{B}}$.

In particular, in view of Remark 2, a basic set $\mathbb{B}$ does not contain any constant polynomial and the elements of $\mathbb{B}$ belong to different equivalence classes in $[\mathscr{P}]$, thus $\|\mathbb{B}\| = |\mathbb{B}|$.

THEOREM 6. *Let $\mathbb{P} \subseteq \mathscr{P}$ and assume $\|\mathbb{P}\| < \infty$. Then there is a basic set $\mathbb{B}$ for $\mathbb{P}$ formed by elements from $\mathbb{P}$.*

Proof. By assumption, there is a finite set $\mathbb{P}' = \{P_1, ..., P_n\} \subseteq \mathbb{P}$ such that $\hat{\mathbb{P}} = \hat{\mathbb{P}}'$. By successively removing elements from $\mathbb{P}'$, one at the time, we finally get a set $\mathbb{B} \subseteq \mathbb{P}'$ which cannot be reduced any further such that the hull is preserved, i.e. a basic set for $\mathbb{P}$. $\square$

DEFINITION 6. Let $\mathbb{P} \subseteq \mathscr{P}$ and assume first that $\|\mathbb{P}\| < \infty$. The *PDE-preserving dimension*, $\dim_{\mathcal{D}} \mathbb{P}$, of $\mathbb{P}$ is defined by $\dim_{\mathcal{D}} \mathbb{P} \equiv \min\{\|\mathbb{B}\| = |\mathbb{B}|\}$ where $\mathbb{B}$ runs through all basic sets for $\mathbb{P}$. A basic set $\mathbb{B}$ for $\mathbb{P}$ with minimal cardinality, i.e., with $|\mathbb{B}| = \dim_{\mathcal{D}} \mathbb{P}$, is called a *minimal basic set* for $\mathbb{P}$. If $\|\mathbb{P}\| = \infty$, then $\dim_{\mathcal{D}} \mathbb{P} \equiv \infty$.

(When $\|\mathbb{P}\| = \infty$ we do not know if there exist any basic sets for $\mathbb{P}$, however, if a basic set $\mathbb{B}$ exists we must have $\|\mathbb{B}\| = \infty$ (Proposition 3), which motivates $\dim_{\mathcal{D}} \mathbb{P} \equiv \infty$. See Section 4 for further remarks on this.)

By Theorem 6 we always have $\dim_{\mathcal{D}} \mathbb{P} \leq \|\mathbb{P}_{\circ\bullet}\|$. We shall see that basic sets for a given set $\mathbb{P}$ with $\|\mathbb{P}\| < \infty$, may indeed contain different number of elements – thus not every basic set for $\mathbb{P}$ is minimal.

EXAMPLE 6. Let $\mathbb{P} = \{P, Q\}$ be formed by two non-constant polynomials that not are associates. Then $\mathbb{P}$ is basic. In view of Example 5, if $\mathbb{B}$ is any other basic set for $\mathbb{P}$, $\mathbb{B}$ is necessarily of the form $\mathbb{B} = \{R, S\}$, where $[P, Q] = [R, S]$. Thus $\dim_{\mathcal{D}} \mathbb{P} = 2$ and, in particular, any two basic sets for $\mathbb{P}$ contain the same number of elements, which is not true in general, see Example 7.

THEOREM 7. *Let $\mathbb{P} = \{P_1, ..., P_n\}$ and assume the polynomials P_i are non-constant and pairwise relatively prime. Then $\mathbb{P}$ is basic and $\hat{\mathbb{P}}$ is formed by constants and associates of distinct products of the elements P_i.*

Proof. That $\mathbb{P}$ is basic is trivial in view of Theorem 5. The description of the hull follows by Proposition 6. Indeed, if the set $\mathbb{P} \cap (I^n)$ is non-empty, it contains one element only and thus forms a pre-D-relatively prime set. $\square$

EXAMPLE 7. Let $\mathbb{P} = \{P, Q, R\}$ and assume the polynomials in $\mathbb{P}$ are non-constant and pairwise D-relatively prime. By Theorem 7, the non-constants in $\hat{\mathbb{P}}$ are formed by $[\mathbb{P}, PQ, PR, QR, PQR]$. Further, $\mathbb{P}$ is basic and in view of Example 5 it is necessarily a minimal basic set. Another basic set for $\mathbb{P}$ is $\mathbb{B} \equiv \{PQ, PR, QR\}$. Indeed, $\mathbb{B} \subseteq \hat{\mathbb{P}}$ and hence $\hat{\mathbb{B}} \subseteq \hat{\mathbb{P}}$. On the other hand, from $P \in \{PQ, PR\}^{\frown} \subseteq \hat{\mathbb{B}}$ and analogously for Q and R, we deduce that $\mathbb{P} \subseteq \hat{\mathbb{B}}$ and hence $\hat{\mathbb{B}} = \hat{\mathbb{P}}$. That $\mathbb{B}$ is basic follows by Example 6 and Theorem 6. Now, based on similar arguments, another basic set for $\mathbb{P}$ is given by $\mathbb{B}' \equiv \{P, Q, PR, QR\}$. Summing up, $\dim_{\mathcal{D}} \mathbb{P} = 3$ and $\mathbb{P}, \mathbb{B}$ are minimal basic sets while $\mathbb{B}'$ is not.

If $\mathbb{P} = \{P_1, ..., P_n\}$ admits a D-prime factorisation (recall that if $d = 1$ this is true for any $\mathbb{P}$), it follows from Proposition 7 that $\dim_{\mathcal{D}} \mathbb{P}$ is the smallest number of elements $\mathbb{B}$ in $\hat{\mathbb{P}}$ (i.e. elements of the form $\vee_i \wedge_j Q_{ij}$, $Q_{ij} \in \mathbb{B}$) such that $\{\vee_i \wedge_j Q_{ij} : Q_{ij} \in \mathbb{B}\} \supseteq \mathbb{P}_{\circ\bullet}$. (Note that the problem to find such minimal set $\mathbb{B}$ and number $|\mathbb{B}|$, has an analogue formulation in any UFD $\mathscr{R}$, say, $\mathscr{R} = \mathbb{Z}$.) Our next objective is to describe this dimension $\dim_{\mathcal{D}} \mathbb{P}$ in the extremal case when $\mathbb{P}$ is formed by n non-constant pairwise D-relatively prime polynomials, and we start by formulating the following corollary of Theorem 7:

COROLLARY 6. *Let $\mathbb{P} = \{P_1, ..., P_n\}$ be a pairwise D-relatively prime set. Then $\hat{\mathbb{P}}$ is formed by constants and associates of distinct products of the elements P_i and, more generally, for any subset $\mathbb{B} \subseteq \hat{\mathbb{P}}$, $\hat{\mathbb{B}}$ is formed by constants and the elements of the form* $\operatorname{lcm}\{\gcd \mathbb{B}_i\}_i$, $\mathbb{B}_i \subseteq \mathbb{B}$ *(i.e., $\vee_i \wedge_j Q_{ij}$, $Q_{ij} \in \mathbb{B}$).*

Now, let $\mathbb{P}$ be formed by n non-constant pairwise D-relatively prime polynomials. We deduce that the dimension of such a set $\mathbb{P}$ only depends on the size $n = |\mathbb{P}| = \|\mathbb{P}\|$, not on the polynomials P_i (and not on d), and we put $d_n \equiv \dim_{\mathcal{D}} \mathbb{P}$. (Clearly $d_1 = 1$ and from Examples 6 and 7, $d_2 = 2$ and $d_3 = 3$.) In fact, let $\mathcal{P}(X)$ denote the powerset of a set X formed by n elements, say, $X = X_n \equiv \{1, ..., n\}$. Then d_n is clearly the smallest number of elements in $\mathcal{P}(X)$ such that the set of finite intersections of the elements contains the single-element sets $\{x\}$. Another description of d_n, that we shall use, is following. We may identify the set of non-zero elements of $\hat{\mathbb{P}}$ with $\mathbb{Z}_2^n$ (binary codes of length n) in a one-to-one way. Indeed, we let $Q = P_1P_2 \in \hat{\mathbb{P}}$ correspond to the element $q = (1, 1, 0, ..., 0)$ etc. and thus, in particular, the P_i:s correspond in this way to the basis elements $e_i = (0, ..., 1, 0...)$ respectively. Moreover, with this identification, the gcd between two elements in $\hat{\mathbb{P}}$ is obtained by the corresponding product in $\mathbb{Z}_2^n$. Thus, d_n is the smallest number of elements in $\mathbb{Z}_2^n$ such that the set obtained by taking distinct products contains the basis elements e_i. Thus to compute d_n is a pure combinatoric problem and we give a proof of how to obtain d_n based on the famous *Sperner's Theorem* (1928). The proof is constructive in the sense that it describes how to obtain corresponding generating and basic sets.

THEOREM 8. $d_n = \min\{m : \binom{m}{k} \geq n$ *for some* $k \leq m\}$. *Thus if $m \geq 1$ is an integer such that $\binom{m}{k} \geq n$ for some $k \leq m$, then there is an m-element set $\mathbb{B} = \{Q_1, ..., Q_m\} \subseteq \hat{\mathbb{P}}$ that generates $\mathbb{P}$. If $m \leq n$, we may choose $\mathbb{B}$ as a basic set.*

Proof. Assume $n \leq \binom{m}{k}$ and let A_k^m denote the subset of $\mathcal{P}(X_m)$ formed by k-element sets. Then $|A_k^m| = \binom{m}{k} \geq n$ so we may choose, and enumerate, n elements $S_1, ..., S_n \in A_k^m$. Now, the idea is to define elements $q_i \in \mathbb{Z}_2^n$, $i = 1, ..., m$, such that if $S_i = \{\alpha_1, ..., \alpha_k\}$ then $q_{\alpha_1} \cdots q_{\alpha_k} = e_i$. But this is simply obtained by letting q_i be one at the j:th coordinate iff $i \in S_j$. (If some $j \notin \cup_i S_i$, $q_j = 0$ and we may "remove" this element.) Thus $d_n \leq m$. We note that we obtain a basic set in this way if the following holds true: For every $x \in X_m$ there is an element S_j such that $x \in S_j$ and $S_j \setminus \{x\} \subseteq S_i$ for some i. In particular, if $m \leq n$ we may construct such a list $S_1, ..., S_n$ by letting the first m sets S_j be $S_1 \equiv \{1, ..., k\}$, $S_2 = \{2, ..., k, k+1\}, ..., S_m \equiv \{m, 1, ..., k-1\}$.

Next, let $q_1, ..., q_m$, $m = d_n$, be a minimal basic set. Then to each basis element e_i we way find a set $S_i = \{\alpha_1, ..., \alpha_l\} \in \mathcal{P}(X_m)$ such that $q_{\alpha_1} \cdots q_{\alpha_l} = e_i$. Clearly, these subsets S_i, $i = 1, ..., n$, must satisfy $S_i \nsubseteq S_j$ when $i \neq j$ (since $e_j \neq q e_i$ for all q if $i \neq j$). Now, by Sperner's Theorem [12, Theorem 6.3], we necessarily have that $n \leq \binom{m}{k}$ where $k = [m/2]$. This completes the proof. □

For the sake of clarity, we illustrate the algorithm, described in the first part of the proof:

EXAMPLE 8. Let $n = 6$, i.e., $\mathbb{P} = \{P_1, ..., P_6\}$. From Theorem 8 we deduce $d_6 = 4$ ($\binom{4}{2} = 6 \geq n$). Let us determine a generating set by applying the algorithm in the proof. Thus, we choose a list: $S_1 = \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, S_6 = \{3,4\}$ from A_2^4. Next we define elements q_i, $i \leq 4$, by $q_1 = (1,1,1,0,0,0)$, $q_2 = (1,0,0,1,1,0)$ etc. The corresponding polynomials $q_i \mapsto Q_i \in \mathscr{P}$ ($Q_1 = P_1P_2P_3$, $Q_2 = P_1P_4P_5, ...$) form then a generating family $\mathbb{B}$ for $\mathbb{P}$. Since $\|\mathbb{B}\| = d_6 = \dim_{\mathcal{D}} \mathbb{P}$, $\mathbb{B}$ is a minimal basic set for $\mathbb{P}$.

Note that if $\mathbb{P}$ is formed by n non-constant pairwise relatively prime polynomials, then $d_n \leq \dim_{\mathcal{D}} \mathbb{P}$.

We conclude by describing what happends with the dimension if we "remove" common factors:

PROPOSITION 8. *Let $\mathbb{P} \subseteq \mathscr{P}$ and assume that $\|\mathbb{P}\| < \infty$ and that S is a non-constant common divisor for $\mathbb{P}_\bullet$. Then*

$$\dim_{\mathcal{D}} \mathbb{P} = \dim_{\mathcal{D}} \mathbb{P}_{\bullet S} \quad \text{if} \quad S \notin \mathbb{P},$$
$$\dim_{\mathcal{D}} \mathbb{P}_{\bullet S} \leq \dim_{\mathcal{D}} \mathbb{P} \leq \dim_{\mathcal{D}} \mathbb{P}_{\bullet S} + 1 \quad \text{if} \quad S \in \hat{\mathbb{P}} \quad (\text{implies } S = \gcd \mathbb{P}_\bullet).$$

Proof. We prove that if $\mathbb{B}$ generates $\mathbb{P}$, then $\mathbb{B}_{\bullet S}$ generates $\mathbb{P}_{\bullet S}$ (note that S is a common divisor for $\mathbb{B}_\bullet$). But, in view of (ii) in Corollary 4,

$S\widehat{\mathbb{B}_{\bullet S}} = S\widehat{\mathbb{P}_{\bullet S}}$ and consequently $\widehat{\mathbb{B}_{\bullet S}} = \widehat{\mathbb{P}_{\bullet S}}$. Hence, we always have that $\dim_{\mathcal{D}} \mathbb{P}_{\bullet S} \leq \dim_{\mathcal{D}} \mathbb{P}$.

On the other hand, assume $\mathbb{B}$ generates $\mathbb{P}_{\bullet S}$. Then, by Corollary 4,

$$\{S\mathbb{B}_{\bullet} \cup \{S\}\}^{\frown} = \widehat{S\mathbb{B}_{\bullet}} \cup [S] = \hat{\mathbb{P}} \cup [S]. \tag{3}$$

Thus if $\mathbb{B}$ is basic and $S \in \hat{\mathbb{P}}$, $S\mathbb{B} \cup \{S\}$ generates $\mathbb{P}$ and consequently, $\dim_{\mathcal{D}} \mathbb{P} \leq \dim_{\mathcal{D}} \mathbb{P}_{\bullet S} + 1$. Thus the second line holds true and it remains to prove that $\dim_{\mathcal{D}} \mathbb{P}_{\bullet S} \geq \dim_{\mathcal{D}} \mathbb{P}$ when $S \notin \mathbb{P}$. So assume $S \notin \mathbb{P}$. It suffices to prove that $S\mathbb{B}_{\bullet}$ generates $\mathbb{P}$. Assume first that $S \in \hat{\mathbb{P}}$. In view of (3) we only have to prove that the hull of $S\mathbb{B}_{\bullet}$ contains S. But $S \in \hat{\mathbb{P}}$ implies that $\mathbb{P}_{\bullet S}$ is D-relatively prime. Further, the assumption $S \notin \mathbb{P}$ implies $\mathbb{P}_{\bullet S \bullet} = \mathbb{P}_{\bullet S}$ and thus $\mathbb{B}_{\bullet}$ is D-relatively prime by Corollary 3. Consequently, $S = \gcd S\mathbb{B}_{\bullet}$ and $S\mathbb{B}_{\bullet}$ is pre-D-relatively prime so by Theorem 4, $S \in \widehat{\mathbb{B}_{\bullet S}}$. But, in the same way, if $S \notin \hat{\mathbb{P}}$ we cannot have that $S\mathbb{B}_{\bullet}$ is pre-D-relatively prime so S is not in the hull of $S\mathbb{B}_{\bullet}$. In view of (3) this implies that $S\mathbb{B}_{\bullet}$ generates $\mathbb{P}$. $\square$

Example 9. Consider the set $\mathbb{P} = \{S, P, Q\}$, where $S = \gcd\{P, Q\}$. Then $\mathbb{P}_{\bullet S} = \{1, P', Q'\}$, where $P' \equiv P/S$ and $Q' \equiv Q/S$. We assume $S, P', Q' \notin [1]$. If P', Q' are D-relatively prime, $\dim_{\mathcal{D}} \mathbb{P} = \dim_{\mathcal{D}} \mathbb{P}_{\bullet S} = 2$. However, if P', Q' not are D-relatively prime, $\dim_{\mathcal{D}} \mathbb{P} = 3 = \dim_{\mathcal{D}} \mathbb{P}_{\bullet S} + 1$.

4. Conclusions and remarks

Basic sets when $\|\mathbb{P}\|$ is infinite: At this point we do not know whether a general set $\mathbb{P} \subseteq \mathscr{P}$ with $\|\mathbb{P}\| = \infty$ admits a basic set or not. A "standard" approach, to prove such an existence, via Zorn's Lemma contains obstacles. Indeed, if $\mathbb{B}$ is a maximal (with respect to inclusion) basic set in $\hat{\mathbb{P}}$, it is not true in general that $\mathbb{B}$ generates $\mathbb{P}$:

Example 10. Assume $P, Q \in \mathscr{P}$ are non-constant and relatively prime. Then with $\mathbb{P} \equiv \{P, Q\}$, $\hat{\mathbb{P}} = [0, 1, P, Q, PQ]$. Now, consider the set $\mathbb{B} \equiv \{P, PQ\}$. Then $\mathbb{B}$ is basic, and is not contained in any other basic set in $\hat{\mathbb{P}}$, but $\hat{\mathbb{B}} = [0, 1, P, PQ] \neq \hat{\mathbb{P}}$.

Example 11 (On basic sets for $\mathbb{H}$). Assume $d = 1$. A basic set for $\mathbb{H}$ is then formed by the monomials $\mathbb{M} \equiv \{\xi^n\}_{n \geq 1}$. Indeed, given $n \geq 1$, then with $T = T_n \equiv zH_{n-1}$, $T \in \mathscr{O}(\mathbb{M} \backslash \{\xi^n\}) \backslash \mathscr{O}(\xi^n)$ so $\mathbb{M}$ is basic. Next, assume $d > 1$ and consider the set $\mathbb{B}$ formed by the elements I^n where $n \geq 1$ and I is an irreducible homogeneous polynomial. Recall that if $0 \neq$

$P \in \mathbb{H}$, then all the irreducible factors of P are homogeneous and thus, $\hat{\mathbb{B}} = \hat{\mathbb{H}}$ (Theorem 4). (In fact, $\mathbb{B}$ *generates* $\mathbb{H}$ *finitely* in the sense that every $P \in \mathbb{H}$ belongs to the hull of finite subset $\mathbb{F} \subseteq \mathbb{B}$.) It is not known whether $\mathbb{B}$ is basic or not. (Note that the set $\mathbb{M}$ of monomials $\{\xi^\alpha\}_{\alpha \in \mathbb{N}^d}$ does not form a basic set for $\mathbb{H}$ since $\xi^\alpha = \mathrm{lcm}\{\xi_i^{\alpha_i}\}_i \in \{\xi_1^{\alpha_1}, \ldots, \xi_d^{\alpha_d}\}\hat{}$ and $\{\xi_i^{\alpha_i}\}_i \subseteq \mathbb{M} \setminus \{\xi^\alpha\}$ if $\alpha_i, \alpha_j \neq 0$ for some $i \neq j$.) However, from Theorem 5 we deduce that $\mathbb{B}$ forms a *finitely basic set* in the sense that every finite subset $\mathbb{F} \subseteq \mathbb{B}$ is basic.

Example 12 (On basic sets for $\mathscr{P}$). Consider the set $\mathbb{B}$ formed by all elements of the form I^n where $n \geq 1$ and I is an irreducible polynomial. Then, as in the previous example, we deduce that $\mathbb{B}$ generates $\mathscr{P}$ finitely and $\mathbb{B}$ forms a finitely basic set. However, again, it is not known if $\mathbb{B}$ is an ordinary basic set.

Let us also note that for any finite subset $\mathbb{F}$ of $\mathscr{P}$, $\{\mathscr{P} \backslash \mathbb{F}\}\hat{} = \hat{\mathscr{P}} = \mathscr{P}$, i.e. $\mathscr{O}(\mathscr{P} \setminus \mathbb{F}) = \mathscr{C}$ (Proposition 2). Indeed, for any given $P \in \mathscr{P}$ there is a polynomial R such that $PR, P(R+1) \in \mathscr{P} \setminus \mathbb{F}$. Since $R, R+1$ are D-relatively prime, $P = \gcd\{PR, P(R+1)\} \in \{\mathscr{P} \setminus \mathbb{F}\}\hat{}$.

Examples 11 and 12 suggest the formulation of the following open problem:

Conjecture 1. *Let $\mathbb{P} \subseteq \mathscr{P}$, is it true that $Q \in \hat{\mathbb{P}}$ implies $Q \in \hat{\mathbb{F}}$ for some finite set $\mathbb{F} \subseteq \mathbb{P}$.*

Note that if the answer is affirmative, then every finitely basic set is in fact basic. Noteworthy is also that if we could prove that Theorem 5 also holds when $||\mathbb{P}|| = \infty$, then Conjecture 1 has indeed an affirmative answer. Conversely, if our conjecture holds true, the equivalencies (i-iii) in Theorem 5 extends to the case when $||\mathbb{P}||$ is infinite.

Extensions: Let us consider how the study and results, can be extended to other spaces (of power-series) and other operator classes.

We denote by $\mathscr{F}$ the entire ring, $\prod_{n \geq 0} \mathscr{P}_n$, of formal power-series in d variables. Thus $\mathscr{F}$ is a reflexive Fréchet space if we equip $\mathscr{F}$ with the product topology. Here we assume that every finite-dimensional space $\mathscr{P}_n$ is endowed with its unique Banach space topology. Next, let us provide $\mathscr{P} = \oplus_{n \geq 0} \mathscr{P}_n$ with the direct sum topology. Since every $\mathscr{P}_n$ is finite-dimensional, the topological- and the algebraic dual of $\mathscr{P}$ coincide. In fact, the Martineau-duality (page 574) extends to the pair $(\mathscr{F}, \mathscr{P})$ and $\mathscr{P}' = \mathscr{P}^* = \mathscr{F}$, see [11, Section 1.7]. Hence, and most importantly, every subspace, and thus every ideal, of $\mathscr{P}$ is closed. As a consequence it follows that every differential operator $P(D) \neq 0$ ($P \in \mathscr{P}$), acting on

$\mathscr{F}$, is surjective since $\mathscr{F}$ is Fréchet and the transpose ${}^tP(D) : \mathscr{P} \to \mathscr{P}$ is "multiplication by P", which thus is a one-to-one closed range operator. We remark that Proposition 4 can be extended by: $\mathbb{P} \subseteq \mathscr{P}$ is D-relatively prime iff $\cap_{\mathbb{P}} \ker P(D) = \{0\}$, where $\ker P(D)$ denotes the kernel of $P(D)$ acting on $\mathscr{F}$.

Now, we define the PDE-preserving hull $\hat{\mathbb{P}}_{\mathscr{F}}$ (with respect to $\mathscr{F}$) of a set $\mathbb{P} \subseteq \mathscr{P}$ in the same way as we defined $\hat{\mathbb{P}}$. Thus, if $\mathscr{O}_{\mathscr{F}}(\mathbb{P})$ denotes the algebra formed by all PDE-preserving operators $T \in \mathscr{L}(\mathscr{F})$ for $\mathbb{P}$, $\hat{\mathbb{P}}_{\mathscr{F}} \equiv \{P \in \mathscr{P} : \mathscr{O}_{\mathscr{F}}(\mathbb{P}) \subseteq \mathscr{O}_{\mathscr{F}}(P)\}$. With arguments as in Section 3 (in particular, the analogue of Lemma 6 extends by the discussion above), we obtain the following analogue of Theorem 5:

Proposition 9. *Let $\mathbb{P} \subseteq \mathscr{P}$ and assume $\|\mathbb{P}\| < \infty$. Let $I_0^{r_0}...I_n^{r_n}$ be the factorisation of $Q \in \mathscr{P} \setminus \mathbb{C}$ into irreducible factors $I_i \in \mathscr{P}$, thus $(Q) = \cap_i(I_i^{r_i})$ (primary decomposition). Then the following are equivalent:*

(i) $Q = I_0^{r_0}...I_n^{r_n} \in \hat{\mathbb{P}}_{\mathscr{F}}$,
(ii) $Q \in (\mathbb{P}_i)$ *for all* i *where* $\mathbb{P}_i \equiv \mathbb{P} \cap (I_i^{r_i}) = \{P \in \mathbb{P} : I_i^{r_i} | P\}$,
(iii) $(Q) = \cap_i(\mathbb{P}_i)$.

Since $(\mathbb{P}) \subseteq \langle\mathbb{P}\rangle$ we deduce that $\hat{\mathbb{P}}_{\mathscr{F}} \subseteq \hat{\mathbb{P}}$. However, if $d = 1$, a polynomial $Q \in \langle\mathbb{P}\rangle$ iff $Q \in (\mathbb{P})$ and consequently $\hat{\mathbb{P}}_{\mathscr{F}} = \hat{\mathbb{P}}$ in this particular case, see also below.

We remark, briefly, that an extension in the same spirit is to deal with the dual pair $\mathrm{EXP} \equiv \cap_{r>0}\mathrm{Exp}_r$ and $\mathscr{O}_0$, formed by zero-exponential type functions respective germs of analytic functions (convergent power-series). (The bilinear form is defined by the formula in [11, p. 28] for the Matrineau-duality.) Recall that EXP, provided with the "standard" topology generated by the semi-norms $\|\!|\cdot|\!\|_r$, is a reflexive Fréchet space and $\mathscr{O}_0$ is a UFD and Noetherian [5, Chapter 6]. It seems as if the analogue of Proposition 9 holds for the PDE-preserving hull of a set $\mathbb{P} \subseteq \mathscr{P}$ with respect to EXP, which is defined as $\hat{\mathbb{P}}_{\mathscr{F}}$, and by which we mean that we use the factorisation in $\mathscr{O}_0$ and replace $(\cdot)$ by the corresponding ideals generated in $\mathscr{O}_0$.

Another type of extension is to study the hull of a set $\mathbb{P} \subseteq \mathscr{P}$ with respect to, say, the Weyl-algebra $\mathscr{A}$. (Recall that $\mathscr{A}$ is the subalgebra of $\mathscr{L} = \mathscr{L}(\mathscr{H})$ formed by the operators of the form $\sum_{\alpha\beta} P_{\alpha\beta} z^\alpha D^\beta$ where the sum is finite, i.e., the operators with symbols $P = P(z,\xi) \in \mathfrak{S}$ that are polynomials in both z and ξ.) More precisely, put $\mathscr{A}(\mathbb{P}) \equiv \mathscr{O}(\mathbb{P}) \cap \mathscr{A}$ and $\hat{\mathbb{P}}_{\mathscr{A}} \equiv \{P \in \mathscr{P} : \mathscr{A}(\mathbb{P}) \subseteq \mathscr{A}(P)\}$. Since, for any set $\mathbb{P}$, $\mathscr{A}(\mathbb{P}) \subseteq \mathscr{O}(\mathbb{P})$ we have that $\hat{\mathbb{P}} \subseteq \hat{\mathbb{P}}_{\mathscr{A}}$ and so, $\hat{\mathbb{P}}_{\mathscr{F}} \subseteq \hat{\mathbb{P}} \subseteq \hat{\mathbb{P}}_{\mathscr{A}}$ – when do we have equalities?

Finally we remark that it is possible to define the hull $\tilde{\mathbb{P}}$, let us call it the smooth hull, of a general set $\mathbb{P} \subseteq \text{Exp}$: $\tilde{\mathbb{P}} \equiv \{\varphi \in \text{Exp} : \mathscr{O}(\mathbb{P}) \subseteq \mathscr{O}(\varphi)\}$. Thus $\hat{\mathbb{P}} = \mathscr{P} \cap \tilde{\mathbb{P}}$, and the following problem was partially posed by the referee: For what sets $\mathbb{P} \subseteq \mathscr{P}$ do we have that $\hat{\mathbb{P}} = \tilde{\mathbb{P}}$?

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