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ASYMPTOTIC DIRICHLET PROBLEM FOR HARMONIC MAPS ON NEGATIVELY CURVED MANIFOLDS

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Abstract. In this paper, we prove the existence of nonconstant bounded harmonic maps on a Cartan-Hadamard manifold of pinched negative curvature by solving the asymptotic Dirichlet problem. To be precise, given any continuous data f on the boundary at infinity with image within a ball in the normal range, we prove that there exists a unique harmonic map from the manifold into the ball with boundary value f.

1. Introduction

In this paper, we study the existence problem of harmonic maps from negatively curved manifolds into a complete Riemannian manifold. Let (M, g) and (N, h) be Riemannian manifolds of dimension m and n, respectively, with local expressions for their metrics $g = g_{ij}dx^i dx^j$ and $h = h_{\alpha\beta}dy^\alpha dy^\beta$, where (x^i) and (y^α) are local coordinates of M and N, respectively. We say that a map $u \in C^1(M, N)$ is harmonic if u is a critical point of the total energy functional

$$E(f) = \int_M e(f)(x)dx,$$

where $f \in C^1(M, N)$ and $e(f)$ is the energy density of f, in local coordinates we have $e(f)(x) = g^{ij}\frac{\partial f^\alpha}{\partial x^i}(x)\frac{\partial f^\beta}{\partial x^j}(x)h_{\alpha\beta}(f(x))$. A straightforward calculation shows that, in terms of local coordinates, harmonic map satisfies the following nonlinear elliptic system of partial differential equations: For each $\alpha = 1, 2, \cdots, n$,

$$\Delta_M u^\alpha(x) + g^{ij}\Gamma^\alpha_{\beta\gamma}\frac{\partial u^\beta}{\partial x^i}\frac{\partial u^\gamma}{\partial x^j} = 0,$$

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where Δ_M is the Laplacian of M and $\Gamma^{\alpha}_{\beta\gamma}$'s are the Cristoffel symbols on N. If the target manifold of a harmonic map is flat, all Christoffel symbols vanish, then the harmonic map equation becomes the Laplace-Beltrami equation. Thus harmonic map is a nonlinear generalization of harmonic function.

The question of existence or nonexistence of nonconstant bounded harmonic functions on a complete Riemannian manifold has long been an interesting topic of study to geometers and analysts. In 1975, Yau [20] gave a result that the Liouville property for positive harmonic function holds on complete Riemannian manifold with nonnegative Ricci curvature. Later, Cheng [6] extended Yau's result to the case of the bounded harmonic maps. He proved that any harmonic map with bounded image from a complete Riemannian manifold with nonnegative Ricci curvature into a Cartan-Hadamard manifold must be constant.

In the direction of proving existence, Greene-Wu [13] introduced the following conjecture:

CONJECTURE 1.1. *Let M be a Cartan-Hadamard manifold. For every point $x \in M$ outside a compact set, suppose that M satisfies an inequality:*

$$K_M(x) \le -\frac{A}{r^2(x)},$$

where $K_M(x)$ is the sectional curvature of M at x, and A is a positive constant, and $r(x) = d(o,x)$ is the distance from a fixed point $o \in M$ to x. Then M would possess enough bounded harmonic functions.

In connection with this conjecture, Choi [8] proposed the asymptotic Dirichlet problem and obtained a sufficient condition for the solvability of the problem in terms of a certain convexity condition at the boundary at infinity $M(\infty)$ when M is a Cartan-Hadamard manifold whose sectional curvature bounded above by a strictly negative constant. (By the asymptotic Dirichlet problem for harmonic functions on a noncompact complete Riemannian manifold, we mean to find a harmonic function satisfying the given Dirichlet boundary condition at the asymptotic boundary.) Later, Choi's convexity condition was verified by Anderson [3] when the sectional curvature of a Cartan-Hadamard manifold is pinched by two strictly negative constants. On the other hand, Ancona [2] showed that the asymptotic Dirichlet problem cannot be solvable in general, if a Cartan-Hadamard manifold has no curvature lower bounds. Partial contributions on this conjecture have been made by Sullivan [19], Ancona [1], Anderson-Schoen [4], Schoen-Yau [18], Cheng [7], Ding-Zhou [10], Hsu-March [16], and Choi and the present authors [9].

In the case of the asymptotic Dirichlet problem for harmonic maps, Avilés-Choi-Micallef [5] extended those works of Anderson [3], of Sullivan [19], and of Anderson-Schoen [4] to the case of harmonic maps. To be precise, let M be a Cartan-Hadamard manifold whose sectional curvature pinched by two strictly negative constants and N be a complete Riemannian manifold. If the image of a map $f \in C^0(M(\infty), N)$ is contained in a geodesic ball of N, which lies within normal range of each of its points, then Avilés-Choi-Micallef proved that there is a harmonic map $u \in C^2(M, N) \cap C^0(M \cup M(\infty), N)$ such that $u = f$ on $M(\infty)$.

In this paper, we solve the asymptotic Dirichlet problem for harmonic maps as follows:

THEOREM 1.2. *Let M be a Cartan-Hadamard manifold of dimension m ($m \geq 2$). For each point $x \in M$ outside a compact set, suppose that the sectional curvature $K_M(x)$ of M at x satisfies the following:*

$$-(\beta \log r(x))^2 \leq K_M(x) \leq -\frac{A}{r^2(x)}, \tag{1}$$

where β is a positive constant to be determined later and A is a positive constant greater than 2, and $r(x) = d(o, x)$ is the distance from a fixed point o in the compact set to x. Let $\mathcal{B}_{r_o}(p)$ be a geodesic ball of a point p, which lies within normal range of p, in a complete Riemannian manifold N. Then the asymptotic Dirichlet problem for harmonic maps is solvable for any $f \in C(M(\infty), \overline{\mathcal{B}}_{r_o}(p))$.

2. Preliminaries

Throughout this paper, M shall always denote a Cartan-Hadamard manifold of dimension m ($m \geq 2$) satisfying the curvature condition (1).(By a Cartan-Hadamard manifold, we mean a complete simply connected manifold with nonpositive sectional curvature.) Fix a point $o \in M$ and write $r(x) = d(o, x)$. Let us denote $M(\infty)$ to be the boundary at infinity of M which is the set of asymptotic classes of unit speed geodesic rays. It is topologized with the cone topology in the sense of [11]. We identify $M(\infty)$ with the unit sphere $\mathbb{S}^{m-1} \subset T_oM$.

Assume that $f : \mathbb{S}^{m-1} \to \mathbb{R}$ is L-Lipschitz. Extend f to a radially constant continuous function h on $M \setminus \{o\}$ in such a way that $h(r, \theta) = f(\theta)$ for every $r > 0$ and $\theta \in \mathbb{S}^{m-1}$. Then for any $x, y \in M$ with

$d(x,y) \le 1$,

$$|h(x) - h(y)| = |f(\theta_x) - f(\theta_y)| \le L|\theta_x - \theta_y|,$$

where θ_z denotes the sphere coordinate of a point z and $|\theta_x - \theta_y|$ denotes the angle at o between the ray from o to x and the ray from o to y. By using the curvature upper bound assumption in (1), Hsu-March [16] estimated the angle as follows: There exist positive constants R_1, C and α such that

$$|\theta_x - \theta_y| \le \frac{C}{(1 + r(x))^{2+\alpha}}$$

for any $x, y \in M$ with $d(x,y) \le 1$ whenever $r(x) > R_1$. Hence we have

$$\operatorname*{osc}_{B_1(x)} h = \sup_{B_1(x)} h - \inf_{B_1(x)} h \le \frac{CL}{(1 + r(x))^{2+\alpha}} \tag{2}$$

for any $r(x) > R_1$, where $B_s(z)$ denotes the metric s-ball centered at z.

A set $P \subset M$ is said to be ε-separated for $\varepsilon > 0$ if $d(p,q) \ge \varepsilon$ for any distinct points p and q in P, and an ε-separated subset is called maximal if it is maximal with respect to the order relation of inclusion. Fix a maximal $(1/3)$-separated subset $P = \{p_1, p_2, \cdots\}$. We may assume that $o \notin P$. Clearly, the balls $B_{1/6}(p_i)$'s are mutually disjoint and $M = \bigcup_{p_i \in P} B_{1/3}(p_i)$. For each $x \in M$, we write $P_x = P \cap B_1(x)$. Then the condition (1) and the comparison theorem imply that there exists a constant C such that

$$\sharp P_x \le C\Big(\frac{r(x)^\beta}{\log r(x)}\Big)^{m-1}, \tag{3}$$

where $\sharp X$ denotes the cardinality of the set X. Define a function $g : M \to \mathbb{R}$ by

$$g(x) = \sum_{p_i \in P} h(p_i)\eta_i(x), \tag{4}$$

where $\{\eta_i\}$ is a partition of unity subordinate to $\{B_1(p_i)\}$ defined as follows: Let $\varphi : [0,\infty) \to [0,1]$ be a C^∞-function such that $\varphi|_{[0,1/3]} = 1$, $\varphi|_{[2/3,\infty)} = 0$, and

$$\max\{|\varphi'(t)|, |\varphi''(t)|\} \le c\chi_{[1/3,2/3]}(t) \tag{5}$$

for some $c > 0$, where χ_X denotes the characteristic function of the set X. For $p_i \in P$ and $x \in M$, let $\xi_i(x) = \varphi(r_i(x))$, where $r_i(x) = d(x, p_i)$. We define $\eta_i(x) = \xi_i(x)/(\sum_j \xi_j(x))$. From the definition of g and the continuity of h near $M(\infty)$, it is easy to see that the function g takes the same value at infinity as that of f, i.e., for each $\mathbf{x} \in M(\infty)$,

$$\lim_{x \to \mathbf{x}} g(x) = f(\mathbf{x}). \tag{6}$$

Since $\langle \nabla r_i(x), \nabla r_i(x) \rangle = 1$, we have

(7)

$$\nabla \xi_i(x) = \varphi'(r_i(x)) \nabla r_i(x) \quad \text{and} \quad \Delta \xi_i(x) = \varphi'(r_i(x)) \Delta r_i(x) + \varphi''(r_i(x)).$$

Thus (5) and (7) imply that $|\nabla \xi_i(x)| \leq c\chi_{\overline{B}_{2/3}(p_i) \setminus B_{1/3}(p_i)}(x)$. By the Hessian comparison theorem, we have $(m-1)/r_i(x) \leq \Delta r_i(x) \leq 2(m-1)\beta \log r(x)$, where $r_i(x) > 0$ and $r(x) > R_1$. Combining this with (5) and (7), we have $|\Delta \xi_i(x)| \leq c \log r(x) \chi_{\overline{B}_{2/3}(p_i) \setminus B_{1/3}(p_i)}(x)$. Since $\sum_j \xi_j(x) \geq 1$, $0 \leq \xi_i(x) \leq 1$ and $\sharp P_x \leq C(r(x)^\beta / \log r(x))^{m-1}$ for every $x \in M$, by a simple computation, we get $|\nabla \eta_i(x)| \leq C(r(x)^\beta / \log r(x))^{m-1}$ and

$$|\Delta \eta_i(x)| \leq C\Big(\frac{r(x)^\beta}{\log r(x)}\Big)^{2(m-1)}. \tag{8}$$

LEMMA 2.1. *Let g be the function given by (4). Then there exists a constant C independent of g such that for any $r(x) > R_1$,*

$$|\Delta g(x)| \ \leq \ \frac{C}{(1 + r(x))^{2+\alpha}} \Big(\frac{r(x)^\beta}{\log r(x)}\Big)^{3(m-1)}. \tag{9}$$

Proof. Since $\sum_{p_i \in P_x} \eta_i \equiv 1$, we have $\nabla g(x) = \sum_{p_i \in P_x} (h(p_i) - h(p)) \nabla \eta_i(x)$, where $p \in P$ is a point such that $x \in B_{1/3}(p)$. Combining this together with (2), (3) and (8), we have

$$\begin{aligned} |\Delta g(x)| \ &\leq \ \sum_{p_i \in P_x} |h(p_i) - h(p)| |\Delta \eta_i(x)| \\ &\leq \ \frac{C \sharp P_x}{(1 + r(x))^{2+\alpha}} \Big(\frac{r(x)^\beta}{\log r(x)}\Big)^{2(m-1)} \\ &\leq \ \frac{C}{(1 + r(x))^{2+\alpha}} \Big(\frac{r(x)^\beta}{\log r(x)}\Big)^{3(m-1)}. \end{aligned}$$

□

Define a C^∞-function $\omega : M \setminus \{o\} \to \mathbb{R}$ by

$$\omega(x) = \frac{1}{(1 + r(x))^\delta}, \tag{10}$$

where $\delta > 0$. It is easy to see that

$$\Delta\omega(x) = -\frac{\delta\Delta r(x)}{(1+r(x))^{1+\delta}} + \frac{\delta(1+\delta)|\nabla r(x)|^2}{(1+r(x))^{2+\delta}}. \tag{11}$$

LEMMA 2.2. *Assume that* $\dim M = m \geq 3$ *and* $0 < \beta < \alpha/(3(m-1))$. *Let* g *be given by* (4) *and* ω *be given by* (10). *Then* $g+\omega$ *is superharmonic and* $g-\omega$ *is subharmonic in* $M \setminus \overline{B}_R(o)$ *whenever* $\delta \in (0, \alpha - 3(m-1)\beta]$, *where* $R = R(\alpha, \beta, \delta) > 0$.

Proof. We may assume that $-(\beta \log r(x))^2 \leq K_M(x) \leq -Ar(x)^{-2}$ on $M \setminus B_{R_0}(o)$ for some $R_0 > 0$. Since M is an m-dimensional Cartan-Hadamard manifold, from the Hessian comparison theorem, $\Delta r \geq (m-1)/r$ for any $r > 0$. Hence by (11),

$$\begin{aligned} \Delta\omega &\leq -\frac{(m-1)\delta}{(1+r)^{1+\delta} r} + \frac{\delta(1+\delta)}{(1+r)^{2+\delta}} \\ &\leq -\frac{\delta(m-2-\delta)}{(1+r)^{2+\delta}} < 0 \end{aligned}$$

whenever $0 < \delta < m-2$ and $r > R_1$ for sufficiently large $R_1 = R_1(\delta) > 0$. Hence by (9),

$$\Delta(g+\omega) \leq \frac{C}{(1+r)^{2+\alpha}} \Big(\frac{r^\beta}{\log r}\Big)^{3(m-1)} - \frac{\delta(m-2-\delta)}{(1+r)^{2+\delta}} < 0, \tag{12}$$

where $0 < \delta \leq \alpha - 3(m-1)\beta$ and $r > R$ for sufficiently large $R = R(\alpha, \beta, \delta) > 0$. Similarly, we obtain an estimate $\Delta(g-\omega) > 0$ for $0 < \delta \leq \alpha - 3(m-1)\beta$ and $r > R$. □

LEMMA 2.3. *Assume that* $\dim M = 2$ *and* $0 < \beta < \alpha/3$. *Let* g *and* ω *be given as in* (4) *and* (10), *respectively. Then* $g+\omega$ *is superharmonic and* $g-\omega$ *is subharmonic in* $M \setminus \overline{B}_R(o)$ *whenever* $\delta \in (0, \alpha - 3\beta]$, *where* $R = R(\alpha, \beta, \delta) > 0$.

Proof. We may assume that $-(\beta \log r(x))^2 \leq K_M(x) \leq -Ar(x)^{-2}$ on $M \setminus B_{R_0}(o)$ for some $R_0 > 0$. From the Hessian comparison theorem, $\Delta r \geq 2/(1+r)$ for any $r > R_0$. Thus by (11), we have

$$\begin{aligned} \Delta\omega &\leq \frac{\delta(1+\delta)}{(1+r)^{2+\delta}} - \frac{2\delta}{(1+r)^{2+\delta}} \\ &= -\frac{\delta(1-\delta)}{(1+r)^{2+\delta}} < 0 \end{aligned}$$

for any $0 < \delta < 1$ and $r > R_0$. Hence by (9),

$$\Delta(g+\omega) \le \frac{C}{(1+r)^{2+\alpha}}\Big(\frac{r^{\beta}}{\log r}\Big)^3 - \frac{\delta(1-\delta)}{(1+r)^{2+\delta}} < 0$$

for any $0 < \delta \le \alpha - 3\beta$ and $r > R$, where $R = R(\alpha, \beta, \delta) > 0$ are given as in Lemma . Similarly, we get $\Delta(g - \omega) > 0$ for any $0 < \delta \le \alpha - 3\beta$ and $r > R$. □

3. Proof of main theorem

In the previous section, we give a superharmonic function and a subharmonic function taking the given boundary data. Using the barriers, we first solve the asymptotic Dirichlet problem for harmonic functions as follows:

THEOREM 3.1. *Let M be a Cartan-Hadamard manifold of dimension m ($m \ge 2$) satisfying the curvature condition* (1). *Then the asymptotic Dirichlet problem for harmonic functions is solvable for any continuous boundary data on $M(\infty)$.*

Proof. If f is a Lipschitz function on $M(\infty)$, then we can choose a constant $0 < \lambda \le 1$ such that

$$\lambda \mathrm{osc}_M\, g \le \frac{1}{(1+R)^{\delta}},$$

where g, R and δ are given as in the previous section. Since $\lambda g + \omega$ is superharmonic and $\lambda g - \omega$ is subharmonic, there is a function $u_i \in C(M)$ such that u_i is harmonic on $B_{2^iR}(o)$ and $u_i \equiv \lambda g$ on $M \setminus B_{2^iR}(o)$ for each $i \in \mathbb{N}$. Since $\lambda g - \omega \le u_i \le \lambda g + \omega$ on $\partial B_R(o) \cup \partial B_{2^iR}(o)$, by the comparison principle, $\lambda g - \omega \le u_i \le \lambda g + \omega$ on $B_{2^iR}(o) \setminus \overline{B}_R(o)$. By the Azela-Ascoli theorem, there are a subsequence $\{u_{i_k}\}$ of $\{u_i\}$ and a limit function $u \in C(M)$ such that $(1/\lambda)u_{i_k}$ converges uniformly to a harmonic function u on any compact subset of M. By (6), for each $\mathbf{x} \in M(\infty)$

$$\lim_{x \to \mathbf{x}} u(x) = f(\mathbf{x}). \tag{13}$$

Let f be a continuous function on $M(\infty)$. Then we can choose a sequence $\{f_i\}$ of Lipschitz functions uniformly converging to f on $M(\infty)$. By the above argument, there exists a sequence $\{u_i\}$ of harmonic functions on M such that $\lim_{x\to\mathbf{x}} u_i(x) = f_i(\mathbf{x})$ for each $\mathbf{x} \in M(\infty)$ and $i \in \mathbb{N}$. Hence there exists a harmonic function u on M such that u_i converges uniformly to u and $u \equiv f$ on $M(\infty)$.

To prove the uniqueness, let u and v be harmonic functions on M satisfying (13). Choose sequences $\{\epsilon_n\}$ and $\{r_n\}$ in such a way that $\epsilon_n \to 0$ and $r_n \to \infty$ as $n \to \infty$, $|u - f| < \epsilon_n$ and $|v - f| < \epsilon_n$ on $M \setminus B_{r_n}(o)$. Then $-2\epsilon_n \leq u - v \leq 2\epsilon_n$ on $\partial B_{r_n}(o)$. By the maximum principle, $-2\epsilon_n \leq u-v \leq 2\epsilon_n$ on $B_{r_n}(o)$. Consequently, $u \equiv v$ on M. □

After submitting this paper, we heard that Hsu [15] also proved that the Dirichlet problem is solvable for the case of harmonic functions in such a curvature condition, but with different approach using probabilistic methods. Now we apply the above result to the case of harmonic maps to prove our main result as follows:

The normal range of a point p in the target manifold N is the domain of the maximal normal coordinate chart on N. Let $\mathcal{B}_{r_o}(p)$ be the geodesic ball in N centered at p with radius r_o. We shall always assume that $r_o < \min\{\pi/(2\sqrt{\kappa})$, injectivity radius of N at $p\}$, where $\kappa \geq 0$ is an upper bound for the sectional curvature of N. This allows us to coordinatize $\overline{\mathcal{B}}_{r_o}(p)$ by means of geodesic normal coordinates centered at p, where $\overline{\mathcal{B}}_{r_o}(p)$ denotes the closure of $\mathcal{B}_{r_o}(p)$ in N. In particular, for a map $f : X \to \overline{\mathcal{B}}_{r_o}(p)$ from a set X into $\overline{\mathcal{B}}_{r_o}(p)$, f can be viewed as being an $\mathbb{R}^n$-valued with respect to the normal coordinates centered at p, where $n = \dim N$.

Proof of Theorem 1.2. Let $f \in C(M(\infty), \overline{\mathcal{B}}_{r_o}(p))$. Then f can be regarded as an $\mathbb{R}^n$-valued map such that

$$f = (f^1, f^2, \cdots, f^n) : M(\infty) \to B_{r_o}(0) \subset \mathbb{R}^n.$$

By Theorem 3.1, we can choose a map $h = (h^1, h^2, \cdots, h^n) : M \to B_{r_o}(0)$ in such a way that for each $l = 1, 2, \cdots, n$, the function h^l is harmonic on M and

$$\lim_{x \to \mathbf{x}} h^l(x) = f^l(\mathbf{x}) \quad \text{for each} \quad \mathbf{x} \in M(\infty).$$

Thus, to prove the existence, it suffices to show that there is a harmonic map $u : M \to B_{r_o}(0)$ such that

$$\rho(u(x), h(x)) \to 0 \quad \text{as} \quad x \to \infty,$$

where $\rho(\cdot, \cdot)$ is a distance function in $\overline{\mathcal{B}}_{r_o}(p)$. Since $\frac{1}{2}\sum_{l=1}^n (f^l)^2$ is also continuous on $M(\infty)$, there exists a function $w \in C^\infty(M) \cap C^0(M \cup M(\infty))$ such that $w \equiv \frac{1}{2}\sum_{l=1}^n (f^l)^2$ on $M(\infty)$. Hence,

$$(w - \frac{1}{2}|h|^2)(x) \to 0 \quad \text{as} \quad x \to \infty. \tag{14}$$

Let v'_R be the harmonic function defined on $B_R(o)$, such that $v'_R = w$ on $\partial B_R(o)$. By the standard Schauder estimates, there exists a subsequence $\{v'_{R_j}\}$ of $\{v'_R\}$ converging uniformly to a harmonic function v' on any compact subset of M. By Theorem 3.1, for $\mathbf{x} \in M(\infty)$, we get

$$\lim_{x \to \mathbf{x}} v'(x) = w(\mathbf{x}). \tag{15}$$

Choose the harmonic functions v_R defined on $B_R(o)$, such that $v_R = \frac{1}{2}|h|^2$ on $\partial B_R(o)$. By (15) and the maximum principle, for any $\epsilon > 0$, there exists $R_0 > 0$ such that $v'_R - \epsilon \le v_R \le v'_R + \epsilon$ on $B_R(o)$ for $R \ge R_0$. By the standard Schauder estimates, there exists a subsequence $\{v_{R_j}\}$ of $\{v_R\}$ converging uniformly to a limit function v on any compact subset of M. In particular, since v is harmonic on M and $v' - \epsilon \le v \le v' + \epsilon$,

$$v \equiv v' \quad \text{on} \quad M. \tag{16}$$

For each $R > 0$, by Theorem 1 in [14], one can find a harmonic map $u_R : B_R(o) \to \overline{\mathcal{B}}_{r_o}(p)$ such that $u_R = h$ on $\partial B_R(o)$. The a priori estimates (Theorem 4) in [12] implies that for sufficiently large $R_0 > 0$ and some $\gamma \in (0,1)$, $|u_R|_{C^{2,\gamma}(B_{R_0}(o))}$ is bounded by a constant depending only on M, $\overline{\mathcal{B}}_{r_o}(p)$ and h, where $R \ge R_0$. Hence by the Azela-Ascoli theorem, there exists a subsequence of $\{u_{R_j}\}$ of $\{u_R\}$ converging uniformly on any compact subset of M. In particular, the limit map $u : M \to \overline{\mathcal{B}}_{r_o}(p)$ is also harmonic. By Lemma 3.1 in [5], there exists a constant $C < \infty$ depending only on the geometry of $\overline{\mathcal{B}}_{r_o}(p)$ such that

$$\rho(u_R(x), h(x))^2 \le C(v_R(x) - \frac{1}{2}|h|^2(x))$$

for all $x \in B_R(o)$. Since $|h|^2$ is subharmonic and $v_R = \frac{1}{2}|h|^2$ on $\partial B_R(o)$, by the maximum principle, the sequence $\{v_R - \frac{1}{2}|h|^2\}$ is increasing. Therefore,

$$v_R(x) - \frac{1}{2}|h|^2(x) \le v(x) - \frac{1}{2}|h|^2(x)$$

for all $x \in B_R(o)$. By a diagonal sequence argument and the Azela-Ascoli theorem,

$$\rho(u(x), h(x))^2 \le C(v(x) - \frac{1}{2}|h|^2(x))$$

for all $x \in M$. (14), (15) and (16) imply that the right hand side of this inequality goes to 0 as x goes to ∞. This proves the existence.

To prove the uniqueness, suppose that there is another harmonic map $\tilde{u}$ such that

$$\rho(\tilde{u}(x), f(x)) \to 0 \text{ as } x \to \infty.$$

Then for any $\epsilon > 0$, there exists $R_0 > 0$ such that $\rho(\tilde{u}(x), u(x)) < \epsilon$ for any $x \in \partial B_R(o)$ whenever $R \geq R_0$. Since both $\tilde{u}$ and u are bounded, by Theorem A in [17], we get $\rho(\tilde{u}(x), u(x)) < \epsilon$ for any $x \in B_R(o)$ whenever $R \geq R_0$. Therefore, letting $R_0 \to \infty$, we have the consequence. □

In the case when the sectional curvature K_M is bounded below by a negative constant, we have that $\sharp P_x$ is uniformly bounded and then

$$|\Delta g(x)| \leq \frac{C}{(1 + r(x))^{2+\alpha}}$$

for any $r(x) > R_1$. Similarly arguing as above, $g + w$ is superharmonic and $g - w$ is subharmonic on $M \setminus \overline{B}_R(o)$ for any $0 < \delta < \alpha - 3(m-1)\beta$. Therefore, we have the following corollary:

COROLLARY 3.2. *Let M be a Cartan-Hadamard manifold of dimension m ($m \geq 2$). For each point $x \in M$ outside a compact set, suppose that the sectional curvature $K_M(x)$ of M at x satisfies the following:*

$$-b^2 \leq K_M(x) \leq -\frac{A}{r^2(x)},$$

where A and b are positive constants with $A > 2$. Let $\mathcal{B}_{r_o}(p)$ be a geodesic ball of a point p, which lies within normal range of p, in a complete Riemannian manifold N. Then the asymptotic Dirichlet problem for harmonic maps is solvable for any $f \in C(M(\infty), \overline{\mathcal{B}}_{r_o}(p))$.

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