

WEAKLY DUO RINGS WITH NIL JACOBSON RADICAL

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ABSTRACT. Yu showed that every right (left) primitive factor ring of weakly right (left) duo rings is a division ring. It is not difficult to show that each weakly right (left) duo ring is abelian and has the classical right (left) quotient ring. In this note we first provide a left duo ring (but not weakly right duo) in spite of it being left Noetherian and local. Thus we observe conditions under which weakly one-sided duo rings may be two-sided. We prove that a weakly one-sided duo ring R is weakly duo under each of the following conditions: (1) R is semilocal with nil Jacobson radical; (2) R is locally finite. Based on the preceding case (1) we study a kind of composition length of a right or left Artinian weakly duo ring R , obtaining that $i(R)$ is finite and $a^{i(R)}R = Ra^{i(R)} = Ra^{i(R)}R$ for all $a \in R$, where $i(R)$ is the index (of nilpotency) of R . Note that one-sided Artinian rings and locally finite rings are strongly π -regular. Thus we also observe connections between strongly π -regular weakly right duo rings and related rings, constructing available examples.

Throughout this note each ring is associative with identity. A ring R is *right (left) duo* if every right (left) ideal of R is two-sided. Chatters and Xue [5] constructed a ring that is right duo but not left duo, in spite of it being Artinian. Courter proved that a right Artinian right duo ring R is left duo if and only if $c({}_R R) = c(R_R)$, where $c(M)$ denotes

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the composition length of a module M ; and obtained as a corollary that a right duo ring is left duo when it is finite dimensional over a field [6, Theorem 2.2 and Corollary 2.3]. A ring R is *weakly right (left) duo* if for each a in R there exists a positive integer $n = n(a)$, depending on a , such that $a^n R (Ra^n)$ is two-sided. A ring is a (weakly) duo ring if it is (weakly) left and right duo. A commutative ring and a direct product of division rings are clearly duo. Right (left) duo rings are obviously weakly right (left) duo but the converse does not hold in general by [12, Example 1]. As we shall see below weak duo-ness is not left-right symmetric; thus in this note we first study conditions under which weakly one-sided duo rings may be two-sided. Next we also observe the connections between weakly right duo rings and related rings, constructing available examples.

EXAMPLE 1. There is a left duo ring that is not weakly right duo. Let S be the quotient field of the polynomial ring $F[t]$ with an indeterminate t over a field F , and define a field monomorphism $\sigma : S \rightarrow S$ by $\sigma\left(\frac{f(t)}{g(t)}\right) = \frac{f(t^2)}{g(t^2)}$. Next consider the skew power series ring $S[[x; \sigma]]$ by σ with an indeterminate x over S , say R , in which every element is of the form $\sum_{n=1}^{\infty} a_n x^n$ with $xa = \sigma(a)x$ for each $a \in S$. Then each coefficient of the elements in $x^n R$ is of the form $\frac{f(t^{2n})}{g(t^{2n})}$ for some positive integer n ; hence $x^m R$ cannot contain Rx^m because $tx^m \notin x^m R$ for any positive integer m . Thus R is not weakly right duo. Next we show that R is left duo. Let $f(x) = a_0 x^k + a_1 x^{k+1} + \cdots + a_h x^{k+h}$ be any polynomial in R such that $a_0 \neq 0$ and h, k are nonnegative integers. Since $a_0 + a_1 x + \cdots + a_h x^h$ is invertible in R , we have $f(x)g(x) = r(x)x^k g(x) = r(x)g_1(x)x^k = r(x)g_1(x)r(x)^{-1}r(x)x^k = r(x)g_1(x)r(x)^{-1}f(x) \in Rf(x)$ for every $g(x) \in R$, where $r(x) = a_0 + a_1 x + \cdots + a_h x^h$ and $x^k g(x) = g_1(x)x^k$ for some $g_1(x) \in R$. Therefore R is left duo (so weakly left duo). \square

The preceding example also demonstrates that the duo condition is not left-right symmetric.

A ring R is called *right Ore* if given $a, b \in R$ with b regular (i.e., not a zero divisor) there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$. It is a well-known fact that R is a right Ore ring if and only if there exists the classical right quotient ring of R . Given a ring R the Jacobson radical of R is denoted by $J(R)$. A ring R is *right (left) quasi-duo* if every maximal right (left) ideal of R is two-sided. A ring is quasi-duo ring if it is both left and right quasi-duo. A ring is *abelian* if every idempotent is central.

LEMMA 2. (1) A ring R is right quasi-duo if and only if $R/J(R)$ is right quasi-duo if and only if every right primitive factor ring is a division ring.

(2) If R is a right or left quasi-duo ring then $J(R)$ contains all nilpotent elements in R .

(3) Weakly right (left) duo rings are abelian and right (left) quasi-duo.

(4) Weakly right (left) duo rings are right (left) Ore.

Proof. (1) By [12, Proposition 1].

(2) By [22] or the result (1).

(3) By [21, Lemma 4] and [22, Proposition 2.2].

(4) By [11, Proposition 12(i)]. □

REMARK. Recall that the ring R in Example 1 is not weakly right duo. But since R is a local ring such that $J(R) = Rx$ and $\frac{R}{J(R)} \cong S$, it is quasi-duo by [12, Proposition 1]; and R is an integral domain (so abelian) by [17, Theorem 1.4.5]. Thus the converse of Lemma 2(3) is not true in general.

In the following arguments we find conditions under which abelian right quasi-duo rings may be weakly duo. A ring R is *semilocal* if $R/J(R)$ is semisimple Artinian. A ring R is *von Neumann regular* if for each $a \in R$ there exist $b \in R$ such that $a = aba$. A ring is *reduced* if it contains no nonzero nilpotent elements.

THEOREM 3. *Let R be a semilocal ring with nil $J(R)$. Then the following conditions are equivalent:*

- (1) R is weakly right duo;
- (2) R is abelian and right quasi-duo;
- (3) R is abelian and left quasi-duo;
- (4) R is weakly left duo.

Proof. (1) \Rightarrow (2) and (4) \Rightarrow (3) are obtained from Lemma 2. To prove (2) \Rightarrow (1) suppose that R is an abelian and right quasi-duo ring. Since R is semilocal, $R/J(R)$ is von Neumann regular by [8, Theorem 1.7]; hence for each $x \in R$ there exists $y \in R$ such that $x - xyx \in J(R)$. Write $x + J(R) = \bar{x}$. Since $J(R)$ is nil, [15, Proposition 3.6.1] implies that there exists $e^2 = e \in R$ such that $\bar{e} = \bar{x}\bar{y}$ and $\bar{x} = \bar{e}\bar{x}$. But $x - ex$ is nilpotent so there exists a positive integer n such that $(x - ex)^n = 0$. Thus $x^n \in eR$ because e is central, i.e., $x^n R \subseteq eR$. Next since $\bar{e} = \bar{x}\bar{y}$ is central in $R/J(R)$, we have $\bar{e} = \bar{x}\bar{y} = \bar{x}\bar{y}\bar{x}\bar{y} = \bar{x}^2\bar{y}^2 = \dots = \bar{x}^n\bar{y}^n$. Thus $e - x^n y^n \in J(R)$, and so $(e - x^n y^n)^m = 0$ for some positive

integer m because $J(R)$ is nil. Consequently $e \in x^n R$, i.e., $eR \subseteq x^n R$; hence we obtain $x^n R = eR$. But eR is a two-sided ideal in R and therefore R is weakly right duo. (3) \Rightarrow (4) is proved similarly. Next if R is one-sided quasi-duo, $R/J(R)$ is reduced by [22, Lemma 2.3]. So if R is semilocal and one-sided quasi-duo, then $R/J(R)$ is a finite direct product of division rings and so $R/J(R)$ is quasi-duo. Consequently we have (2) \Leftrightarrow (3). \square

REMARK. (1) In the hypothesis of Theorem 3, the condition “semilocal” is not superfluous. The ring R in Example 4 below is an abelian right quasi-duo ring with $J(R)$ nil; however R is not weakly right duo. But $R/J(R)$ is isomorphic to a polynomial ring over a field whence R is not semilocal.

(2) In the hypothesis of Theorem 3, the condition “nil $J(R)$ ” is not superfluous. For, the ring R in Example 1 is quasi-duo by [12, Proposition 4] and abelian; but R is not weakly right duo such that $J(R) = Rx$ is non-nil.

(3) In the conditions (2) and (3) of Theorem 3, the condition “abelian” is not superfluous as can be seen by the 2 by 2 upper triangular matrix ring over a division ring.

Recall that a ring R is right quasi-duo if and only if $R/J(R)$ is right quasi-duo [12, Proposition 1]. So it is also natural to ask whether a ring R is weakly right duo when R is an abelian right quasi-duo ring such that $J(R)$ is nil and $R/J(R)$ is weakly right duo, based on Theorem 3. However the answer is negative by the following.

EXAMPLE 4. Let F be a field and $S = F[t]$ be the polynomial ring with an indeterminate t over F . Define a ring homomorphism $\sigma : S \rightarrow S$ by $\sigma(f(t)) = f(t^2)$. Consider the skew power series ring $T = S[[y; \sigma]]$ over S by σ , every element of which is of the form $\sum_{n=0}^{\infty} y^n a_n$, only subject to $ay = y\sigma(a)$ for each $a \in S$. Next let R be the subring $\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in S \text{ and } b \in T \right\}$ of the 2 by 2 upper triangular matrix ring over T . Since S is a domain, $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are the only idempotents in R ; hence R is abelian. Every maximal right ideal of R is of the form $\left\{ \begin{pmatrix} m & b \\ 0 & m \end{pmatrix} \in R \mid m \in M \text{ and } b \in T \right\}$, where M is a maximal ideal of S ; but it is two-sided so R is right quasi-duo. Note that S is semiprimitive, so $J(R) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in R \mid b \in T \right\}$. Thus $J(R)$

is nil, and $R/J(R)$ is isomorphic to S so it is clearly weakly right duo. However the right ideal $f^n R$ of R cannot contain the left ideal Rf^n of R , using a computation similar to Example 1, where $f = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ and n is any positive integer. Actually $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} f^n = \begin{pmatrix} 0 & yt^n \\ 0 & 0 \end{pmatrix} \notin f^n R$ since $\begin{pmatrix} t^n & 0 \\ 0 & t^n \end{pmatrix} \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & yt^{2n} \\ 0 & 0 \end{pmatrix}$. Thus R is not weakly right duo. \square

A ring R is *strongly π -regular* if for every a in R there exist a positive integer n , depending on a , and an element b in R satisfying $a^n = a^{n+1}b$. It is obvious that a ring R is strongly π -regular if and only if R satisfies the descending chain condition on principal right ideals of the form $aR \supseteq a^2R \supseteq \dots$, for every a in R . The strong π -regularity is left-right symmetric by Dischinger [7]; hence one-sided perfect (so one-sided Artinian) rings are strongly π -regular.

Let R be a ring. According to Azumaya [1], an element $a \in R$ is called *strongly π -regular* if there exist positive integers m and n , depending on a , and $b, c \in R$ such that $a^n = a^{n+1}b$ and $a^m = ca^{m+1}$. For any strongly π -regular $a \in R$ there is the least positive integer n by Azumaya [1, Lemma 3] such that $a^n R = a^{n+1} R = \dots$ and $Ra^n = Ra^{n+1} = \dots$; in this situation, n is called the π -index of a due to Tominaga [20], write $\pi i(a)$ for n . The supremum of all π -indices in R is called π -index of R also due to Tominaga [20], write $\pi i(R)$. It is showed by Dischinger [7] that a ring is strongly π -regular if and only if each element is strongly π -regular. Thus we can compute the $\pi i(a)$ for any $a \in R$ when a ring R is strongly π -regular.

The *index* (of nilpotency) of a nilpotent element x in a ring R is the least positive integer n such that $x^n = 0$, write $i(a)$ for n . The *index* (of nilpotency) of a subset I of R is the supremum of the indices (of nilpotency) of all nilpotent elements in I , write $i(I)$. If such a supremum is finite, then I is said to be *of bounded index* (of nilpotency).

NOTE. Note that each nilpotent element is strongly π -regular and the index coincides with the π -index; hence the index of a ring R coincides with the π -index of R as the aleph-zero when R is not of bounded index. Moreover the index of a ring R coincides with the π -index of R by [1, Theorem 4] when R is of bounded index.

Next consider the set $\{a^{\pi i(a)} R, Ra^{\pi i(a)} \mid a \in R\}$, say $S(R)$. In this situation we can compute the composition length of $a^{\pi i(a)} R (Ra^{\pi i(a)})$,

considering only the composition series generated by the principal right (left) ideals in $S(R)$, denote it by $wc(a^{\pi i(a)}R_R)$ ($wc({}_R Ra^{\pi i(a)})$).

THEOREM 5. *Let R be a right or left Artinian weakly right duo ring. Then we have the following assertions.*

- (1) $\pi i(R) = i(R)$ is finite (i.e., R is of bounded index).
- (2) $a^{\pi i(a)}R = Ra^{\pi i(a)} = Ra^{\pi i(a)}R$ (i.e., any one-sided ideal in $S(R)$ is two-sided) and $a^{\pi i(a)}R = a^{i(R)}R = Ra^{i(R)} = Ra^{i(R)}R$ for all $a \in R$.
- (3) R satisfies each of the following equivalent conditions:
 - (a) $wc(R_R) = wc({}_R R)$;
 - (b) If $I_1 \subseteq I_2$ are right ideals in $S(R)$ with $(I_2/I_1)_R$ simple, then ${}_R(I_2/I_1)$ is also simple;
 - (c) $wc(I_R) = wc({}_R I)$ for any right ideal I in $S(R)$.

Proof. (1) $J(R)$ contains all nilpotent elements by Lemma 2(2, 3) and $J(R)$ is nilpotent since R is right or left Artinian; hence R is of bounded index. Thus we have finite $\pi i(R) = i(R)$ by the Note above.

(2) Since R is right or left Artinian and weakly right duo, R is weakly duo by Theorem 3. Let $a \in R$. If $a^k R$ is two-sided for some $k \geq \pi i(a)$ then $Ra^k R = a^k R = a^{\pi i(a)}R$ by the choice of $\pi i(a)$ and thus $a^{\pi i(a)}R = Ra^{\pi i(a)}R$. If $a^k R$ is two-sided for some $k < \pi i(a)$ then for any $r \in R$, $ra^{mk} = a^k r_1 a^{k(m-1)} = a^{2k} r_2 a^{k(m-2)} = \dots = a^{k(m-1)} r_{m-1} a^k = a^{mk} r_m \in a^{mk} R$, where $r_0 = r$, $r_i a^k = a^k r_{i+1}$ for $i = 0, 1, 2, \dots, m - 1$ and $r_i \in R$; hence $a^{mk} R$ is two-sided for any positive integer m . Consequently we have $Ra^{tk} R = a^{tk} R = a^{\pi i(a)}R$, by the choice of $\pi i(a)$, for some positive integer t with $tk \geq \pi i(a)$. Thus $a^{\pi i(a)}R = Ra^{\pi i(a)}R$. The proof of the left case is symmetric and thus we obtain $a^{\pi i(a)}R = Ra^{\pi i(a)} = Ra^{\pi i(a)}R$. We get $a^{\pi i(a)}R = a^{i(R)}R = Ra^{i(R)} = Ra^{i(R)}R$ immediately since $\pi i(a) \leq i(R)$.

(3) By the result (2), each one-sided ideal in $S(R)$ is two-sided and then we obtain the condition (a) $wc(R_R) = wc({}_R R)$. Next we apply the method of the proof of [6, Theorem 2.2] to prove the equivalences. (a) \Rightarrow (b): $I_1 \subseteq I_2$ can be extended to a composition series for R_R in $S(R)$, and this series is also a normal series for R_R in $S(R)$ with nonzero factors because $a^{\pi i(a)}R = Ra^{\pi i(a)} = Ra^{\pi i(a)}R$. By the condition (a), I_2/I_1 must be a simple left R -module when it is a simple right R -module. (b) \Rightarrow (c): Let C be a composition series for I_R in $S(R)$. Then C is a normal series for ${}_R I$ in $S(R)$ such that each factor is simple by the condition (b), since $a^{\pi i(a)}R = Ra^{\pi i(a)} = Ra^{\pi i(a)}R$. (c) \Rightarrow (a) is obvious. \square

REMARK. (1) The converse of the second statement of Theorem 5 is not true in general by the following. Let R be the n by n matrix

ring over a division ring with $n \geq 2$, then R is semisimple Artinian with $wc(R_R) = wc({}_R R) = n$ but R is not weakly right duo by Lemma 2.

(2) By Theorem 5, one may conjecture that weakly one-sided duo rings are weakly duo when they are right or left Noetherian. But the answer is negative by Example 1. Indeed the ring R in Example 1 is left Noetherian by [17, Theorem 1.4.5].

Following Bass [3], we call a ring R *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. The class of semiperfect rings contains semilocal rings with nil Jacobson radical (so one-sided Artinian rings) by [15, Proposition 3.6.1]. Due to Thrall [19], a ring R of finite rank is *QF-3* if it has a unique (up to isomorphism) minimal faithful left R -module. It is well-known that the QF-3 condition is left-right symmetric. Given a ring R the right (left) annihilator in R is denoted by $r_R(-)$ ($l_R(-)$).

PROPOSITION 6. (1) *A ring R is a semiperfect abelian ring if and only if R is a finite direct sum of local rings.*

(2) *A ring R is a semiperfect weakly right duo ring if and only if R is a finite direct sum of local weakly right duo rings.*

(3) *Let R be a right Artinian weakly right duo ring. If R is QF-3 and $a \in R$, then $a^{\pi i(a)} R (= R a^{\pi i(a)})$ is both a right and a left annihilator in R .*

Proof. (1) (\Rightarrow): Since R is semiperfect, R has a finite orthogonal set of local idempotents whose sum is 1 by [15, Proposition 3.7.2], say $R = \sum_{i=1}^n e_i R$ such that each $e_i R e_i$ is a local ring. Because R is abelian, they are central; hence $e_i R$'s are ideals of R and consequently each $e_i R = e_i R e_i$ is a local ring. (\Leftarrow): Note that 0 and 1 are all idempotents in a local ring (so local rings are abelian), and that local rings are semiperfect. Thus R is abelian and semiperfect when it is a finite direct sum of local rings.

(2) It is easily checked that every homomorphic image of a weakly right duo ring is weakly right duo, and direct sums of weakly right duo rings are weakly right duo. By Lemma 2(3) weakly right duo rings are abelian, so we obtain the proof by the result (1).

(3) Suppose that R is a right Artinian weakly right duo ring. By Lemma 2(3) and the result (1), we put $R = \sum_{i=1}^n e_i R$ such that $\{e_1, \dots, e_n\}$ is a finite orthogonal set of local idempotents whose sum is 1. Every $e_i R$ is also right Artinian obviously. It is well-known that R is QF-3 if and only if the injective hull of R_R is projective, so every $e_i R$ is

also QF-3 since $e_i R R = e_i R (\sum_{j=1}^n e_j R) = e_i R e_i R$. It is also well-known that a ring is QF-3 if and only if it contains a faithful injective left ideal; but $e_i R$ is local and so $e_i R$ must be injective since e_i is the only nonzero idempotent, i.e., $e_i R$ is right self-injective. Consequently $e_i R$ is a quasi-Frobenius ring and so is R . By Theorem 5(2) we have $a^{\pi i(a)} R = R a^{\pi i(a)} = R a^{\pi i(a)} R$ for all $a \in R$. Thus $r_R l_R (a^{\pi i(a)} R) = a^{\pi i(a)} R = R a^{\pi i(a)} = l_{R R} (R a^{\pi i(a)})$ since R is quasi-Frobenius. \square

EXAMPLE 7. (1) The result (2) in Proposition 6 does not apply to quasi-duo rings by the 2 by 2 upper triangular matrix ring over a division ring.

(2) There is an Artinian weakly duo ring R such that R is not QF-3 and $Rx^{\pi i(x)}$ is neither a left nor a right annihilator in R for some $x \in R$.

Let R be the subring $\left\{ \begin{pmatrix} a & a_{12} & a_{13} \\ 0 & a & a_{23} \\ 0 & 0 & a \end{pmatrix} \mid a, a_{ij} \in D \right\}$ of the 3 by 3

upper triangular matrix ring over a division ring D . Then R is weakly right duo since a matrix of the form $\begin{pmatrix} a & a_{12} & a_{13} \\ 0 & a & a_{23} \\ 0 & 0 & a \end{pmatrix}$ with $a \neq 0$ is

invertible and a matrix of the form $\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$ is nilpotent. Note

that R is Artinian (hence R is weakly duo by Theorem 3). Assume that R is QF-3, then R is quasi-Frobenius by the proof of Proposition

6(3). Let $x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R$. Since $Rx = l_{R R} (Rx) = l_{R R} (x)$

($xR = r_{R R} (xR) = r_{R R} (x)$), we have

$$\begin{aligned} \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} &= l_{R R} \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \\ &= l_{R} \left(\begin{pmatrix} 0 & D & D \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \left(\begin{pmatrix} 0 & 0 & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) &= r_R l_R \left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \\ &= r_R \left(\begin{pmatrix} 0 & D & D \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & D & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

a contradiction; hence R is not QF-3. In fact $Rx = RxR = xR$. Thus $\pi i(x) = 1$ and $Rx^{\pi i(x)}$ is neither a left nor a right annihilator in R . \square

Next we study some special conditions under which the above concepts coincide, concerning the primitivity of polynomial rings over division rings. Due to Jacobson [14], a ring is called *strongly right (left) bounded* if every nonzero right (left) ideal contains a nonzero ideal and a ring is called *right (left) bounded* if every essential right (left) ideal contains a nonzero ideal. It is obvious that right duo rings are strongly right bounded and strongly right bounded rings are right bounded. A ring R is *right Ore* if given $a, b \in R$ with b regular there exist $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$, and the left case can be defined analogously. A ring is Ore if it is both right and left Ore. In the following we obtain equivalence relations between weakly duo-ness of a kind of polynomial ring and related conditions.

PROPOSITION 8. *Let D be a division ring such that the center F of D is algebraically closed. Then the following conditions are equivalent:*

- (1) $D[x]$ is weakly right duo;
- (2) $D[x]$ is right duo;
- (3) $D[x]$ is right quasi-duo;
- (4) D is algebraic over F ;
- (5) $D = F$;
- (6) $D[x]$ is strongly right bounded;
- (7) $D[x]$ is right bounded;
- (8) $D[x]$ is not right primitive;
- (9) Every right primitive ideal of $D[x]$ is maximal;
- (10) The left versions of the conditions (1)–(3) and (6)–(9).

Proof. (5) \Rightarrow (3), (6) \Rightarrow (7), and (5) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3) are straightforward. (7) \Rightarrow (8) \Rightarrow (4) and (8) \Rightarrow (7) are obtained by [9, Theorem 15.2].

(4) \Rightarrow (5): Since F is algebraically closed by hypothesis, we have $D = F$.

(3) \Rightarrow (8): By [12, Proposition 1], $D[x]$ cannot be right primitive.

(8) \Leftrightarrow (9) is obtained by [16, Propositions 1 and 19].

(7) \Rightarrow (6): $D[x]$ is an Ore domain by [9, Corollary 5.16], so every nonzero right (left) ideal is essential; hence $D[x]$ is strongly right (left) bounded if and only if it is right (left) bounded. The equivalences among (4), (5) and (10) are proved symmetrically. \square

For the rings D and F in Proposition 8, we have $D[x] \cong D \otimes_F F[x]$ by [18, Corollary 1.7.20]; hence we can provide an available example for tensor products of weakly duo rings as a byproduct of Proposition 8.

EXAMPLE 9. The tensor product of weakly right duo rings need not be weakly right duo. Let F be an algebraically closed field of characteristic zero, W be the first Weyl algebra over the field F , and R be the right quotient division ring of W . Clearly the center of R is F . $R[x]$ is right primitive by Proposition 8 since $F \subsetneq R$; hence $R[x]$ is not weakly right duo. It is obvious that $R, F[x]$ are duo, and note that $R[x] \cong R \otimes_F F[x]$ by [18, Corollary 1.7.20].

A ring R is π -regular if for each $a \in R$ there exists a positive integer n , depending on a , and $b \in R$ such that $a^n = a^n b a^n$. Strongly π -regular rings are π -regular by Azumaya [1] and clearly von Neumann regular rings are π -regular. A ring R is *right (left) weakly π -regular* if for each a in R there exists a positive integer $n = n(a)$, depending on a , such that $a^n \in a^n R a^n R$ ($a^n \in R a^n R a^n$). π -regular rings are clearly both right and left weakly π -regular. A ring is *strongly regular* if it is abelian and von Neumann regular.

PROPOSITION 10. *Let R be a semiprimitive right quasi-duo ring. Then the following conditions are equivalent:*

- (1) R is a right (or left) weakly π -regular ring.
- (2) R is a strongly regular ring.
- (3) R is a von Neumann regular ring.
- (4) R is a strongly π -regular ring.
- (5) R is a π -regular ring.

Proof. It suffices to show (1) \Rightarrow (2). Since R is semiprimitive right quasi-duo, R is reduced by [22, Lemma 2.3] and so [4, Theorem 8] implies that every prime ideal of R is maximal. Thus every prime factor ring of R is simple (so primitive); hence every prime factor ring of R is a division ring by [12, Proposition 1]. Consequently every prime ideal of R is completely prime (i.e., every prime factor ring of R is a domain) and so [8, Theorem 1.21] implies that R is von Neumann regular. Therefore R is strongly regular since it is reduced. \square

REMARK. By [10, Theorem 7], a ring R is right (or left) weakly π -regular if and only if R is strongly π -regular if and only if R is π -regular when R is right quasi-duo. But the semiprimitivity of the hypothesis is essential for the conditions (2) and (3) in Proposition 9 by the following. Let R be the 2 by 2 upper triangular matrix ring over a division ring D , say $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$. Then R is quasi-duo and strongly π -regular; however it is not von Neumann regular since $J(R) = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \neq 0$.

COROLLARY 11. *Let R be a right (or left) weakly π -regular ring. Then R is abelian and right quasi-duo if and only if R is weakly right duo.*

Proof. Suppose that R is a right weakly π -regular ring. We first show that $J(R)$ is nil. Let $r \in J(R)$, then $r^n R = r^n R r^n R$ for some positive integer n and so $r^n(1 - s) = 0$ for some $s \in R r^n R$. But since $s \in J(R)$, $1 - s$ is invertible and thus we have $r^n = 0$, showing that $J(R)$ is nil. The proof for the left weakly π -regular case is symmetric. Note that $R/J(R)$ is also right weakly π -regular. By Lemma 2 it suffices to prove the necessity. Let R be abelian and right quasi-duo. Then $R/J(R)$ is also right quasi-duo, and so $R/J(R)$ is strongly regular by Proposition 10. Next the proof of (2) \Rightarrow (1) in Theorem 3 applies to show that R is weakly right duo. \square

Observing Theorem 5 and Corollary 11, one may suspect that weakly right duo rings may be right duo when they are right Artinian. However the following erases the possibility.

EXAMPLE 12. Let R be the ring in Example 7. Note that R is an Artinian weakly duo ring by Example 7. However R is not right duo:

for, letting $a = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, we have

$$Ra = \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix} \not\subseteq aR = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}. \quad \square$$

We use $N(R)$ to denote the set of all nilpotent elements in a ring R .

COROLLARY 13. *Let R be an abelian right quasi-duo ring. Then the following conditions are equivalent:*

- (1) R is a strongly π -regular ring.
- (2) R is a π -regular ring.
- (3) R is a right (or left) weakly π -regular ring.
- (4) $R/J(R)$ is a von Neumann regular ring with $\text{nil } J(R)$.
- (5) $R/J(R)$ is a strongly regular ring with $\text{nil } J(R)$.

Proof. By [10, Theorem 7], a ring R is right (or left) weakly π -regular if and only if R is strongly π -regular if and only if R is π -regular when R is right quasi-duo. Thus we will show (3) \Rightarrow (4), (4) \Rightarrow (5), and (5) \Rightarrow (2). (3) \Rightarrow (4): Note that $R/J(R)$ is right quasi-duo by hypothesis and is right (or left) weakly π -regular by the condition. Thus $R/J(R)$ is von Neumann regular by Proposition 10. The Jacobson radicals of right (or left) weakly π -regular rings are nil by the proof of Corollary 11. (4) \Rightarrow (5): Since R is right quasi-duo by hypothesis, $R/J(R)$ is reduced by [22, Lemma 2.3]. (5) \Rightarrow (1): $R/J(R)$ is a reduced ring with $\text{nil } J(R)$ by the condition, so we have $N(R) = J(R)$ (i.e., $N(R)$ is an ideal of R); hence R is π -regular by [2, Theorem 3]. \square

The condition “ $\text{nil } J(R)$ ”, in (4) and (5) of Corollary 13, is essential by a power series ring over a von Neumann regular ring. Weakly right duo rings are abelian and right quasi-duo by Lemma 2, so we have the following as a corollary.

COROLLARY 14. [10, Proposition 14] *For a weakly right duo ring R , the following conditions are equivalent:*

- (1) R is a strongly π -regular ring.
- (2) R is a π -regular ring.
- (3) R is a right (left) weakly π -regular ring.
- (4) $R/J(R)$ is a strongly regular ring with $\text{nil } J(R)$.

A ring R is called *locally finite* if every finite subset of R generates a finite multiplicative semigroup of R . Finite rings are clearly locally finite. An algebraic closure of a finite field is locally finite but not finite. Notice that every locally finite ring R is strongly π -regular by the proof of [13, Proposition 16] (in fact, there is a positive integer m such that $a^m = a^{2m}$, so we have $a^m R = a^{m+1} R = \dots = a^{2m} R = \dots$ for all $a \in R$).

PROPOSITION 15. *Let R be a locally finite ring. Then the following conditions are equivalent:*

- (1) R is abelian;
- (2) R is weakly right duo;
- (3) R is weakly left duo;

(4) R is weakly duo.

Proof. Let $a \in R$. Since R is locally finite, there exists positive integer m such that a^m is an idempotent by the proof of [13, Proposition 16]. If a^m is central, then Ra^m and a^mR are two-sided ideals of R . Thus we proved (1) \Rightarrow (4). (4) \Rightarrow (3) and (4) \Rightarrow (2) are obvious. (3) \Rightarrow (1) and (2) \Rightarrow (1) are obtained by Lemma 2(3). \square

COROLLARY 16. *Let R be a finite ring. Then the following conditions are equivalent:*

- (1) R is abelian;
- (2) R is weakly right duo;
- (3) R is weakly left duo;
- (4) R is weakly duo;
- (5) R is a finite direct sum of local weakly duo rings with nilpotent Jacobson radical.

Proof. Finite rings are Artinian (so semiperfect), hence we get the corollary by Propositions 6 and 15. \square

Quasi-duo rings need not be a condition of Corollary 16, considering the 2 by 2 upper triangular matrix ring over a finite field.

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