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# MATRIX PRESENTATIONS OF THE TEICHMÜLLER SPACE OF A PAIR OF PANTS

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Abstract. A pair of pants $\Sigma(0,3)$ is a building block of oriented surfaces. The purpose of this paper is to formulate the matrix presentations of elements of the Teichmüller space of a pair of pants. In the level of the matrix group $\mathbf{SL}(2,\mathbb{R})$, we shall show that an odd number of traces of matrix presentations of the generators of the fundamental group of $\Sigma(0,3)$ should be negative.

## 1. Introduction

A *hyperbolic* structure on a smooth surface $M$ is a representation of $M$ as a quotient $\Omega/\Gamma$ of a strictly convex domain $\Omega \subset \mathbb{H}^2$ by a discrete group $\Gamma \subset \mathbf{PSL}(2,\mathbb{R})$ acting properly and freely. If $\chi(M) < 0$, then the equivalence classes of hyperbolic structures on $M$ form a deformation space $\mathfrak{T}(M)$ called the *Teichmüller space.*

Let $M$ be a compact connected smooth surface and $\pi = \pi_1(M)$ the fundamental group of $M$. Given a hyperbolic structure on $M$, the action of $\pi$ by deck transformation on the universal covering space $\tilde{M}$ of $M$ determines a homomorphism $\pi \to \mathbf{PSL}(2,\mathbb{R})$ called the *holonomy homomorphism* and it is well-defined up to conjugation in $\mathbf{PSL}(2,\mathbb{R})$. Thus the Teichmüller space $\mathfrak{T}(M)$ has a natural topology which identified with an open subset of the space $\mathrm{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))/\mathbf{PSL}(2,\mathbb{R})$ the orbit space of homomorphisms $\pi \to \mathbf{PSL}(2,\mathbb{R})$. Since holonomy homomorphisms $\pi \to \mathbf{PSL}(2,\mathbb{R})$ are isomorphic to their images, the generators of $\pi$ can be presented by the conjugacy classes of matrices in $\mathbf{PSL}(2,\mathbb{R})$.

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Let $M = \Sigma(g,n)$ be a compact connected oriented surface with $g$-genus and $n$-boundary components. Then $M$ can be decomposed as a disjoint union of $2g-2+n$ pairs of pants $\Sigma(0,3)$. Thus a pair of pants $\Sigma(0,3)$ is a building block of an oriented surface $M$. The purpose of this paper is to formulate the matrix presentations of elements of the Teichmüller space of a pair of pants $\Sigma(0,3)$.

In Section 2, we recall some preliminary definitions and describe the relation between the deformation space $\mathfrak{D}(M)$ of $(G,X)$-structures on a smooth manifold $M$ and the orbit space $\mathrm{Hom}(\pi,G)/G$. In Section 3, we define the hyperbolic elements of $\mathbf{SL}(2,\mathbb{R})$ and $\mathbf{PSL}(2,\mathbb{R})$ and classify the locations of fixed points and principal lines of hyperbolic elements. In Section 4, we calculate the matrix presentations of elements of the Teichmüller space $\mathfrak{T}(\Sigma(0,3))$. In terms of $\mathbf{SL}(2,\mathbb{R})$, we shall show some relations among the traces of the matrix presentations of the generators of the fundamental group of $\Sigma(0,3)$.

## 2. Deformation space of $(G,X)$-structures

Let $X$ be a smooth manifold and $G$ a connected Lie group. An action of $G$ on $X$ is called *strongly effective* if $g_1, g_2 \in G$ agree on a nonempty open set of $X$, then $g_1 = g_2$. By this requirement, for any nontrivial $g \in G$, the set of fixed points $X_g = \{x \in X \mid g \cdot x = x\}$ is nowhere dense in $X$. Each element $g$ of $G$ is called a *$(G,X)$-transformation.* Let $\Omega$ be an open subset of $X$. A map $\phi : \Omega \to X$ is called *locally-$(G,X)$* if for each component $W \subset \Omega$, there exists a $(G,X)$-transformation $g \in G$ such that $\phi|_W = g|_W$. Since $G$ acts strongly effectively on $X$, above element $g$ is unique for each component. Clearly a locally-$(G,X)$ map is a local diffeomorphism.

Let $M$ be a connected smooth $n$-manifold. A *$(G,X)$-structure* on $M$ is a maximal collection of coordinate charts $\{(U_\alpha, \psi_\alpha)\}$ such that

1. $\{U_\alpha\}$ is an open covering of $M$.
2. For each $\alpha$, $\psi_\alpha : U_\alpha \to X$ is a diffeomorphism onto its image.
3. If $(U_\alpha, \psi_\alpha)$ and $(U_\beta, \psi_\beta)$ are two coordinate charts with $U_\alpha \cap U_\beta \neq \emptyset$, then the transition function

$$\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$$

is locally-$(G,X)$.

Now we give two examples of $(G,X)$-structures.

EXAMPLE 2.1. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper half complex plane. Then $\mathbf{SL}(2, \mathbb{R})$ acts on $\mathbb{H}^2$ by

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot z = \frac{az+b}{cz+d}\,. \tag{2.1}$$

Since we have $A \cdot z = (-A) \cdot z$ for any $A \in \mathbf{SL}(2, \mathbb{R})$ and $z \in \mathbb{H}^2$, the Lie group $\mathbf{PSL}(2, \mathbb{R}) = \mathbf{SL}(2, \mathbb{R})/\pm I$ acts strongly effectively on $\mathbb{H}^2$. A $(\mathbf{PSL}(2, \mathbb{R}), \mathbb{H}^2)$-structure on a surface $M$ is called a *hyperbolic structure* on $M$.

EXAMPLE 2.2. Let $\mathbb{RP}^2$ be the space of all lines through the origin in $\mathbb{R}^3$. For a nonzero vector $v$ in $\mathbb{R}^3$, $[v]$ denotes the corresponding point in $\mathbb{RP}^2$. Let $B$ be an element of $\mathbf{GL}(3, \mathbb{R})$, the group of linear transformations of $\mathbb{R}^3$. Then $B$ preserves lines through the origin and induces a projective transformation of $\mathbb{RP}^2$. Thus $\mathbf{GL}(3, \mathbb{R})$ acts on $\mathbb{RP}^2$ by

$$B \cdot [v] = [Bv]. \tag{2.2}$$

Since the scalar matrices $\mathbb{R}^* \subset \mathbf{GL}(3, \mathbb{R})$ acts trivially on $\mathbb{RP}^2$, the Lie group $\mathbf{PGL}(3, \mathbb{R}) = \mathbf{GL}(3, \mathbb{R})/\mathbb{R}^*$ acts strongly effectively on $\mathbb{RP}^2$. A $(\mathbf{PGL}(3, \mathbb{R}), \mathbb{RP}^2)$-structure on a surface $M$ is called a *real projective structure* on $M$.

A manifold with a $(G, X)$-structure is called a *$(G, X)$-manifold.* Let $N$ be a $(G, X)$-manifold. If $f : M \to N$ is a local diffeomorphism of smooth manifolds, then we can give the induced $(G, X)$-structure on $M$ via $f$. In particular every covering space of a $(G, X)$-manifold has the canonically induced $(G, X)$-structure.

Let $M$ and $N$ be $(G, X)$-manifolds and $f : M \to N$ a smooth map. Then $f$ is called a *$(G, X)$-map* if for each coordinate chart $(U, \psi_U)$ on $M$ and $(V, \psi_V)$ on $N$, the composition $\psi_V \circ f \circ \psi_U^{-1} : \psi_U(f^{-1}(V) \cap U) \to \psi_V(f(U) \cap V)$ is locally-$(G, X)$.

A $(G, X)$-manifold $M$ can be *developed* into $X$ as follows. For more detail, see Thurston's book [8]. Let $p : \tilde{M} \to M$ denote a universal covering map of $M$ and $\pi$ the covering transformation group of $\tilde{M}$. We shall identify $\pi$ with the fundamental group $\pi_1(M)$ of $M$. Since $\tilde{M}$ is simply connected, the coordinate charts on $\tilde{M}$ can globalize to define a $(G, X)$-map $\mathbf{dev} : \tilde{M} \to X$, called the *developing map.* The covering transformation $\gamma \in \pi$ defines an automorphism of $\tilde{M}$. The corresponds to coordinate changes in the atlas for the $(G, X)$-structure result a homomorphism $h : \pi \to G$ such that the following diagram commutes.

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle h(\gamma)} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X \end{array} \tag{2.3}$$

The homomorphism $h : \pi \to G$ is called the *holonomy homomorphism.* The image $\Gamma = h(\pi) \subset G$ is called the *holonomy group.* The image $\Omega = \mathbf{dev}(\tilde{M}) \subset X$ is called the *developing image.* The pair $(\mathbf{dev}, h)$ consisting of the developing map and the holonomy homomorphism is called a *developing pair.*

Suppose $(\mathbf{dev}', h')$ is another developing pair commuting above diagram (2.3). Then there exists $g \in G$ such that $\mathbf{dev}' = g \circ \mathbf{dev}$ and $h' = \iota_g \circ h$ where $\iota_g : G \to G$ denotes the inner automorphism defined by $g$ ; that is, $h'(\gamma) = (\iota_g \circ h)(\gamma) = g \circ h(\gamma) \circ g^{-1}$.

$$\begin{array}{ccccc} \tilde{M} & \xrightarrow{\mathbf{dev}} & X & \xrightarrow{g} & X \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle h(\gamma)} & & \downarrow{\scriptstyle h'(\gamma)} \\ \tilde{M} & \xrightarrow[\mathbf{dev}]{} & X & \xrightarrow[g]{} & X \end{array} \tag{2.4}$$

Thus the developing pair $(\mathbf{dev}, h)$ is unique up to the $G$-action by composition and conjugation respectively.

Consider a pair $(f, N)$ where $N$ is a $(G, X)$-manifold and $f : M \to N$ is a diffeomorphism. Then $M$ admits the induced $(G, X)$-structure via $f$. The set of all such pairs $(f, N)$ is denoted by $\mathcal{A}(M)$. Then $\mathcal{A}(M)$ is the space of all $(G, X)$-structures on $M$. We say two pairs $(f', N')$ and $(f, N)$ in $\mathcal{A}(M)$ are *equivalent* if there exists a $(G, X)$-diffeomorphism $g' : N' \to N$ such that $g' \circ f'$ is isotopic to $f$ ; that is, there exists a diffeomorphism $g : M \to M$, which is isotopic to the identity map $I_M$ such that the following diagram commutes :

$$\begin{array}{ccc} M & \xrightarrow{f'} & N' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ M & \xrightarrow[f]{} & N \end{array}$$

The set of equivalence classes $\mathcal{A}(M)/\sim$ will be denoted by $\mathfrak{D}(M)$ and called the *deformation space* of $(G, X)$-structures on $M$.

DEFINITION 2.3. Let $M$ be a connected smooth 2-manifold. The deformation space of the hyperbolic structures on $M$ is called the *Teichmüller space* and denoted by $\mathfrak{T}(M)$. The deformation space of real projective structures on $M$ is denoted by $\mathbb{RP}^2(M)$.

The deformation space $\mathfrak{D}(M)$ is closely related to $\mathrm{Hom}(\pi, G)/G$ the orbit space of homomorphisms $\phi : \pi \to G$. Let $M$ be a compact connected smooth manifold. Since $M$ is compact, the fundamental group $\pi$ of $M$ admits finite generators $\gamma_1, \cdots, \gamma_m$ with finite relations $R_1, \cdots, R_k$. For example if $M$ is $\Sigma(g, n)$, that is a compact connected smooth surface with $g$-genus and $n$-boundary components, then $\pi$ admits $2g + n$ generators $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_n$ with a single relation

$$R = C_n \cdots C_1 B_g^{-1} A_g^{-1} B_g A_g \cdots B_1^{-1} A_1^{-1} B_1 A_1 = I.$$

From the correspondence of the homomorphism $\phi : \pi \to G$ to the image of generators $g_1 = \phi(\gamma_1), \cdots, g_m = \phi(\gamma_m)$, $\mathrm{Hom}(\pi, G)$ may be identified with the collection of all $m$-tuples $(g_1, \ldots, g_m) \subset G^m$ satisfying

$$R_1(g_1, \ldots, g_m) = I, \ \cdots, \ R_k(g_1, \ldots, g_m) = I.$$

The group $G$ acts on $\mathrm{Hom}(\pi, G)$ by conjugation ; that is, for $g \in G$ and $\phi \in \mathrm{Hom}(\pi, G)$, the action $g \cdot \phi$ is defined by

$$(g \cdot \phi)(\gamma) = g \circ \phi(\gamma) \circ g^{-1}$$

where $\gamma \in \pi$. Taking the holonomy homomorphism of a $(G, X)$-structure defines a map

$$\mathbf{hol} : \mathfrak{D}(M) \longrightarrow \mathrm{Hom}(\pi, G)/G$$

which is a local diffeomorphism. See Goldman [3] and Johnson [5] for details.

Let $M$ be a hyperbolic surface. Then the developing map $\mathbf{dev}$ is a diffeomorphism from $\tilde{M}$ onto a convex domain $\Omega = \mathbf{dev}(\tilde{M}) \subset \mathbb{H}^2$ and the holonomy homomorphism $h$ is an isomorphism from $\pi$ onto a discrete subgroup $\Gamma = h(\pi) \subset \mathbf{PSL}(2, \mathbb{R})$ which acts properly and freely on $\Omega$. Thus if a compact connected smooth surface $M$ has a hyperbolic structure, the $M$ is diffeomorphic to the quotient $\Omega/\Gamma$. Therefore the element of the Teichmüller space $\mathfrak{T}(M)$ will be identified with a conjugacy class of $\mathrm{Hom}(\pi, \mathbf{PSL}(2, \mathbb{R}))$.

If $M$ has a real projective structure, then generally the developing map is just a local diffeomorphism and the developing image may be not convex. A domain $\Omega \subset \mathbb{RP}^2$ is called *convex* if there exist a projective line $\ell \subset \mathbb{RP}^2$ such that $\Omega \subset (\mathbb{RP}^2 - \ell)$ and $\Omega$ is a convex subset of the affine plane $\mathbb{RP}^2 - \ell$ ; that is, if $x, y \in \Omega$, then the line segment

$\overline{xy}$ lies in $\Omega$. By definition, $\mathbb{RP}^2$ itself is not convex. A real projective structure on $M$ is called *convex* if the developing map $\mathbf{dev} : \tilde{M} \to \mathbb{RP}^2$ is a diffeomorphism onto a convex domain in $\mathbb{RP}^2$. The following fundamental theorem is from Goldman's paper [4].

THEOREM 2.4. *Let $M$ be a real projective surface. Then the following statements are equivalent.*

1. *$M$ has a convex real projective structure.*
2. *$M$ is projectively diffeomorphic to a quotient $\Omega/\Gamma$ where $\Omega \subset \mathbb{RP}^2$ is a convex domain and $\Gamma \subset \mathbf{PGL}(3, \mathbb{R})$ is a discrete group acting properly and freely on $\Omega$.*

DEFINITION 2.5. The *Goldman space* $\mathcal{G}(M)$ is the subset of $\mathbb{RP}^2(M)$ consisting of the equivalence classes of convex real projective structures.

The Goldman space $\mathcal{G}(M)$ is an analogue of the Teichmüller space $\mathfrak{T}(M)$. The Goldman space $\mathcal{G}(M)$ is a component of $\mathbb{RP}^2(M)$ and the restriction of $\mathbf{hol} : \mathbb{RP}^2(M) \to \mathrm{Hom}(\pi, \mathbf{PGL}(3, \mathbb{R}))/\mathbf{PGL}(3, \mathbb{R})$ to $\mathcal{G}(M)$ is an embedding onto an open subset.(Choi and Goldman [2]) It is known that $\mathfrak{T}(M)$ embeds into $\mathcal{G}(M)$. That means every hyperbolic structure on $M$ defines a convex real projective structure on $M$. Similarly as the Teichmüller space $\mathfrak{T}(M)$, the element of the Goldman space $\mathcal{G}(M)$ will be identified with a conjugacy class of $\mathrm{Hom}(\pi, \mathbf{PGL}(3, \mathbb{R}))$.

## 3. Matrix presentations of a pair of pants

An element $A$ of $\mathbf{SL}(2, \mathbb{R})$ is said to be *hyperbolic* if $A$ has two distinct real eigenvalues. Since the characteristic polynomial of $A$ is $f(\lambda) = \lambda^2 - t\lambda + 1$ where $t = \mathrm{tr}(A)$, $A$ is hyperbolic if and only if $\mathrm{tr}(A)^2 > 4$. Thus a hyperbolic element $A$ in $\mathbf{SL}(2, \mathbb{R})$ can be expressed by the diagonal matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \tag{3.1}$$

via an $\mathbf{SL}(2, \mathbb{R})$-conjugation where $\alpha^2 > 1$.

Let $A$ be an element of $\mathbf{PSL}(2, \mathbb{R})$. Since the absolute value of trace is still defined, $A \in \mathbf{PSL}(2, \mathbb{R})$ is said to be *hyperbolic* if $|\mathrm{tr}(A)| > 2$. It is known that $A$ is hyperbolic if and only if $A$ has two distinct fixed points on $\partial\mathbb{H}^2$. The following theorem is due to Kuiper [7].

THEOREM 3.1. *Let $M$ be a compact connected oriented hyperbolic surface. Then every nontrivial element of holonomy group $\Gamma \subset \mathbf{PSL}(2, \mathbb{R})$ is hyperbolic.*

Let $M = \Sigma(g,n)$ be a compact connected oriented surface with $g$-genus and $n$-boundary components. If $\chi(M) = 2-2g-n < 0$, then there exist $3g-3+n$ nontrivial homotopically-distinct disjoint simply-closed curves on $M$ such that they decompose $M$ as the disjoint union of $2g-2+n$ pairs of pants $\Sigma(0,3)$. Thus a pair of pants $\Sigma(0,3)$ is a building block of an oriented surface $M$. For more detail, see Wolpert's paper [9].

The goal of this section is to find an expression of the elements of the Teichmüller space $\mathfrak{T}(\Sigma(0,3))$ of a pair of pants. Since $\mathfrak{T}(\Sigma(0,3))$ embeds into $\mathrm{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))/\mathbf{PSL}(2,\mathbb{R})$, we should calculate the matrix presentations of the conjugacy classes of $\mathrm{Hom}(\pi, \mathbf{PSL}(2,\mathbb{R}))$.

First we consider the positions of fixed points of hyperbolic elements in $\mathbf{SL}(2,\mathbb{R})$.

LEMMA 3.2. *Suppose $A, B \in \mathbf{SL}(2,\mathbb{R})$ and $P \in \mathbf{GL}(2,\mathbb{R})$ satisfying $B = PAP^{-1}$. If $z \in \mathbb{H}^2$ is a fixed point of $A$, then $w = Pz \in \mathbb{H}^2$ is a fixed point of $B$.*

*Proof.* Since we have $Bw = (PAP^{-1})(Pz) = P(Az) = Pz = w$, $w = Pz$ is a fixed point of $B$. □

The *principal line* of a hyperbolic element $A \in \mathbf{SL}(2,\mathbb{R})$ or $\mathbf{PSL}(2,\mathbb{R})$ is the $A$-invariant unique geodesic in $\mathbb{H}^2$. And it is the line joining two fixed points of $A$. Since the principal line has a distinct direction, one of the fixed points of $A$ is called the *repelling* fixed point and the other is called the *attracting* fixed point. For more easy understanding, see Figure 1, or Beardon's book [1].

PROPOSITION 3.3. *Suppose* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $B = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ *are hyperbolic elements of $\mathbf{SL}(2,\mathbb{R})$. Then we have the following relations.*

1. $\mathrm{Det}(A) = \mathrm{Det}(B)$
2. $\mathrm{Tr}(A) = \mathrm{Tr}(B)$
3. *If $z$ is a fixed point of $A$, then $-z$ is a fixed point of $B$.*

*Proof.* Let $P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we can calculate

$$PAP^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = B.$$

Therefore the point $w = Pz = \frac{1\cdot z+0}{0\cdot z-1} = -z$ is a fixed point of $B$. □

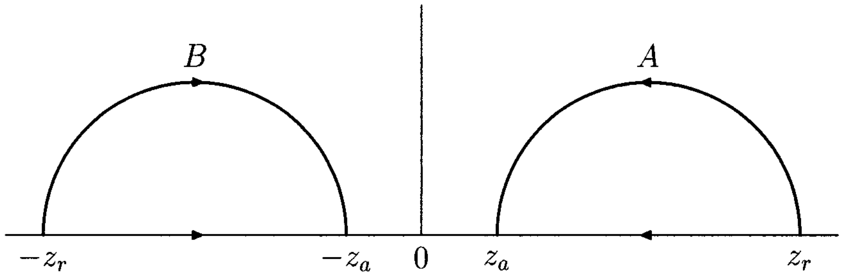


FIGURE 1. The fixed points of the matrices $A$ and $B$

Thus the principal lines of $A$ and $B$ are symmetric with respect to the imaginary axis.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}(2,\mathbb{R})$ be a hyperbolic element. We now consider the location of the principal line of $A$ and the relations of entries of $A$.

THEOREM 3.4. *Suppose that $A$ is a hyperbolic element of $\mathbf{SL}(2,\mathbb{R})$ and $z_r, z_a$ are the repelling and attracting fixed points of $A$. Then*

1. $0 < z_a,\ z_r < \infty$ *if and only if* $(a-d)\,c > 0,\ b\,c < 0$.
2. $z_a < z_r$ *if and only if* $(a+d)\,c < 0$.

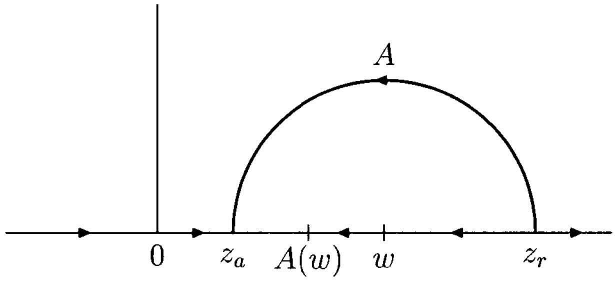


FIGURE 2. The principal line with $0 < z_a < z_r < \infty$

*Proof.* Since $z_a, z_r$ are the fixed points of the hyperbolic transformation $A(z) = \frac{az+b}{cz+d}$, they are the roots of the equation

$$c\,z^2 + (d-a)\,z - b = 0. \tag{3.2}$$

Suppose $0 < z_a,\ z_r < \infty$. First we claim that $c \neq 0$. If $c = 0$, then $1 = \det(A) = ad$. Thus $d = a^{-1}$ and $A(z) = a^2 z + ab$. This yields that

$\infty$ is a fixed point of $A(z)$ since $a \neq 0$. It contradicts the assumption. Since $z_a + z_r = \frac{a-d}{c}$ and $z_a \cdot z_r = \frac{-b}{c}$, it proves $0 < z_a,\, z_r < \infty$ if and only if $(a-d)\,c > 0$ and $b\,c < 0$.

Since we have $c \neq 0$, the roots $z_a, z_r$ of the Equation (3.2) can be expressed by

$$z_a,\, z_r = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}. \tag{3.3}$$

Suppose that the attracting fixed point $z_a$ is smaller than the repelling fixed point $z_r$ ; i.e. $z_a < z_r$. Let $w$ be the mid point of the fixed points $z_a$ and $z_r$ ; i.e. $w = (z_a + z_r)/2 = (a-d)/(2c)$. Then the condition $z_a < z_r$ is equivalent to $A(w) < w$. Since we can compute

$$\begin{aligned} A(w) - w &= \frac{a(\frac{a-d}{2c}) + b}{c(\frac{a-d}{2c}) + d} - \left(\frac{a-d}{2c}\right) \\ &= \frac{a(a-d) + 2bc}{c(a+d)} - \left(\frac{a-d}{2c}\right) = \frac{(a+d)^2 - 4}{2(a+d)c}, \end{aligned}$$

and $(a+d)^2 > 4$, it proves $z_a < z_r$ if and only $(a+d)\,c < 0$. This completes the proof. □

Corollary 3.5. *Let $A \in \mathbf{SL}(2, \mathbb{R})$ representing a hyperbolic transformation of $\mathbb{H}^2$ and $z_r, z_a$ the repelling and attracting fixed points of $A$. Suppose $0 < z_a < z_r < \infty$, then $a^2 < d^2$ and $b\,d > 0$.*

*Proof.* From the Theorem 3.4, we have the relations $(a-d)\,c > 0$ and $(a+d)\,c < 0$. Thus $(a-d)(a+d)c^2 = (a^2 - d^2)c^2 < 0$ implies $a^2 < d^2$. Since $z_a < z_r$, the image of the origin under $A$ should be positive as in the Figure 2. That means $A(0) = b/d > 0$. Thus we have $b\,d > 0$. This also implies $b \neq 0$ and $d \neq 0$. □

Corollary 3.6. *Let $A \in \mathbf{SL}(2, \mathbb{R})$ representing a hyperbolic transformation of $\mathbb{H}^2$.*

1. *Suppose that $b > 0$. Then $0 < z_a < z_r < \infty$ if and only if $c < 0,\ d > 0,\ |a| < d$.*
2. *Suppose that $b < 0$. Then $0 < z_a < z_r < \infty$ if and only if $c > 0,\ d < 0,\ |a| < (-d)$.*

*Proof.* Suppose $0 < z_a < z_r < \infty$ and $b > 0$. Since we have the relations $b\,c < 0$, $b\,d > 0$ and $a^2 < d^2$, the condition $b > 0$ yields that $c < 0$, $d > 0$, and $|a| < |d| = d$. Conversely the condition $|a| < d$ derives $(a-d) < 0$, and $(a+d) > 0$. Since $c < 0$ we get $(a-d)\,c > 0$ and $(a+d)\,c < 0$ . Since $b\,c < 0$, this induces $0 < z_a < z_r < \infty$. We can prove similarly for the case $b < 0$. □

## 4. Teichmüller space of a pair of pants $\Sigma(0,3)$

Recall that a pair of pants $M = \Sigma(0,3)$ is a sphere with three holes. Suppose $M$ is equipped with a hyperbolic structure. Since the holonomy homomorphism is isomorphic to its image, the fundamental group $\pi$ of $M$ will be identified with

$$\pi = \langle A, B, C \in \mathbf{PSL}(2,\mathbb{R}) \mid R = CBA = I \rangle.$$

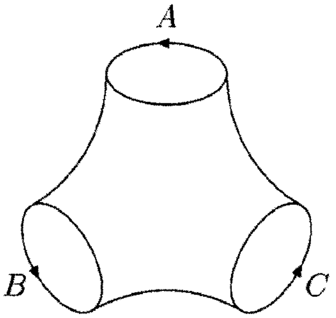


FIGURE 3. A pair of pants $M = \Sigma(0,3)$

Let $A, B, C \in \mathbf{PSL}(2,\mathbb{R})$ represent the boundary components of $M$. We will find the expression of the generators $A, B$ and $C$ of $\pi$ in terms of $\mathbf{SL}(2,\mathbb{R})$ instead of $\mathbf{PSL}(2,\mathbb{R})$ because $\mathbf{SL}(2,\mathbb{R})$ is easier to compute and understand than $\mathbf{PSL}(2,\mathbb{R})$. Since the matrices $A, B, C \in \mathbf{SL}(2,\mathbb{R})$ are hyperbolic and represented up to conjugate, without loss of generality, we can assume

$$B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}$$

with $\mu^2 > 1$. Then $B(0) = 0$ since

$$B(z) = \frac{\mu \cdot z + 0}{0 \cdot z + \mu^{-1}} = \mu^2 z.$$

Thus 0 is the repelling fixed point and $\infty$ is the attracting fixed point of $B$ since $\mu^2 > 1$. By the discreteness of holonomy group, $A(0) \neq 0$. Suppose

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $b \neq 0$. If $b = 0$, then

$$A(0) = \frac{a \cdot 0 + b}{c \cdot 0 + d} = 0,$$

contradicting for $A(0) \neq 0$. Suppose $\mathrm{tr}(A) = \lambda + \lambda^{-1}$ where $\lambda^2 > 1$. Since $a + d = \mathrm{tr}(A) = \lambda + \lambda^{-1}$, we have $d = -a + \lambda + \lambda^{-1}$. Since $\det(A) = ad - bc = 1$, we obtain

$$bc = ad - 1 = a(-a + \lambda + \lambda^{-1}) - 1 = -(a - \lambda)(a - \lambda^{-1}).$$

Thus we have $c = -(a - \lambda)(a - \lambda^{-1})b^{-1}$ since $b \neq 0$. Therefore

$$A = \begin{pmatrix} a & b \\ -(a-\lambda)(a-\lambda^{-1})b^{-1} & -a+\lambda+\lambda^{-1} \end{pmatrix}.$$

Suppose $b > 0$. Let

$$P = \begin{pmatrix} \sqrt{b^{-1}} & 0 \\ 0 & \sqrt{b} \end{pmatrix},$$

then

$$PAP^{-1} = \begin{pmatrix} a & 1 \\ -(a-\lambda)(a-\lambda^{-1}) & -a+\lambda+\lambda^{-1} \end{pmatrix},$$

$$PBP^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} = B.$$

Similary if $b < 0$, then there exist

$$Q = \begin{pmatrix} \sqrt{-b^{-1}} & 0 \\ 0 & \sqrt{-b} \end{pmatrix}$$

such that

$$QAQ^{-1} = \begin{pmatrix} a & -1 \\ (a-\lambda)(a-\lambda^{-1}) & -a+\lambda+\lambda^{-1} \end{pmatrix},$$

$$QBQ^{-1} = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} = B.$$

Since $R = CBA = I$, we can get $C = A^{-1}B^{-1}$. Therefore the generators $A, B$ and $C$ of $\pi$ are expressed by

(4.1)
$$A = \begin{pmatrix} a & 1 \\ -(a-\lambda)(a-\lambda^{-1}) & -a+\lambda+\lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix},$$

and

$$C = \begin{pmatrix} \mu^{-1}(-a+\lambda+\lambda^{-1}) & -\mu \\ \mu^{-1}(a-\lambda)(a-\lambda^{-1}) & a\mu \end{pmatrix} \tag{4.2}$$

or

$$A = \begin{pmatrix} a & -1 \\ (a-\lambda)(a-\lambda^{-1}) & -a+\lambda+\lambda^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix}, \tag{4.3}$$

and

$$C = \begin{pmatrix} \mu^{-1}(-a+\lambda+\lambda^{-1}) & \mu \\ -\mu^{-1}(a-\lambda)(a-\lambda^{-1}) & a\mu \end{pmatrix}. \tag{4.4}$$

As a result, the trace of $C$ is the same for the both cases ; that is

$$\operatorname{tr}(C) = \mu^{-1}(-a+\lambda+\lambda^{-1}) + a\mu.$$

Suppose $\operatorname{tr}(C) = \nu + \nu^{-1}$ with $\nu^2 > 1$. After some simple computations we have

$$a = \frac{\mu}{\mu^2-1}\left((\nu+\frac{1}{\nu}) - \frac{1}{\mu}(\lambda+\frac{1}{\lambda})\right). \tag{4.5}$$

Therefore $\{\lambda, \mu, \nu\}$ is a coordinate for the Teichmüller space $\mathfrak{T}(\Sigma(0,3))$, i.e., the dimension of the Teichmüller space $\mathfrak{T}(\Sigma(0,3))$ is 3.

COROLLARY 4.1. *Suppose $z_r, z_a$ are the repelling and attracting fixed points of the hyperbolic matrix*

$$A = \begin{pmatrix} a & 1 \\ -(a-\lambda)(a-\lambda^{-1}) & -a+\lambda+\lambda^{-1} \end{pmatrix}$$

*with $\lambda^2 > 1$. Then $0 < z_a < z_r$ if and only if $a < \lambda^{-1} < 1 < \lambda$.*

*Proof.* By Corollary 3.6, we have the relations $(a-\lambda)(a-\lambda^{-1}) > 0$ and $a \le |a| < -a+\lambda+\lambda^{-1}$. Suppose $(a-\lambda) > 0$ and $(a-\lambda^{-1}) > 0$. Then $2a > \lambda + \lambda^{-1}$. It contradicts the result $a < -a+\lambda+\lambda^{-1}$. Thus the inequalities should be $(a-\lambda) < 0$ and $(a-\lambda^{-1}) < 0$. Also we have $-a \le |a| < -a+\lambda+\lambda^{-1}$. Thus we obtain $0 < \lambda+\lambda^{-1}$. The assumption $\lambda^2 > 1$ yields that $a < \lambda^{-1} < 1 < \lambda$. Conversely if $a < \lambda^{-1} < 1 < \lambda$, then we can easily show that $A_{21} < 0$, $A_{22} > 0$ and $|A_{11}| < A_{22}$ where $A_{ij}$ is the $(i,j)$-th entry of the matrix $A$. □

REMARK 4.2. Thus above matrix $A$ has positive valued trace $\lambda+\lambda^{-1}$.

COROLLARY 4.3. *Suppose $z_r, z_a$ are the repelling and attracting fixed points of the hyperbolic matrix*

$$A = \begin{pmatrix} a & -1 \\ (a-\lambda)(a-\lambda^{-1}) & -a+\lambda+\lambda^{-1} \end{pmatrix}$$

*with $\lambda^2 > 1$. Then $0 < z_a < z_r$ if and only if $\lambda < -1 < \lambda^{-1} < a$.*

*Proof.* It can be proved in the same way in Corollary 4.1. □

Since $A, B, C$ are hyperbolic elements and the holonomy group is discrete, the locations of the principal lines of $A, B, C$ should be one of the the following figures. (Keen [6])

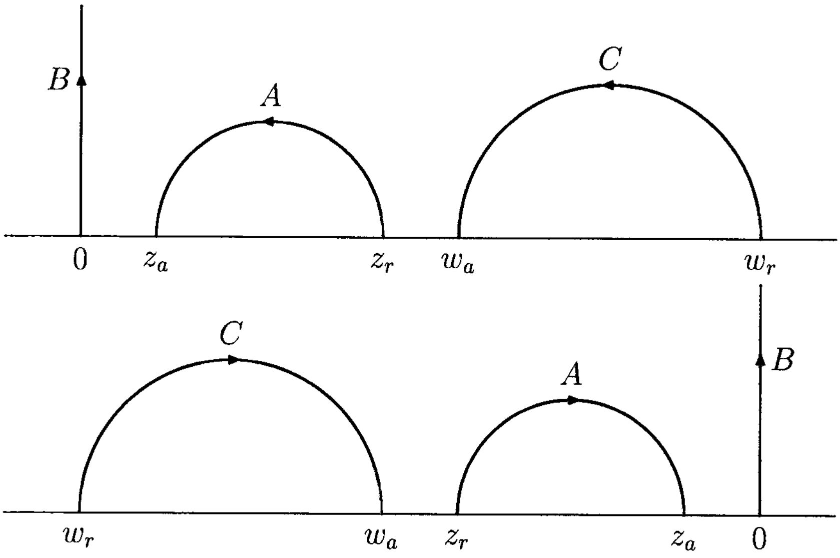


FIGURE 4. The locations of the principal lines of $A, B, C$

The relation of matrices between two diagrams is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff A = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.$$

Thus without loss of generality, we may assume that $0 < z_a < z_r$.

THEOREM 4.4. *Suppose $z_r, w_r, z_a, w_a$ are the repelling and attracting fixed points of the hyperbolic matrices $A$ in (4.1) and $C$ in (4.2) with $\mu^2 > 1$ respectively. Suppose we have $0 < z_a < z_r$ and $0 < w_a < w_r$, then $a < 0$, $\lambda > 1$ and $\lambda + \lambda^{-1} < (-a)(\mu^2 - 1)$.*

*Proof.* Let $C_{ij}$ stand for the $(i, j)$-th entry of the matrix $C$. Since $0 < w_a < w_r$, we get $C_{12}C_{22} = (-\mu)(a\mu) > 0$. Thus we get $a < 0$. $C_{12}C_{21} < 0$ implies $(a - \lambda)(a - \lambda^{-1}) > 0$. And the condition $(C_{11} + C_{22})C_{21} = \left[\mu^{-1}(-a + \lambda + \lambda^{-1}) + a\mu\right] \mu^{-1}(a - \lambda)(a - \lambda^{-1}) < 0$ implies $(-a + \lambda + \lambda^{-1}) < -a\mu^2$. Thus we have $(\lambda + \lambda^{-1}) < (-a)(\mu^2 - 1)$. $(C_{11} - C_{22})C_{21} = \left[\mu^{-1}(-a + \lambda + \lambda^{-1}) - a\mu\right] \mu^{-1}(a - \lambda)(a - \lambda^{-1}) > 0$ implies $(\lambda + \lambda^{-1}) > a(\mu^2 + 1)$. Since $\lambda > 1$ and $a < 0$, above condition trivially holds. Therefore the conditions for $0 < z_a < z_r$ and $0 < w_a < w_r$ are $a < 0$, $\lambda > 1$ and $\lambda + \lambda^{-1} < (-a)(\mu^2 - 1)$. □

Since the matrix $C$ is representing a boundary component a pair of pants, $C$ is hyperbolic. Thus

$$\operatorname{tr}(C)^2 = \left(\mu^{-1}(-a+\lambda+\lambda^{-1})+a\mu\right)^2 > 4\,.$$

Multiply both sides by $\mu^2 > 1$ induces $\left((-a+\lambda+\lambda^{-1})+a\mu^2\right)^2 = \left((\lambda+\lambda^{-1})+a(\mu^2-1)\right)^2 > 4\mu^2$. Since $(\lambda+\lambda^{-1})+a(\mu^2-1) < 0$, we have $-(\lambda+\lambda^{-1})-a(\mu^2-1) > 2|\mu|$. Therefore the hyperbolic condition for the matrix $C$ in (4.2) is

$$(-a)(\mu^2-1) > (\lambda+\lambda^{-1})+2|\mu|, \tag{4.6}$$

where $a < 0$, $\lambda > 1$, and $\mu^2 > 1$.

Now we consider the position of fixed points of the matrix $A$ and $C$.

THEOREM 4.5. *Suppose that $A$ is the hyperbolic matrix in* (4.1) *with* $0 < z_a < z_r$. *Then the fixed points of $A$ are*

$$z_a = \frac{1}{\lambda-a} \quad \text{and} \quad z_r = \frac{1}{\lambda^{-1}-a}. \tag{4.7}$$

*Proof.* By the Equation (3.3),

$$\begin{aligned} z_a,\, z_r &= \frac{(2a-\lambda-\lambda^{-1}) \pm \sqrt{(\lambda+\lambda^{-1})^2-4}}{-2(a-\lambda)(a-\lambda^{-1})} \\ &= \frac{(2a-\lambda-\lambda^{-1}) \pm |\lambda-\lambda^{-1}|}{-2(a-\lambda)(a-\lambda^{-1})} \\ &= \frac{(2a-\lambda-\lambda^{-1}) \pm (\lambda-\lambda^{-1})}{-2(a-\lambda)(a-\lambda^{-1})} \\ &= \frac{2(a-\lambda^{-1})}{-2(a-\lambda)(a-\lambda^{-1})} \quad \text{or} \quad \frac{2(a-\lambda)}{-2(a-\lambda)(a-\lambda^{-1})} \\ &= \frac{1}{(\lambda-a)} \quad \text{or} \quad \frac{1}{(\lambda^{-1}-a)}. \end{aligned}$$

Since $(\lambda-a) > (\lambda^{-1}-a)$, the attracting fixed point $z_a$ of $A$ is $1/(\lambda-a)$ and the repelling fixed point $z_r$ of $A$ is $1/(\lambda^{-1}-a)$. □

THEOREM 4.6. *Suppose that $C$ is the hyperbolic matrix in* (4.2) *with* $0 < w_a < w_r$. *Then the fixed points of $C$ are*

$$w_a = \frac{E-\sqrt{D}}{2(\lambda-a)(\lambda^{-1}-a)} \quad \text{and} \quad w_r = \frac{E+\sqrt{D}}{2(\lambda-a)(\lambda^{-1}-a)},$$

*where* $E = -a(\mu^2+1)+\lambda+\lambda^{-1}$ *and* $D = \left(a(\mu^2-1)+\lambda+\lambda^{-1}\right)^2-4\mu^2$.

*Proof.* By the Equation (3.3), the fixed points $w_a$, $w_r$ of $C$ is

$$\frac{[\mu^{-1}(-a+\lambda+\lambda^{-1})-a\mu]\pm\sqrt{[\mu^{-1}(-a+\lambda+\lambda^{-1})+a\mu]^2-4}}{2\mu^{-1}(a-\lambda)(a-\lambda^{-1})}$$
$$=\frac{[(-a+\lambda+\lambda^{-1})-a\mu^2]\pm\sqrt{[(-a+\lambda+\lambda^{-1})+a\mu^2]^2-4\mu^2}}{2(a-\lambda)(a-\lambda^{-1})}$$
$$=\frac{[-a(\mu^2+1)+\lambda+\lambda^{-1}]\pm\sqrt{[a(\mu^2-1)+\lambda+\lambda^{-1})]^2-4\mu^2}}{2(\lambda-a)(\lambda^{-1}-a)}.$$

The fact $(\lambda-a)(\lambda^{-1}-a)>0$ proves the theorem. □

THEOREM 4.7. *Suppose the matrices $A, B, C$ in (4.1) and (4.2) have the relation $a<0$, $\lambda>1$, $\mu^2>1$ and $(-a)(\mu^2-1)>(\lambda+\lambda^{-1})+2|\mu|$. Then $\{A,B,C\}$ forms generators of the fundamental group $\pi$ of a pair of pants $\Sigma(0,3)$.*

*Proof.* We should show that $0<z_a<z_r<w_a<w_r$. By Theorem 4.4, it is enough to show that $z_r<w_a$. Theorems 4.5 and 4.6 and the facts $(\lambda-a)>0$ and $(\lambda^{-1}-a)>0$ yield that $z_r<w_a$ if and only if $2(\lambda-a)<E-\sqrt{D}$ ; that is

$$\sqrt{D}<E-2(\lambda-a)=(-a)(\mu^2-1)-\lambda+\lambda^{-1}.$$

Since $(-a)(\mu^2-1)-\lambda+\lambda^{-1}>(-a)(\mu^2-1)-\lambda-\lambda^{-1}>0$, it is equivalent to show that

$$D=\left((-a)(\mu^2-1)-\lambda-\lambda^{-1}\right)^2-4\mu^2<\left((-a)(\mu^2-1)-\lambda+\lambda^{-1}\right)^2.$$

After some calculations we can get $\left((-a)(\mu^2-1)-\lambda\right)>-\mu^2\lambda$. This is equivalent to $a(\mu^2-1)<\lambda(\mu^2-1)$. Since $a<0$, $\lambda>1$ and $\mu^2>1$, it proves the theorem. □

THEOREM 4.8. *Suppose the matrices $A, B, C$ in (4.3) and (4.4) have the relation $a>0$, $\lambda<-1$, $\mu^2>1$ and $a(\mu^2-1)>-(\lambda+\lambda^{-1})+2|\mu|$. Then $\{A,B,C\}$ forms generators of the fundamental group $\pi$ of a pair of pants.*

*Proof.* This can be proved by the same way in the Theorem 4.7. □

Finally we consider the relations of traces of $A, B$, and $C$ in $\mathbf{SL}(2,\mathbb{R})$.

THEOREM 4.9. *Suppose the matrices $\{A,B,C\}$ in (4.1) and (4.2) forms generators of the fundamental group $\pi$ of a pair of pants. Then*

1. *$\mu$ is positive if and only if $\nu<-1$.*
2. *$\mu$ is negative if and only if $\nu>1$.*

*Proof.* Recall the Equation (4.5) that is the relation among the traces of the matrices $A, B, C$ and the value $a$. If we plug in the Equation (4.5) to the inequality (4.6) representing the hyperbolic condition of the matrix $C$, then we obtain $(\lambda + \lambda^{-1}) + 2|\mu| < -\mu(\nu + \nu^{-1}) + (\lambda + \lambda^{-1})$. Hence we get the inequality

$$2|\mu| < -\mu(\nu + \nu^{-1}). \tag{4.8}$$

If $\mu$ is positive, then above inequality (4.8) becomes $2\mu < -\mu(\nu + \nu^{-1})$. Since $-\mu$ is negative, we have $-2 > \nu + \nu^{-1}$. Therefore $\nu < -1$. Similarly if $\mu$ is negative, then we have $-2\mu < -\mu(\nu + \nu^{-1})$. Since $-\mu$ is positive, we get $2 < \nu + \nu^{-1}$. Therefore $\nu > 1$. □

REMARK 4.10. Since $A, B, C$ should satisfy the condition (4.6),

$$\mu > \frac{1 + \sqrt{1 - a(\lambda + \lambda^{-1}) + a^2}}{-a} > 1 \quad \text{if } \mu \text{ is positive,}$$

$$\mu < \frac{1 + \sqrt{1 - a(\lambda + \lambda^{-1}) + a^2}}{a} < -1 \quad \text{if } \mu \text{ is negative.}$$

THEOREM 4.11. *Suppose the matrices $\{A, B, C\}$ in (4.3) and (4.4) forms a generator of the fundamental group $\pi$ of a pair of pants. Then*

1. *$\mu$ is positive if and only if $\nu > 1$.*
2. *$\mu$ is negative if and only if $\nu < -1$.*

*Proof.* This can be proved by the same way in the Theorem 4.9. □

Since $\mathrm{tr}(A) > 2$ in (4.1) and $\mathrm{tr}(A) < -2$ in (4.3), we conclude the following result.

COROLLARY 4.12. *Suppose the matrices $\{A, B, C\}$ are in (4.1) and (4.2) or in (4.3) and (4.4) which forms generators of the fundamental group $\pi$ of a pair of pants. Then $\mathrm{tr}(A) \cdot \mathrm{tr}(B) \cdot \mathrm{tr}(C) < -8$.*

Therefore we cannot have the matrices $A, B, C \in \mathbf{SL}(2, \mathbb{R})$ which are representing the boundary components of a pair of pants with all three positive traces. An odd number of traces must be negative.

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