J. Korean Math. Soc. **42** (2005), No. 3, pp. 405–434

L^p ESTIMATES FOR A ROUGH MAXIMAL OPERATOR ON PRODUCT SPACES

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ABSTRACT. We establish appropriate L^p estimates for a class of maximal operators $\mathcal{S}_\Omega^{(\gamma)}$ on the product space $\mathbf{R}^n \times \mathbf{R}^m$ when Ω lacks regularity and $1 \le \gamma \le 2$. Also, when $\gamma = 2$, we prove the L^p $(2 \le p < \infty)$ boundedness of $\mathcal{S}_\Omega^{(2)}$ whenever Ω is a function in a certain block space $B_q^{(0,0)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ (for some $q > 1$). Moreover, we show that the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is nearly optimal in the sense that the operator $\mathcal{S}_\Omega^{(2)}$ may fail to be bounded on L^2 if the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is replaced by the weaker conditions $\Omega \in B_q^{(0,\varepsilon)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for any $-1 < \varepsilon < 0$.

1. Introduction

Throughout this article, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n \backslash \{0\}$ and p' denote the exponent conjugate to p, that is $1/p + 1/p' = 1$. Let $n, m \ge 2$. Suppose that $\mathbf{S}^{d-1}$ $(d = n$ or $m)$ is the unit sphere of $\mathbf{R}^d$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$.

In 1990, L. K. Chen and H. Lin [4], investigated the L^p boundedness of a class of maximal operators $\mathcal{M}_\Omega^{(\gamma)}$ and obtained the following result:

THEOREM A [4]. *Assume $\Omega \in C(\mathbf{S}^{n-1})$ with $\int_{\mathbf{S}^{n-1}} \Omega(x) d\sigma(x) = 0$. Denote*

$$(1.1) \qquad \mathcal{M}_\Omega^{(\gamma)} f(x) = \sup_h \left| \int_{\mathbf{R}^n} f(x-y) h(|y|) \Omega(y') |y|^{-n} dy \right|,$$

Received September 16, 2004.

2000 Mathematics Subject Classification: Primary 42B20; Secondary 42B25, 42B30.

Key words and phrases: rough kernel, block space, singular integral, maximal operator, product domains.

where the supremum is taken over all measurable radial functions $h(t)$ with $\|h\|_{L^\gamma(\mathbf{R}_+,dr/r)} \leq 1$ and $1 \leq \gamma \leq 2$. Then

$$\left\| \mathcal{M}_\Omega^{(\gamma)}(f) \right\|_{L^p(\mathbf{R}^n)} \leq C_p \|f\|_{L^p(\mathbf{R}^n)}$$

for $(\gamma n)' < p < \infty$ and $f \in L^p$. Moreover, the range of p is the best possible.

In this paper, we are interested in studying the L^p boundedness of the corresponding maximal operator $\mathcal{S}_\Omega^{(\gamma)}$ on product spaces $\mathbf{R}^n \times \mathbf{R}^m$, which is defined by

$$(1.2) \qquad \mathcal{S}_\Omega^{(\gamma)} f(x,y) = \sup_{h \in \mathcal{H}_\gamma} \left| \int_{\mathbf{R}^n \times \mathbf{R}^m} f(x-u, y-v) K(u,v)\,du dv \right|,$$

where

$$K(u,v) = h(|u|, |v|)\Omega\left(u', v'\right) |u|^{-n} |v|^{-n},$$

$\gamma \geq 1$, $\mathcal{H}_\gamma = \mathcal{H}_\gamma(\mathbf{R}_+ \times \mathbf{R}_+)$ denote the set of all measurable radial functions $h(s,t)$ defined on $\mathbf{R}_+ \times \mathbf{R}_+$ and satisfying $\|h\|_{L^\gamma(\mathbf{R}_+ \times \mathbf{R}_+,\ dsdt/(st))} \leq 1$ and Ω is a function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ with $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and satisfying the vanishing conditions

$$(1.3) \qquad \int_{\mathbf{S}^{n-1}} \Omega\left(u', \cdot\right) d\sigma\left(u'\right) = \int_{\mathbf{S}^{m-1}} \Omega\left(\cdot, v'\right) d\sigma\left(v'\right) = 0.$$

We focus our attention in this paper on investigating the L^p boundedness of $\mathcal{S}_\Omega^{(\gamma)}$ whenever Ω lacks regularity. In fact, our main goal in this paper is to seek a solution to the following problem.

PROBLEM. *Determine whether the L^p boundedness of the operator $\mathcal{S}_\Omega^{(\gamma)}$ holds under a condition in the form of $\Omega \in B_q^{(0,v)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, $v > -1$, and if so, what is the best possible value of v.*

Here, $B_q^{(0,v)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for $v > -1$ and $q > 1$ is a special class of block spaces which was introduced in [10] and can be traced back to [13] and [12]. We point out that on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, for any $q > 1$ and $v > -1$, the following inclusions hold and are proper:

$$(1.4) \qquad \bigcup_{r>1} L^r(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \subset B_q^{(0,v)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1});$$

$$(1.5) \qquad B_{q_2}^{(0,v)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \subset B_{q_1}^{(0,v)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \text{ if } 1 < q_1 < q_2;$$

$$(1.6) \quad B_q^{(0,v_2)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \subset B_q^{(0,v_1)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \text{ if } v_2 > v_1 > -1.$$

With regard to the relationship between the spaces $B_q^{(0,\alpha-1)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and $L(\log^+ L)^\alpha(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ (for $\alpha > 0$) remains open.

We shall obtain a solution to the above problem. The method we use in this paper is flexible enough to treat a more general class of maximal operators than those given by (1.2). In order to state our main theorem, we now introduce some notations and definitions.

For suitable functions Φ and Ψ defined on $\mathbf{R}_+$ and an Ω satisfying (1.3), we define the maximal operator $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}$ on the product space $\mathbf{R}^n \times \mathbf{R}^m$ by

$$\begin{aligned}(\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}f)(x,y) \\ (1.7) \quad = \sup_{h\in\mathcal{H}_\gamma}\left|\int_{\mathbf{R}^n\times\mathbf{R}^m} f(x-\Phi(|u|)u', y-\Psi(|v|)v')K(u,v)dudv\right|.\end{aligned}$$

Denote $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi} = \mathcal{S}^{(\gamma)}_{\Omega}$ if $\Phi(t) = \Psi(t) = t$. Also, denote $\mathcal{S}^{(\gamma)}_{\Omega} = \mathcal{S}_{\Omega}$ and $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi} = \mathcal{S}_{\Omega,\Phi,\Psi}$ if $\gamma = 2$.

We shall need the following definitions from [7]:

Definition. We say that a function Γ satisfies "hypothesis I" if

(*a*) Γ is a nonnegative C^1 function on $(0,\infty)$,
(*b*) Γ is strictly increasing, $\Gamma(2t) \geq \lambda\Gamma(t)$ for some fixed $\lambda > 1$ and $\Gamma(2t) \leq c\Gamma(t)$ for some constant $c \geq \lambda > 1$,
(*c*) $\Gamma'(t) \geq \alpha\Gamma(t)/t$ on $(0,\infty)$ for some fixed $\alpha \in (0, \log_2 c]$ and $\Gamma'(t)$ is monotone on $(0,\infty)$.

Definition. We say that Γ satisfies "hypothesis D" if

(a') Γ is a nonnegative C^1 function on $(0,\infty)$,
(b') Γ is strictly decreasing, $\Gamma(t) \geq \lambda\Gamma(2t)$ for some fixed $\lambda > 1$ and $\Gamma(t) \leq c\Gamma(2t)$ for some constant $c \geq \lambda > 1$,
(c') $|\Gamma'(t)| \geq \alpha\Gamma(t)/t$ on $(0,\infty)$ for some fixed $\alpha \in (0, \log_2 c]$ and $\Gamma'(t)$ is monotone on $(0,\infty)$.

Model functions for the Γ satisfy hypothesis I are $\Gamma(t) = t^d$ with $d > 0$, and their linear combinations with positive coefficients. Model functions for the Γ satisfy hypothesis D are $\Gamma(t) = t^r$ with $r < 0$, and their linear combinations with positive coefficients.

Our principal results in this paper are the following:

Theorem 1.1. *Let* $\mathcal{S}_{\Omega,\Phi,\Psi}$ *be given as in* (1.7). *Assume that* Φ *and* Ψ *satisfy either hypothesis I or hypothesis D. Then* *(a) If* $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ *and satisfies* (1.3), *then* $\mathcal{S}_{\Omega,\Phi,\Psi}$ *is bounded on* $L^p(\mathbf{R}^n \times \mathbf{R}^m)$ *for* $2 \le p < \infty$;

(b) There exists an Ω *which lies in* $B_q^{(0,v)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ *for all* $-1 < v < 0$ *and satisfies* (1.3) *such that* $\mathcal{S}_\Omega$ *is not bounded on* $L^2(\mathbf{R}^n \times \mathbf{R}^m)$.

THEOREM 1.2. *Let* $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}$ *be given as in* (1.7). *Assume that* Φ *and* Ψ *satisfy either hypothesis I or hypothesis D. Then if* $\Omega \in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ *for some* $q > 1$, $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}$ *is bounded on* $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ *for* $\max\{(\gamma' n\delta)/(\gamma' n + n\delta - \gamma'), (\gamma' m\delta)/(\gamma' m + m\delta - \gamma')\} < p < \infty$ *and* $1 \le \gamma \le 2$, *where* $\delta = \max\{2, q'\}$.

The paper is organized as follows. In Section 2, a few definitions and lemmas will be recalled or proved. Section 3 will contain the proof of the optimality of the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for the L^2 boundedness of $\mathcal{S}_\Omega$. In Sections 4 and 5, we shall present the proof of the main theorems.

Throughout the rest of the paper the letter C will stand for a constant but not necessarily the same one each time it appears and it is independent of the essential variables.

ACKNOWLEDGMENT. The author wish to thank very much the referee for his helpful comments.

2. Some definitions and lemmas

The block spaces originated in the work of M. H. Taibleson and G. Weiss on the convergence of the Fourier series (see [16]) in connection with the developments of the real Hardy spaces. Below we shall recall the definition of block spaces on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$. For further background information about the theory of spaces generated by blocks and its applications to harmonic analysis one can consult the book [12].

The special class of block spaces $B_q^{(0,v)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ (for $v > -1$ and $q > 1$) was introduced by Jiang and Lu with respect to the study of singular integral operators on product domains [10].

DEFINITION 2.1. A *q-block* on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ is an $L^q (1 < q \le \infty)$ function $b(x, y)$ that satisfies

$$\text{(i) } \operatorname{supp}(b) \subset I; \qquad \text{(ii) } \|b\|_{L^q} \le |I|^{-1/q'},$$

where $|\cdot|$ denotes the product measure on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, and I is an interval on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, i.e.,

$$I = \left\{x' \in \mathbf{S}^{n-1} : \left|x' - x_0'\right| < \alpha\right\} \times \left\{y' \in \mathbf{S}^{m-1} : \left|y' - y_0'\right| < \beta\right\}$$

for some $\alpha, \beta > 0$, $x_0' \in \mathbf{S}^{n-1}$ and $y_0' \in \mathbf{S}^{m-1}$.

DEFINITION 2.2. *The block space* $B_q^{(0,v)} = B_q^{(0,v)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ is defined by

$$B_q^{(0,v)} = \left\{ \Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}) \colon \Omega = \sum_{\mu=1}^{\infty} \vartheta_\mu b_\mu,\ M_q^{(0,v)}\left(\{\vartheta_\mu\}\right) < \infty \right\},$$

where each ϑ_μ is a complex number; each b_μ is a q -block supported on an interval I_μ on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$, $v > -1$ and

$$M_q^{(0,v)}\left(\{\vartheta_\mu\}\right) = \sum_{\mu=1}^{\infty} |\vartheta_\mu| \left\{1 + \log^{(v+1)}(|I_\mu|^{-1})\right\}. \tag{2.1}$$

Let $\|\Omega\|_{B_q^{(0,v)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} = N_q^{(0,v)}(\Omega) = \inf\left\{M_q^{(0,v)}\left(\{\vartheta_\mu\}\right)\right\}$, where the infimum is taken over all q-block decompositions of Ω.

We remark that the definition of $B_q^{(0,v)}([a,b] \times [c,d])$ for $a,b,c,d \in \mathbf{R}$ will be the same as that of $B_q^{(0,v)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ except for minor modifications.

By following similar arguments as proving Lemma 5.1 in [1], we get the following.

LEMMA 2.3. *For any* $v > -1, a, b, c, d \in \mathbf{R}$,

(i) $N_q^{(0,v)}$ *is a norm on* $B_q^{(0,v)}([a,b] \times [c,d])$ *and* $(B_q^{(0,v)}([a,b] \times [c,d]), N_q^{(0,v)})$ *is a Banach space;*

(ii) *If* $f \in B_q^{(0,v)}([a,b] \times [c,d])$ *and* g *is a measurable on* $[a,b]$ *with* $|g| \leq |f|$ *then* $g \in B_q^{(0,v)}([a,b] \times [c,d])$ *with*

$$N_q^{(0,v)}(g) \leq N_q^{(0,v)}(f);$$

(iii) *Let* I_1 *and* I_2 *be two disjoint intervals in* $[a,b]\times[c,d]$ *with* $|I_1|, |I_2| < 1$ *and* $\alpha_1, \alpha_2 \in \mathbf{R}_+$. *Then*

$$N_q^{(0,v)}(\alpha_1\chi_{I_1} + \alpha_2\chi_{I_2}) \geq N_q^{(0,v)}(\alpha_1\chi_{I_1}) + N_q^{(0,v)}(\alpha_2\chi_{I_2});$$

(iv) *Let I be a interval in $[a,b]\times[c,d]$ with $|I|<1$. Then*

$$N_q^{(0,v)}(\chi_I) \geq |I|\,(1+\log^{v+1}(|I|^{-1})).$$

Let $\mu\in\mathbf{N}\cup\{0\}$ and I_μ be an interval on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ with $|I_\mu|<e^{-1}$. Also, let $\theta_\mu=[\log|I_\mu|^{-1}]$ and $\omega_\mu=2^{\theta_\mu}$, where $[\cdot]$ is the greatest integer function. Now, set $a_{\mu,k,\Phi}=\Phi(\omega_\mu^k)$ if Φ satisfies hypothesis I and $a_{\mu,k,\Phi}=\left(\Phi(\omega_\mu^k)\right)^{-1}$ if Φ satisfies hypothesis D.

LEMMA 2.4. *Let $\mu\in\mathbf{N}\cup\{0\}, q>1$ and $\tilde{b}_\mu$ be a function on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ satisfying*

(i) $\left\|\tilde{b}_\mu\right\|_q\leq|I_\mu|^{-1/q'}$ *for some interval I_μ on $\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}$ with $|I_\mu|<e^{-1}$;*

(ii) $\left\|\tilde{b}_\mu\right\|_1\leq 1$ *and*

(iii) $\tilde{b}_\mu$ *satisfies the vanishing conditions in (1.3) with Ω replaced by $\tilde{b}_\mu$. Let*

$$J_{\mu,k,j}(\xi,\eta)=\left(\int_{[\omega_\mu^k,\omega_\mu^{k+1})\times[\omega_\mu^j,\omega_\mu^{j+1})}|F_{k,j,\mu}(t,s)|^2\,\frac{dtds}{ts}\right)^{1/2},$$

where

$$F_{k,j,\mu}(t,s)=\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}b_\mu(x,y)e^{-i(\Phi(t)\xi\cdot x+\Psi(s)\eta\cdot y)}d\sigma(x)d\sigma(y).$$

Then if Φ satisfies hypothesis I,

$$|J_{\mu,k,j}(\xi,\eta)| \leq C\theta_\mu; \tag{2.2}$$

$$|J_{\mu,k,j}(\xi,\eta)| \leq C\theta_\mu\,(a_{\mu,k,\Phi})^{\frac{\beta}{\theta_\mu}}\,|\xi|^{\frac{\beta}{\theta_\mu}}; \tag{2.3}$$

$$|J_{\mu,k,j}(\xi,\eta)| \leq C\theta_\mu\,(a_{\mu,k,\Phi})^{-\frac{\beta}{\theta_\mu}}\,|\xi|^{-\frac{\beta}{\theta_\mu}}; \tag{2.4}$$

and if Φ satisfies hypothesis D,

$$|J_{\mu,k,j}(\xi,\eta)| \leq C\theta_\mu; \tag{2.5}$$

$$|J_{\mu,k,j}(\xi,\eta)| \leq C\theta_\mu\,(a_{\mu,k,\Phi})^{-\frac{\beta}{\theta_\mu}}\,|\xi|^{\frac{\beta}{\theta_\mu}}; \tag{2.6}$$

$$|J_{\mu,k,j}(\xi,\eta)| \leq C\theta_\mu\,(a_{\mu,k,\Phi})^{\frac{\beta}{\theta_\mu}}\,|\xi|^{-\frac{\beta}{\theta_\mu}}, \tag{2.7}$$

for some positive constants C and β independent of k,j,ξ,η and θ_μ with β satisfying $0<\beta q'<1$. If Ψ satisfies either hypothesis I or hypothesis

D, then the same estimates in (2.2)–(2.7) *hold except that we replace* k *with* j, ξ *with* η *and* Φ *with* Ψ.

Proof. We shall only present the proof of the lemma if Φ satisfies hypothesis I, since the proof for the other cases will be essentially the same. By condition (ii) on b_μ it is easy to see that (2.2) holds. Next, by the vanishing conditions of b_μ and changing variables we have

$$|J_{\mu,k,j}(\xi,\eta)|^2 \le \int_{[1,\omega_\mu)\times[1,\omega_\mu)} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |b_\mu(x,y)| \times \left| e^{-i\Phi(\omega_\mu^k t)\xi\cdot x} - 1 \right| d\sigma(x)d\sigma(y) \right)^2 \frac{dtds}{ts}.$$

By using the assumptions on Φ we get

$$|J_{\mu,k,j}(\xi,\eta)| \le C\theta_\mu \, |a_{\mu,k,\Phi}\xi| \, .$$

By combining the preceding estimate with the trivial estimate $|J_{\mu,k,j}(\xi,\eta)| \le C\theta_\mu$ we obtain

$$|J_{\mu,k,j}(\xi,\eta)| \le C\theta_\mu \, |a_{\mu,k+1,\Phi}\xi|^{\frac{\beta}{\theta_\mu}} \text{ for any } \beta \text{ with } 0<\beta q'<1. \tag{2.8}$$

On the other hand, by Schwarz's inequality we have

$$\begin{aligned} |F_{k,j,\mu}(t,s)|^2 &\le \int_{\mathbf{S}^{m-1}} \left| \int_{\mathbf{S}^{n-1}} b_\mu(x,y) e^{-i\Phi(\omega_\mu^k t)\xi\cdot x} d\sigma(x) \right|^2 d\sigma(y) \\ &= \int_{\mathbf{S}^{m-1}} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} b_\mu(x,y)\overline{b_\mu(u,y)} \times e^{-i\Phi(\omega_\mu^k t)\xi\cdot(x-u)} d\sigma(x)d\sigma(u) \right) d\sigma(y). \end{aligned}$$

Thus,

$$|J_{\mu,k,j}(\xi,\eta)|^2 \le C\theta_\mu \int_{\mathbf{S}^{m-1}} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} b_\mu(x,y)\overline{b_\mu(u,y)} \times H_{\mu,k}(\xi,x,u) d\sigma(x)d\sigma(u) \right) d\sigma(y), \tag{2.9}$$

where

$$H_{\mu,k}(\xi,x,u) = \int_1^{\omega_\mu} e^{-i\Phi(\omega_\mu^k t)\xi\cdot(x-u)} \frac{dt}{t}.$$

Note that

$$H_{\mu,k}(\xi,x,u)=\int_1^{\omega_\mu} e^{-i\Phi(\omega_\mu^k t)\xi\cdot(x-u)}\frac{dt}{t}\equiv\int_1^{\omega_\mu} g'(t)\frac{dt}{t},$$

where

$$g(t)=\int_1^t e^{-i\Phi(\omega_\mu^k r)\xi\cdot(x-u)}dr,\ 1\le t\le\omega_\mu.$$

Now, using the assumptions on Φ, we obtain

$$\frac{d}{dr}\left(\Phi(\omega_\mu^k r)\right)=\omega_\mu^k\Phi'(\omega_\mu^k r)\ge\alpha\frac{\Phi(\omega_\mu^k r)}{r}\ge\alpha\frac{a_{\mu,k,\Phi}}{t}\text{ for }1\le r\le t\le\omega_\mu.$$

By applying van der Corput's lemma,

$$|g(t)|\le\alpha^{-1}\left|a_{\mu,k,\Phi}\xi\right|^{-1}\left|\xi'\cdot(x-u)\right|^{-1}t,\text{ for }1\le t\le\omega_\mu.$$

Thus by integrating by parts, we have

$$|H_{\mu,k}(\xi,x,u)|\le C\theta_\mu\left|a_{\mu,k,\Phi}\xi\right|^{-1}\left|\xi'\cdot(x-u)\right|^{-1}.$$

By combining the last estimate with the trivial estimate $|H_{\mu,k}(\xi,x,u)|\le(\log 2)\theta_\mu$ we get

(2.10)

$$|H_{\mu,k}(\xi,x,u)|\le C\theta_\mu\left|a_{\mu,k,\Phi}\xi\right|^{-\beta}\left|\xi'\cdot(x-u)\right|^{-\beta}\text{ for any }0<\beta\le 1.$$

By Schwarz's inequality, condition (i) on b_μ and (2.9)–(2.10) we get

$$|J_{\mu,k,j}(\xi,\eta)|^2\le C\theta_\mu^2\left|I_\mu\right|^{-2/q'}\left|a_{\mu,k,\Phi}\xi\right|^{-\beta}$$
$$\times\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\left|\xi'\cdot(x-u)\right|^{-\beta q'}d\sigma(x)d\sigma(u)\right)^{1/q'}$$

Since the last integral is finite we get

$$|J_{\mu,k,j}(\xi,\eta)|\le C\theta_\mu\left|I_\mu\right|^{-1/q'}\left|a_{\mu,k,\Phi}\xi\right|^{-\beta/2}$$

which when combined with the trivial estimate $|J_{\mu,k,j}(\xi,\eta)|\le C\theta_\mu$ yields

$$|J_{\mu,k,j}(\xi,\eta)|\le C\theta_\mu\left|a_{\mu,k,\Phi}\xi\right|^{-\frac{\beta}{2\theta_\mu}}. \tag{2.11}$$

Combining the trivial estimate $|J_{\mu,k,j}(\xi,\eta)|\le C\theta_\mu$ with the estimates (2.8) and (2.11) we obtain the desired estimates (2.2)–(2.4). The lemma is proved. □

For $\mu\in\mathbf{N}\cup\{0\}$ and $y\in\mathbf{S}^{n-1}$, let $\mathcal{M}_{\Phi,y}^{(\mu)}(f)$ denote the maximal function defined by

$$\mathcal{M}_{\Phi,y}^{(\mu)}f(x)=\sup_{k\in\mathbf{Z}}\left|\int_{\omega_\mu^k}^{\omega_\mu^{k+1}}f(x-\Phi(t)y)\frac{dt}{t}\right|.$$

LEMMA 2.5. *Assume that Φ satisfies either hypothesis I or hypothesis D. Then*

$$\left\|\mathcal{M}_{\Phi,y}^{(\mu)}(f)\right\|_p \leq C\theta_\mu \|f\|_p \tag{2.12}$$

for $1 < p \leq \infty$ and $f \in L^p$, where C_p is a positive constant independent of θ_μ and y.

Proof. Let $s = \Phi(t)$. Assume first that Φ satisfies hypothesis I. By the assumptions on Φ, we have $\frac{dt}{t} \leq \frac{ds}{\alpha s}$. So, by a change of variable we have

$$\begin{aligned}
\mathcal{M}_{\Phi,y}^{(\mu)}f(x) &\leq \sum_{l=1}^{[\log|I_\mu|^{-1}]} \sup_{k\in\mathbf{Z}} \left(\int_{\omega_\mu^k 2^{l-1}}^{\omega_\mu^k 2^l} |f(x-\Phi(t)y)| \frac{dt}{t}\right) \\
&\leq \frac{1}{\alpha} \sum_{l=1}^{[\log|I_\mu|^{-1}]} \sup_{k\in\mathbf{Z}} \left(\int_{\Phi(\omega_\mu^k 2^{l-1})}^{\Phi(\omega_\mu^k 2^l)} |f(x-sy)| \frac{ds}{s}\right) \\
&\leq \frac{1}{\alpha}[\log|I_\mu|^{-1}] \sup_{r>0} \left(\int_{\Phi(r/2)}^{\Phi(r)} |f(x-sy)| \frac{ds}{s}\right).
\end{aligned}$$

Now, by conditions (a) and (a) we have

$$\begin{aligned}
\mathcal{M}_{\Phi,y}^{(\mu)}f(x) &\leq \frac{1}{\alpha}[\log|I_\mu|^{-1}] \sup_{r>0} \left(\int_{\Phi(r/2)}^{c\Phi(r/2)} |f(x-sy)| \frac{ds}{s}\right) \\
&\leq C\left(\log|I_\mu|^{-1}\right) M_y^* f(x), \qquad (2.13)
\end{aligned}$$

where

$$M_y^* f(x) = \sup_{R>0} R^{-1} \int_0^R |f(x-sy)|\, ds$$

is the Hardy-Littlewood maximal function of f in the direction of y. On the other hand, if Φ satisfies hypothesis D, as above we have $\frac{dt}{t} \leq -\frac{ds}{\alpha s}$ and

$$\begin{aligned}
\mathcal{M}_{\Phi,y}^{(\mu)}f(x) &\leq \frac{1}{\alpha} \sum_{l=1}^{[\log|I_\mu|^{-1}]} \sup_{r>0} \left(\int_{\Phi(r)}^{\Phi(r/2)} |f(x-sy)| \frac{ds}{s}\right) \\
&\leq \frac{1}{\alpha} \sum_{l=1}^{[\log|I_\mu|^{-1}]} \sup_{r>0} \left(\int_{\frac{1}{c}\Phi(r/2)}^{\Phi(r/2)} |f(x-sy)| \frac{ds}{s}\right) \\
&\leq C\log(|I_\mu|^{-1}) M_y^* f(x). \qquad (2.14)
\end{aligned}$$

By (2.13)–(2.14) and the L^p boundedness of $M_y^* f$ with bound independent of y we get (2.12). This concludes the proof of the lemma. $\square$

For any $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, we define the maximal operator

$$\sigma^*_{\Omega,\mu} f(x,y) = \sup_{k,j\in\mathbf{Z}} |\sigma_{k,j,\Omega,\mu} * f(x,y)|, \tag{2.15}$$

where

$$\begin{aligned}
&\sigma_{k,j,\Omega,\mu} * f(x,y) \\
&= \int_{\omega_\mu^j \le |v| < \omega_\mu^{j+1}} \int_{\omega_\mu^k \le |u| < \omega_\mu^{k+1}} \left| f(x - \Phi(|u|)u', y - \Psi(|v|)v') \right| \\
&\times \frac{|\Omega(u',v')|}{|u|^n |v|^m} du dv.
\end{aligned}$$

Lemma 2.6. *Let $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Assume that Φ and Ψ satisfy either hypothesis I or hypothesis D. Then*

$$\left\| \sigma^*_{\Omega,\mu}(f) \right\|_p \le C_p \theta_\mu^2 \|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \|f\|_p \tag{2.16}$$

for $1 < p \le \infty$ and $f \in L^p$, where C_p is independent of Ω, μ, and f.

Proof. Using polar coordinates we get

$$\begin{aligned}
&|\sigma_{k,j,\Omega,\mu} * f(x,y)| \\
&\le \int_{[\omega_\mu^k, \omega_\mu^{k+1}) \times [\omega_\mu^j, \omega_\mu^{j+1})} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(u',v')| \\
&\times \left| f(x - \Phi(t)u', y - \Psi(s)v') \right| d\sigma(u') d\sigma(v') \frac{dtds}{ts}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sigma^*_{\Omega,\mu} f(x,y) \\
&\le C \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(u',v')| \left(\mathcal{M}^{(\mu)}_{\Psi,v'} \circ \mathcal{M}^{(\mu)}_{\Phi,u'} \right)(f)(x,y) d\sigma(u') d\sigma(v'),
\end{aligned}$$

where "$\circ$" denotes the composition of operators. By Lemma 2.5 and noticing that

$$\begin{aligned}
&\left\| \sigma^*_{\Omega,\mu}(f) \right\|_p \\
&\le C \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(u',v')| \left\| \left(\mathcal{M}^{(\mu)}_{\Psi,v'} \circ \mathcal{M}^{(\mu)}_{\Phi,u'} \right)(f) \right\|_p d\sigma(u') d\sigma(v'),
\end{aligned}$$

we get (2.16) which ends the proof of the lemma.

Let M_S be the spherical maximal operator defined by

$$M_S f(x) = \sup_{r>0} \int_{\mathbf{S}^{n-1}} |f(x - r\theta)|\, d\sigma(\theta).$$

By the results of E. M. Stein [14] and J. Bourgain [3] we have the following:

Lemma 2.7. *Suppose that $n \geq 2$ and $p > n/(n-1)$. Then $M_S(f)$ is bounded on $L^p(\mathbf{R}^n)$.*

We shall need the spherical maximal operator M_{SP} defined on functions $f(x,y)$ on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$M_{SP} f(x,y) = \sup_{r,s>0} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |f(x - r\theta, y - sv)|\, d\sigma(\theta) d\sigma(v). \tag{2.17}$$

Lemma 2.8. *Suppose that n, $m \geq 2$ and $p > \max\{n/(n-1), m/(m-1)\}$. Then $M_{SP}(f)$ is bounded on $L^p(\mathbf{R}^n \times \mathbf{R}^m)$.*

The proof of this lemma follows easily by using Lemma 2.7 and noticing that

$$M_{SP} f(x,y) \leq \left(M_S^{(2)} \circ M_S^{(1)}\right) f(x,y),$$

where the maximal operators $M_S^{(1)}$ and $M_S^{(2)}$ are defined on functions f on $\mathbf{R}^n\times \mathbf{R}^m$ by $(M_S^{(1)} f)(x,y) = (M_S^{(1)}) f(\cdot, y))\,(x)$ and $(M_S^{(2)} f)(x,y) = (M_S^{(2)}) f\,(x,\cdot))(y)$.

Lemma 2.9. *Suppose that $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$. Then for some positive constant C, we have*

$$\begin{aligned} &\left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega(\xi,\eta) f(\xi,\eta) d\sigma(\xi) d\sigma(\eta)\right|^2 \\ &\leq C \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(\xi,\eta)|^{\max\{0,2-q\}}\, |f(\xi,\eta)|^2\, d\sigma(\xi) d\sigma(\eta) \end{aligned} \tag{2.18}$$

for arbitrary functions f.

Proof. When $q \geq 2$ (so that $q' \leq 2$), from Hölder's inequality we have

$$\begin{aligned}&\left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega(\xi,\eta)f(\xi,\eta)d\sigma(\xi)d\sigma(\eta)\right|^2\\ \leq\ & \|\Omega\|_q^2\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |f(\xi,\eta)|^{q'}\, d\sigma(\xi)d\sigma(\eta)\right)^{2/q'}\\ \leq\ & \|\Omega\|_q^2\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |f(\xi,\eta)|^2\, d\sigma(\xi)d\sigma(\eta)\end{aligned}$$

which is the statement of the lemma for the case $q \geq 2$.

When $1 < q < 2$ (so that $q' > 2$), the conclusion of the lemma follows from Schwarz's inequality and the fact that $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. This finishes the proof of the lemma. □

LEMMA 2.10. *Suppose that $\Omega \in L^q(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$. Let $\delta = \max\{2, q'\}$. Then for some positive constant C, we have*

$$\begin{aligned}&\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(\xi,\eta)|^{\max\{0,2-q\}}\, |\omega(x-t\xi, y-r\eta)|\, d\sigma(\xi)d\sigma(\eta)\\ &\leq C\left(\mathcal{M}_{SP}\left(|\omega|^{\delta/2}\right)(x,y)\right)^{2/\delta}\end{aligned} \tag{2.19}$$

for all positive real numbers t and r, $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$ and arbitrary functions ω

Proof. As in the proof of Lemma 2.9, we shall consider the cases $q \geq 2$ and $1 < q < 2$ separately. We notice that if $q \geq 2$, the inequality (2.19) is obvious. However, if $1 < q < 2$, (2.19) follows easily from Hölder's inequality and noticing that $(\frac{q}{2-q})' = q'/2$. The lemma is proved. □

3. Proof of the Theorem 1.1(b)

By duality, the operator $\mathcal{S}_\Omega$ is simply

$$\mathcal{S}_\Omega f(x,y) = \left(\int_{(0,\infty)\times(0,\infty)} |L_{r,t}f(x,y)|^2\, \frac{drdt}{rt}\right)^{1/2},$$

where

$$L_{r,t}f(x,y) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-r\xi, y-t\eta)\Omega(\xi,\eta)d\sigma(\xi)d\sigma(\eta).$$

It is easy to see that

$$\widehat{(L_{r,t}f)}(x,y) = \hat{f}(x,y)\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} e^{-2\pi i(t\xi\cdot u+s\eta\cdot v)}\Omega\,(u,v)\,d\sigma(u)d\sigma(v)\right).$$

By straightforward calculations, it is easy to verify that $\mathcal{S}_\Omega$ is bounded on $L^2(\mathbf{R}^n\times\mathbf{R}^m)$ if and only if the multiplier

$$\begin{aligned}&\mathcal{M}(\xi,\eta)\\ &=\left(\int_{(0,\infty)\times(0,\infty)}\left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} e^{-2\pi i(t\xi\cdot u+s\eta\cdot v)}\right.\right.\\ &\qquad\left.\left.\times\,\Omega\,(u,v)\,d\sigma(u)d\sigma(v)\right|^2\frac{dtds}{ts}\right)^{1/2}\end{aligned}\tag{3.1}$$

is an L^∞ function. It is easy to see that

$$\begin{aligned}&(\mathcal{M}(\xi,\eta))^2\\ &=\lim_{M\to\infty,\varepsilon_2\to 0}\ \lim_{N\to\infty,\varepsilon_1\to 0}\int_{(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})^2}\Omega\,(u,v)\,\overline{\Omega\,(x,y)}\\ &\quad\times\left(\int_{\varepsilon_2|\eta|}^{M|\eta|} e^{-2\pi is\eta'\cdot(v-y)}\frac{ds}{s}\right)\\ &\quad\times\left(\int_{\varepsilon_1|\xi|}^{N|\xi|} e^{-2\pi it\xi'\cdot(u-x)}\frac{dt}{t}\right)d\sigma(u)d\sigma(v)d\sigma(x)d\sigma(y),\end{aligned}$$

where $\xi'=\xi/|\xi|$ and $\eta'=\eta/|\eta|$. Note that

$$\begin{aligned}&\int_{\varepsilon_1|\xi|}^{N|\xi|}\left(e^{-2\pi it\xi'\cdot(u-x)}-\cos(2\pi t)\right)\frac{dt}{t}\\ &\to\ \log\left|\xi'\cdot(u-x)\right|^{-1}-i\frac{\pi}{2}sgn(\xi'\cdot(u-x))\end{aligned}$$

as $N\to\infty$ and $\varepsilon_1\to 0$, and the integral is bounded, uniformly in ε_1 and N, by $C\left(1+\left|\log\left|\xi'\cdot(u-x)\right|\right|\right)$. Thus, using (1.3) and the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
&(\mathcal{M}(\xi,\eta))^2 \\
&= \int_{(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})^2} \Omega\,(u,v)\,\overline{\Omega\,(x,y)} \\
&\times \left(\log\left|\xi'\cdot(u-x)\right|^{-1} - i\frac{\pi}{2}sgn(\xi'\cdot(u-x))\right) \\
&\times \left(\log\left|\eta'\cdot(v-y)\right|^{-1} - i\frac{\pi}{2}sgn(\eta'\cdot(v-y))\right) d\sigma(u)d\sigma(v)d\sigma(x)d\sigma(y).
\end{aligned}$$

If Ω is a real-valued function, we get

$$\begin{aligned}
&(\mathcal{M}(\xi,\eta))^2 \\
&= \int_{(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})^2} \Omega\,(u,v)\,\Omega\,(x,y) \left(\log\left|\xi'\cdot(u-x)\right|^{-1}\log\left|\eta'\cdot(v-y)\right|^{-1}\right) \\
&- \left(\frac{\pi^2}{4}sgn(\xi'\cdot(u-x))sgn(\eta'\cdot(v-y))\right) d\sigma(u)d\sigma(v)d\sigma(x)d\sigma(y).
\end{aligned}$$

Now, we are ready to prove Theorem 1.1 (b). For the sake of simplicity we shall present the construction of our Ω only in the cas $n=m=2$ and $q=\infty$. Other cases can be obtained by making minor modifications. Also, we shall work on $[-1,1]^2$ instead of $\mathbf{S}^1\times\mathbf{S}^1$. We notice that the proof Theorem 1.1 (b) will be completed if we can construct an Ω on $[-1,1]^2$ with the following properties:

$$\int_{-1}^{1}\Omega\,(u,\cdot)\,du = \int_{-1}^{1}\Omega\,(\cdot,v)\,dv = 0; \tag{3.1}$$

$$\Omega \in B_{\infty}^{(0,v)}([-1,1]^2) \text{ for each } v, -1<v<0; \tag{3.2}$$

$$\Omega \notin B_{\infty}^{(0,0)}([-1,1]^2); \tag{3.3}$$

$$\int_{[0,1]^2}\int_{[0,1]^2} G(u,v,x,y)dudvdxdy = \infty; \tag{3.4}$$

$$\int_{[-1,1]^2\setminus[0,1]^2}\int_{[0,1]^2} |G(u,v,x,y)|\,dudvdxdy < \infty; \tag{3.5}$$

$$\int_{[0,1]^2}\int_{[-1,1]^2\setminus[0,1]^2} |G(u,v,x,y)|\,dudvdxdy < \infty; \tag{3.6}$$

$$\int_{[-1,1]^2\setminus[0,1]^2}\int_{[-1,1]^2\setminus[0,1]^2} |G(u,v,x,y)|\,dudvdxdy < \infty, \tag{3.7}$$

where

$$G(u,v,x,y) = \Omega\,(u,v)\,\Omega(x,y)(\log|x-u|^{-1})(\log|y-v|^{-1}).$$

To this end, for integers $k, j \geq 3$, let $I_j = [\frac{1}{j+1}, \frac{1}{j})$ and

$$\begin{aligned}\alpha_k &= (\frac{k}{\log k})\sum_{j=3}^{\infty}\frac{1}{(j+1)(\log j)(\log j+\log k)},\\ \beta_{k,j} &= \frac{jk}{(\log j)(\log k)(\log k+\log j)}.\end{aligned}$$

Now, by definition of α_k, we notice that

$$\begin{aligned}\alpha_k &= \frac{k}{\log k}\left(\sum_{j=3}^{k}\frac{1}{(j+1)(\log j)(\log k+\log j)}+\right.\\ &\qquad \left.\sum_{j=k+1}^{\infty}\frac{1}{(j+1)(\log j)(\log k+\log j)}\right)\\ &\leq \frac{k}{(\log k)^2}\left(\sum_{j=3}^{k}\frac{1}{(j+1)(\log j)}\right)+\frac{k}{\log k}\left(\sum_{j=k+1}^{\infty}\frac{1}{(j+1)(\log j)^2}\right)\\ &\leq \frac{2k\log(\log k)}{(\log k)^2}+\frac{k}{(\log k)^2}.\end{aligned}$$

Therefore, for some positive constant C independent of k,

$$\alpha_k \leq C\frac{k\log(\log k)}{(\log k)^2} \tag{3.8}$$

and hence

$$\sum_{k=3}^{\infty}\frac{\alpha_k}{k(k+1)} < \infty.$$

Define Ω on $[-1,1]^2$ by

$$\Omega(u,v) = \sum_{j=2}^{\infty}\sum_{k=2}^{\infty}C_{k,j}\chi_{I_k\times I_j}(u,v),$$

where, for integers $k, j \geq 3$,

$$C_{2,2} = \left(\sum_{k=3}^{\infty}\frac{\alpha_k}{k(k+1)}\right), C_{2,k} = C_{k,2} = -\alpha_k\left|I_k\right|, C_{k,j} = \beta_{k,j}\left|I_k\times I_j\right|,$$

$$\begin{aligned}b_{2,2}(u,v) &= \chi_{[-1,0]^2}(u,v), b_{2,k}(u,v) = \left|I_k\right|^{-1}\chi_{[-1,0]\times I_k}(u,v),\\ b_{k,2}(u,v) &= \left|I_k\right|^{-1}\chi_{I_k\times[-1,0]}(u,v),\ b_{k,j}(u,v) = \left|I_k\times I_j\right|^{-1}\chi_{I_k\times I_j}(u,v)\end{aligned}$$

and χ_A represents the characteristic function of a set A.

By straightforward calculations, it is easy to see that (3.1)–(3.2) hold. To prove (3.3) we invoke Lemma 2.3. In fact, we notice that each $b_{k,j}$ is an ∞-block supported on the interval $I_k \times I_j$. So to prove (3.3), it suffices to show that $N_\infty^{(0,0)}(\Omega) = \infty$. To this end, by Lemma 2.3 we have for each M, N,

$$\begin{aligned}
&N_\infty^{(0,0)}\left(\Omega + \sum_{k=3}^{\infty} C_{2,k} b_{2,k} + \sum_{k=3}^{\infty} C_{k,2} b_{k,2} - C_{2,2} b_{2,2}\right) \\
\geq\ & \sum_{j=3}^{M} \sum_{k=3}^{N} C_{k,j} \left|I_k \times I_j\right|^{-1} N_\infty^{(0,0)}(\chi_{I_k \times I_j}) \\
\geq\ & \sum_{j=3}^{M} \sum_{k=3}^{N} C_{k,j} \left(1 + \log(|I_k \times I_j|^{-1})\right) \\
\geq\ & \sum_{j=3}^{M} \sum_{k=3}^{N} \frac{1}{(k+1)(j+1)(\log k + \log j)} \\
\geq\ & C \sum_{j=3}^{M} \sum_{k=3}^{N} \frac{1}{(k+1)(j+1)(\log k + \log j)^2}
\end{aligned}$$

for some positive constant C independent of M and N. Letting $M, N \to \infty$, and since

$$\sum_{j=3}^{\infty} \sum_{k=3}^{\infty} \frac{1}{kj(\log k + \log j)^2} \geq \sum_{k=3}^{\infty} \frac{1}{k} \sum_{j \geq k}^{\infty} \frac{1}{j(\log j)^2} = \infty$$

we get

$$N_\infty^{(0,0)}\left(\Omega + \sum_{k=3}^{\infty} C_{2,k} b_{2,k} + \sum_{k=3}^{\infty} C_{k,2} b_{k,2} - C_{2,2} b_{2,2}\right) = \infty.$$

Since $N_\infty^{(0,0)}\left(\sum\limits_{k=3}^{\infty} C_{2,k} b_{2,k}\right)$, $N_\infty^{(0,0)}\left(\sum\limits_{k=3}^{\infty} C_{k,2} b_{k,2}\right)$, $N_\infty^{(0,0)}\left(C_{2,2} b_{2,2}\right)$ are finite numbers, we obtain $N_\infty^{(0,0)}(\Omega) = \infty$.

Now, we prove (3.4). Notice that integral in (3.4) equals to

$$\mathcal{I} = \sum_{j=3}^{\infty}\sum_{k=3}^{\infty}\sum_{s=3}^{\infty}\sum_{l=3}^{\infty}\beta_{k,j}\beta_{l,s} \times \int_{I_k\times I_j}\int_{I_l\times I_s}\log|x-u|^{-1}\log|y-v|^{-1}\,dxdydudv.$$

Now, for $(u,x)\in I_k\times I_l$ with $l\geq 2(k+1)$, we have $u\geq 2x$ and hence $\log|x-u|^{-1}\geq\log k$. Similarly, $\log|y-v|^{-1}\geq\log j$ for $(v,y)\in I_j\times I_s$ with $s\geq 2(j+1)$. Therefore, we have

$$\begin{aligned}
&\mathcal{I}\\
&\geq C\sum_{j=3}^{\infty}\sum_{s\geq 2(j+1)}\sum_{k=3}^{\infty}\sum_{l\geq 2(k+1)}\frac{1}{lkjs(\log k+\log j)(\log l)(\log s)(\log l+\log s)}\\
&\geq C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s\geq 2(j+1)}\frac{\log(\frac{\log s}{\log k})}{kjs(\log k+\log j)(\log s)^2}\\
&\geq C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\frac{\log(\frac{\log j}{\log k})}{kj(\log k+\log j)(\log j)}\\
&\geq C\sum_{k=3}^{\infty}\sum_{j\geq 2k}\frac{\log(\frac{\log j}{\log k})}{kj(\log k+\log j)(\log j)}\\
&\geq C\sum_{k=3}^{\infty}\frac{1}{k}\sum_{j\geq 2k}\frac{1}{j\,(\log j)^2}\\
&\geq C\sum_{k=3}^{\infty}\frac{1}{k\log k}=\infty.
\end{aligned}$$

To prove (3.5), we divide $[-1,1]^2\backslash[0,1]^2$ into three parts: $[-1,0]\times[0,1]$, $[0,1]\times[-1,0]$, and $[-1,0]\times[-1,0]$. First the integral over $[-1,0]\times[0,1]\times[0,1]^2$ is dominated from above by

$$I_* = \sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s=3}^{\infty}\beta_{k,j}\alpha_s\,|\mathcal{I}(k)|\,\mathcal{J}(j,s), \tag{3.9}$$

where

$$\mathcal{J}(j,s) = \int_{I_j\times I_s}\log|y-v|^{-1}\,dvdy,$$

and

$$\mathcal{I}(k) = \int_{I_k} \int_{-1}^{0} \log |x-u|^{-1}\, dxdu.$$

By straightforward calculations, it is easy to show that

$$|\mathcal{I}(k)| \leq C\frac{1}{k^2}; \tag{3.10}$$

$$\mathcal{J}(j,s) \leq C\frac{\log j}{j^2 s^2} \quad \text{if } s > 2j; \tag{3.11}$$

$$\mathcal{J}(j,s) \leq C\frac{\log s}{j^2 s^2} \quad \text{if } j > 2s; \tag{3.12}$$

$$\mathcal{J}(j,s) \leq C\frac{\log s}{s^4} \quad \text{if } j/2 \leq s \leq 2j \tag{3.13}$$

for some positive constant C independent of k, j and s. By (3.9)–(3.10), we have

$$I_* \leq I_1 + I_2 + I_3, \tag{3.14}$$

where

$$\begin{aligned}
I_1 &= \sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{s>2j} \frac{js\log(\log s)}{k(\log j)(\log k)(\log k+\log j)(\log s)^2}\mathcal{J}(j,s),\\
I_2 &= \sum_{k=3}^{\infty}\sum_{s=3}^{\infty}\sum_{j>2s} \frac{js\log(\log s)}{k(\log j)(\log k)(\log k+\log j)(\log s)^2}\mathcal{J}(j,s),\\
I_3 &= \sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\sum_{j/2\leq s\leq 2j} \frac{js\log(\log s)}{k(\log j)(\log k)(\log k+\log j)(\log s)^2}\mathcal{J}(j,s).
\end{aligned}$$

Thus, by (3.11) we get

$$\begin{aligned}
I_1 &\leq C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty} \frac{1}{kj(\log k)(\log k+\log j)} \sum_{s>2j}\frac{\log(\log s)}{s(\log s)^2}\\
&\leq C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty} \frac{\log(\log j)}{kj(\log k)(\log j)(\log k+\log j)}\\
&\leq C\left(\sum_{k=3}^{\infty}\frac{1}{k(\log k)^{3/2}}\right)\left(\sum_{j=3}^{\infty}\frac{\log(\log j)}{j(\log j)^{3/2}}\right) < \infty.
\end{aligned}$$

Similarly, by (3.12) $I_2 < \infty$. For I_3, by (3.13) we have

$$\begin{aligned} S_3 &\leq C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\frac{j}{k(\log j)(\log k)(\log k+\log j)}\left(\sum_{j/2\leq s\leq 2j}\frac{\log(\log s)}{s^3(\log s)}\right) \\ &\leq C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\frac{\log(\log j)}{kj(\log j)^2(\log k)(\log k+\log j)} \\ &\leq C\left(\sum_{k=3}^{\infty}\frac{1}{k(\log k)^2}\right)\left(\sum_{j=3}^{\infty}\frac{\log(\log j)}{j(\log j)^2}\right) < \infty. \end{aligned}$$

Thus, we get $I_* < \infty$ which in turn proves the finiteness of the integral over $[-1,0] \times [0,1] \times [0,1]^2$. Similarly, the integral over $[0,1] \times [-1,0] \times [0,1]^2$ is finite. Also, the integral over $[-1,0] \times [-1,0] \times [0,1]^2$ is bounded from above by

$$\begin{aligned} &C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}a_{k,j}\,|\mathcal{I}(k)|\,|\mathcal{I}(j)| \\ &\leq C\sum_{k=3}^{\infty}\sum_{j=3}^{\infty}\frac{1}{kj(\log j)(\log k)(\log k+\log j)} \\ &\leq C\left(\sum_{k=3}^{\infty}\frac{1}{k(\log k)^{3/2}}\right)\left(\sum_{j=3}^{\infty}\frac{1}{k(\log j)^{3/2}}\right) < \infty \end{aligned}$$

which ends the proof of (3.5).

Similarly, (3.6) holds. Finally, we need to verify (3.7). As above, we divide $[-1,1]^2\backslash[0,1]^2$ into three parts: $[-1,0]\times[0,1]$, $[0,1]\times[-1,0]$, and $[-1,0]\times[-1,0]$. We shall only present the proof of the finiteness of the integral over $[-1,0]\times[0,1]\times[-1,0]\times[0,1]$ and over $[-1,0]\times[0,1]\times[0,1]\times[-1,0]$. The proof of the other cases either will be similar or easier. To this end, we notice that the integral over $[-1,0]\times[0,1]\times[-1,0]\times[0,1]$ is bounded from above by

$$\begin{aligned} &C\sum_{k=3}^{\infty}\sum_{l=3}^{\infty}\frac{kl\log(\log k)\log(\log l)}{(\log k)^2(\log l)^2}\mathcal{J}(k,l)\left(\int_{-1}^{0}\int_{-1}^{0}\log|y-v|^{-1}\,dvdy\right) \\ &\leq C\sum_{k=3}^{\infty}\sum_{l=3}^{\infty}\frac{kl\log(\log k)\log(\log l)}{(\log k)^2(\log l)^2}\mathcal{J}(k,l) = I^* \end{aligned}$$

Now, as above we split I^* as

$$I^* = I_1^* + I_2^* + I_3^*,$$

where

$$\begin{aligned}
I_1^* &= \sum_{k=3}^{\infty} \sum_{l>2k} \frac{kl \log(\log k) \log(\log l)}{(\log k)^2 (\log l)^2} \mathcal{J}(k,l); \\
I_2^* &= \sum_{l=3}^{\infty} \sum_{k>2l} \frac{kl \log(\log k) \log(\log l)}{(\log k)^2 (\log l)^2} \mathcal{J}(k,l); \\
I_3^* &= \sum_{k=3}^{\infty} \sum_{k/2 \le l \le 2k} \frac{kl \log(\log k) \log(\log l)}{(\log k)^2 (\log l)^2} \mathcal{J}(k,l).
\end{aligned}$$

By (3.11), we have

$$\begin{aligned}
I_1^* &\le C \sum_{k=3}^{\infty} \frac{\log(\log k)}{k\,(\log k)} \left(\sum_{l>2k} \frac{\log(\log l)}{l(\log l)^2} \right) \\
&\le C \sum_{k=3}^{\infty} \frac{(\log(\log k))^2}{k\,(\log k)^2} < \infty.
\end{aligned}$$

Similarly, by (3.12) $I_2^* < \infty$. By (3.13),

$$\begin{aligned}
I_3^* &\le C \sum_{k=3}^{\infty} \frac{k \log(\log k)}{(\log k)^2} \sum_{k/2 \le l \le 2k} \frac{\log(\log l)}{l^3 (\log l)} \\
&\le C \sum_{k=3}^{\infty} \frac{(\log(\log k))^2}{k\,(\log k)^3} < \infty.
\end{aligned}$$

This finishes the proof of the finiteness of the integral over $[-1,0] \times [0,1] \times [-1,0] \times [0,1]$. Now, we turn to the proof of the finiteness of the integral over $[-1,0] \times [0,1] \times [0,1] \times [-1,0]$. We notice that the integral over $[-1,0] \times [0,1] \times [0,1] \times [-1,0]$ is bounded from above by

$$\begin{aligned}
& C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \frac{kj \log(\log k) \log(\log j)}{(\log k)^2 (\log j)^2} |\mathcal{I}(k)|\, |\mathcal{I}(j)| \\
\le\ & C \sum_{k=3}^{\infty} \sum_{j=3}^{\infty} \frac{\log(\log k) \log(\log j)}{kj\,(\log k)^2 (\log j)^2} \\
=\ & \left(\sum_{k=3}^{\infty} \frac{\log(\log k)}{k(\log k)^2} \right) \left(\sum_{j=3}^{\infty} \frac{\log(\log j)}{j(\log j)^2} \right) < \infty.
\end{aligned}$$

This ends the proof of (3.7) and hence the proof of Theorem 1.1 (b) is completed.

4. Proof of Theorem 1.1(a)

We shall only present the proof of Theorem 1.1 (a) for the case Φ and Ψ satisfy hypothesis I, since the proofs of the remaining cases are essentially the same. Assume that $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$ and satisfies (1.3). Thus Ω can be written as $\Omega = \sum_{\mu=1}^{\infty} \vartheta_\mu b_\mu$, where $\vartheta_\mu \in \mathbf{C}$, b_μ is a q-block supported on an interval I_μ on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ and $M_q^{(0,0)}(\{\vartheta_\mu\}) < \infty$. To each block function $b_\mu(\cdot,\cdot)$, let $\tilde{b}_\mu(\cdot,\cdot)$ be a function defined by

$$\begin{aligned}\tilde{b}_\mu(x,y) &= b_\mu(x,y) - \int_{\mathbf{S}^{n-1}} b_\mu(u,y)d\sigma(u) \\ &\quad - \int_{\mathbf{S}^{m-1}} b_\mu(x,v)d\sigma(v) + \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} b_\mu(u,v)d\sigma(u)d\sigma(v).\end{aligned}$$

Let $\mathbf{D} = \{\mu \in \mathbf{N} : |I_\mu| < e^{-1}\}$. Let $\tilde{b}_0 = \Omega - \sum_{\mu\in\mathbf{D}} \vartheta_\mu \tilde{b}_\mu$. Then it is easy to verify that, for all $\mu \in \mathbf{D} \cup \{0\}$, $\tilde{b}_\mu$ satisfies the vanishing conditions (1.3) and

$$\|\tilde{b}_\mu\|_q \le C|I_\mu|^{-1/q'}; \tag{4.1}$$

$$\|\tilde{b}_\mu\|_1 \le C; \tag{4.2}$$

$$\Omega = \sum_{\mu\in\mathbf{D}\cup\{0\}} \vartheta_\mu \tilde{b}_\mu, \tag{4.3}$$

where I_0 is any interval on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ with $|I_0| = e^{-2}$ and C is a positive constant independent of μ. By (4.3) we have

$$\mathcal{S}_{\Omega,\Phi,\Psi} f(x,y) \le \sum_{\mu\in\mathbf{D}\cup\{0\}} |\vartheta_\mu| \mathcal{S}_{b_\mu,\Phi,\Psi} f(x,y). \tag{4.4}$$

Therefore, the proof of Theorem 1.1 (a) is completed if we can show that

$$\left\|\mathcal{S}_{b_\mu,\Phi,\Psi} f\right\|_p \le C_p \log|I_\mu|^{-1} \|f\|_p \text{ for all } 2 \le p < \infty \text{ and } f \in L^p. \tag{4.5}$$

By the conditions on Φ, the sequence $\{a_{\mu,k,\Phi} : k \in \mathbf{Z},\ \mu \in \mathbf{D} \cup \{0\}\}$ is a lacunary sequence with $\frac{a_{\mu,k+1,\Phi}}{a_{\mu,k,\Phi}} \geq \lambda^{[\log|I_\mu|^{-1}]}$. Let $\left\{\Lambda_{k,\Phi}^{(\mu)}\right\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\ \infty)$ adapted to the intervals $E_{k,\Phi}^{(\mu)} = [a_{\mu,k+1,\Phi}^{-1},\ a_{\mu,k-1,\Phi}^{-1}]$. To be precise, we require the following:

$$\Lambda_{k,\Phi}^{(\mu)} \in C^\infty,\ 0 \leq \Lambda_{k,\Phi}^{(\mu)} \leq 1,\ \sum_k \Lambda_{k,\Phi}^{(\mu)}(t) = 1 \text{ for all } t,$$

$$\operatorname{supp} \Lambda_{k,\Phi}^{(\mu)} \subseteq E_{k,\Phi}^{(\mu)},\quad \left|\frac{d^s \Lambda_{k,\Phi}^{(\mu)}(t)}{dt^s}\right| \leq \frac{C_s}{t^s},$$

where C_s is independent of the lacunary sequence $\{a_{\mu,k,\Phi} : k \in \mathbf{Z}\}$. (We remark at this point that if Φ satisfies hypothesis D, the partition of unity needed in our proof should have the same properties as above except that we need it to be adapted to the intervals $[a_{\mu,k-1,\Phi},\ a_{\mu,k+1,\Phi}]$. Let $I_{k,\mu} = \left[\omega_\mu^k, \omega_\mu^{k+1}\right)$, $\widehat{\Gamma_{k,\Phi}^{(\mu)}}(\xi) = \Lambda_{k,\Phi}^{(\mu)}(|\xi|)$ and $\widehat{\Gamma_{j,\Psi}^{(\mu)}}(\eta) = \Lambda_{j,\Psi}^{(\mu)}(|\eta|)$ for $(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m$.

By duality and Minkowski's inequality we have

$$\begin{aligned}
&\mathcal{S}_{b_\mu,\Phi,\Psi} f(x,y) \\
&= \left(\sum_{k,j\in\mathbf{Z}} \int_{I_{k,\mu}\times I_{j,\mu}} \left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(t)\xi, y-\Psi(r)\eta)\right.\right. \\
&\quad \left.\left. \times\ b_\mu(\xi,\eta) d\sigma(\xi) d\sigma(\eta)\right|^2 \frac{dtdr}{tr}\right)^{1/2} \\
&\leq \left(\sum_{k,j\in\mathbf{Z}} \int_{I_{k,\mu}\times I_{j,\mu}} \left|\sum_{l,s\in\mathbf{Z}} F_{k+l,j+s,t,r,b_\mu}^{(\mu)} f(x,y)\right|^2 \frac{dtdr}{tr}\right)^{1/2} \\
&\leq \sum_{l,s\in\mathbf{Z}} \left(\sum_{k,j\in\mathbf{Z}} \int_{I_{k,\mu}\times I_{j,\mu}} \left|F_{k+l,j+s,t,r,b_\mu}^{(\mu)} f(x,y)\right|^2 \frac{dtdr}{tr}\right)^{1/2} \\
&\equiv \sum_{l,s\in\mathbf{Z}} T_{l,s,b_\mu}^{(\mu)} f(x,y),
\end{aligned}$$

where

$$\begin{aligned} &F^{(\mu)}_{k+l,j+s,t,r,\Omega}f(x,y) \\ &= \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega(\xi,\eta) \\ &\quad \times \left((\Gamma^{(\mu)}_{k+l,\Phi}\otimes\Gamma^{(\mu)}_{j+s,\Psi}) * f\right)(x-\Phi(t)\xi, y-\Psi(r)\eta)\, d\sigma(\xi)d\sigma(\eta). \end{aligned}$$

Therefore, to prove (4.5), it suffices to show that

$$\left\|T^{(\mu)}_{l,s,b_\mu}(f)\right\|_p \le C_p\theta_\mu \lambda^{-\beta_p|l|}\lambda^{-\beta_p|s|}\left\|f\right\|_p \tag{4.6}$$

for some positive constants C_p, β_p and for all $2 \le p < \infty$, where $\theta_\mu = [\log|I_\mu|^{-1}]$.

Let us first prove (4.6) for $p = 2$. By Plancherel's theorem, we have

$$\begin{aligned} &\left\|T^{(\mu)}_{l,s,b_\mu}(f)\right\|_2^2 \\ &= \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,j\in\mathbf{Z}} \int_{I_{k,\mu}\times I_{j,\mu}} \left|F^{(\mu)}_{k+l,j+s,t,r,b_\mu}f(x,y)\right|^2 \frac{dtdr}{tr}dxdy \\ &\le \sum_{k,j\in\mathbf{Z}} \int_{\Delta^{(\mu)}_{k+l,j+s}} (J_{\mu,k,j}(\xi,\eta))^2\left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta, \end{aligned} \tag{4.7}$$

where

$$\Delta^{(\mu)}_{k,j} = \left\{(\xi,\eta)\in\mathbf{R}^n\times\mathbf{R}^m : (|\xi|,|\eta|)\in E^{(\mu)}_{k,\Phi}\times E^{(\mu)}_{j,\Psi}\right\}$$

and $J_{\mu,k,j}(\xi,\eta)$ is defined as in Lemma 2.4. By Lemma 2.4, we have

$$|J_{\mu,k,j}(\xi,\eta)| \le C\theta_\mu; \tag{4.8}$$

$$|J_{\mu,k,j}(\xi,\eta)| \le C\theta_\mu \left|a_{\mu,k,\Phi}\xi\right|^{\frac{\beta}{\theta_\mu}}\left|a_{\mu,j,\Psi}\eta\right|^{\frac{\beta}{\theta_\mu}}; \tag{4.9}$$

$$|J_{\mu,k,j}(\xi,\eta)| \le C\theta_\mu \left|a_{\mu,k,\Phi}\xi\right|^{-\frac{\beta}{\theta_\mu}}\left|a_{\mu,j,\Psi}\eta\right|^{\frac{\beta}{\theta_\mu}}; \tag{4.10}$$

$$|J_{\mu,k,j}(\xi,\eta)| \le C\theta_\mu \left|a_{\mu,k,\Phi}\xi\right|^{\frac{\beta}{\theta_\mu}}\left|a_{\mu,j,\Psi}\eta\right|^{-\frac{\beta}{\theta_\mu}}; \tag{4.11}$$

$$|J_{\mu,k,j}(\xi,\eta)| \le C\theta_\mu \left|a_{\mu,k,\Phi}\xi\right|^{-\frac{\beta}{\theta_\mu}}\left|a_{\mu,j,\Psi}\eta\right|^{-\frac{\beta}{\theta_\mu}}. \tag{4.12}$$

Therefore, by (4.8)–(4.12), we have

$$\begin{aligned} \left\|T^{(\mu)}_{l,s,b_\mu}(f)\right\|_2^2 &\le C\theta_\mu^2\lambda^{-2\beta|l|}\lambda^{-2\beta|s|}\sum_{k,j\in\mathbf{Z}}\int_{\Delta^{(\mu)}_{k+l,j+s}}\left|\hat{f}(\xi,\eta)\right|^2 d\xi d\eta \\ &\le C\theta_\mu^2\lambda^{-2\beta|l|}\lambda^{-2\beta|s|}\left\|f\right\|_2^2, \end{aligned}$$

which obviously implies

$$\left\|T^{(\mu)}_{l,s,b_\mu}(f)\right\|_2 \le C\theta_\mu \lambda^{-\beta|l|}\lambda^{-\beta|s|}\,\|f\|_2\,. \tag{4.13}$$

Now, we compute the L^p-norm of $T^{(\mu)}_{l,s,b_\mu}(f)$ for $p > 2$. By duality, there is a function g in $L^{(p/2)'}(\mathbf{R}^n \times \mathbf{R}^m)$ with $\|g\|_{(p/2)'} \le 1$ such that

$$\begin{aligned}
&\left\|T^{(\mu)}_{l,s,b_\mu}(f)\right\|_p^2 \\
&= \sum_{k,j\in\mathbf{Z}} \int_{\mathbf{R}^n\times\mathbf{R}^m} \int_{I_{k,\mu}\times I_{j,\mu}} \left|F^{(\mu)}_{k+l,j+s,t,r,b_\mu} f(x,y)\right|^2 \frac{dtdr}{tr}\,|g(x,y)|\,dxdy \\
&\le \left\|\tilde{b}_\mu\right\|_1 \sum_{k,j\in\mathbf{Z}} \int_{\mathbf{R}^n\times\mathbf{R}^m} \int_{I_{k,\mu}\times I_{j,\mu}} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |b_\mu(\xi,\eta)| \\
&\quad \times \left|(\Gamma^{(\mu)}_{k+l,\Phi}\otimes\Gamma^{(\mu)}_{j+s,\Psi}) * f(x,y)\right|^2 \\
&\quad \times |g(x+\Phi(t)\xi, y+\Psi(r)\eta)|\,d\sigma(\xi)d\sigma(\eta)\frac{dtdr}{tr}dxdy \\
&\le C\sum_{k,j\in\mathbf{Z}} \int_{\mathbf{R}^n\times\mathbf{R}^m} \left|(\Gamma^{(\mu)}_{k+l,\Phi}\otimes\Gamma^{(\mu)}_{j+s,\Psi}) * f(x,y)\right|^2 \sigma^*_{b_\mu,\mu}(\tilde{g})(-x,-y)dxdy \\
&\le C\left\|\sum_{k,j\in\mathbf{Z}} \left|(\Gamma^{(\mu)}_{k+l,\Phi}\otimes\Gamma^{(\mu)}_{j+s,\Psi}) * f\right|^2\right\|_{p/2} \left\|\sigma^*_{b_\mu,\mu}(\tilde{g})\right\|_{(p/2)'},
\end{aligned}$$

where $\tilde{g}(x,y) = g(-x,-y)$.

By using Lemma 2.6, (4.2), the Littlewood-Paley theory and Theorem 3 along with the remark that follows its statement in [14], p. 96, we have

$$\left\|T^{(\mu)}_{l,s,b_\mu}(f)\right\|_p \le C_p\theta_\mu \|f\|_p \ \text{ for } 2 < p < \infty. \tag{4.14}$$

By interpolation between (4.13) and (4.14) we get (4.6) for $2 < p < \infty$ which ends the proof of Theorem 1.1 (a).

5. Proof of Theorem 1.2

Assume that $\Omega \in L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})$ for some $q > 1, 1 \le \gamma \le 2$ and $\max\{(\gamma' n\delta)/(\gamma' n + n\delta - \gamma'), (\gamma' m\delta)/(\gamma' m + m\delta - \gamma')\} < p < \infty$, where $\delta = \max\{2, q'\}$. The proof of Theorem 1.2 will be divided into three steps.

Step 1. $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $\gamma=2$, and $\nu(n,m)<p<\infty$, where $\nu(n,m)=\max\{(2n\delta)/(2n+n\delta-2),(2m\delta)/(2m+m\delta-2)\}$.

By (1.4) and Theorem 1.1 (a), we need to prove that $\mathcal{S}^{(2)}_{\Omega,\Phi,\Psi}=\mathcal{S}_{\Omega,\Phi,\Psi}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ only for p belongs to the open interval $(\nu(n,m),2)$. To this end, by the arguments employed in the proof of Theorem 1.1 (a) and (4.6), we notice that it suffices to show that

$$\left\|T^{(0)}_{l,s,\Omega}(f)\right\|_p\le C_p\lambda^{-\beta_p|l|}\lambda^{-\beta_p|s|}\left\|f\right\|_p \tag{5.1}$$

for $p\in(\nu(n,m),2)$. We notice that by a simple change of variables we have

$$T^{(0)}_{l,s,\Omega}f(x,y)=\left(\sum_{k,j\in\mathbf{Z}}\int_{J_\Phi\times\ J_\Psi}\left|Y_{k,j,l,s,t,r}f(x,y)\right|^2\frac{dtdr}{tr}\right)^{1/2},$$

where

$$\begin{aligned}&Y_{k,j,l,s,t,r}f(x,y)\\&=\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega(\xi,\eta)((\Gamma^{(0)}_{k+l,\Phi}\otimes\Gamma^{(0)}_{j+s,\Psi})*f)\\&\quad\times(x-2^kt\xi,y-2^jr\eta)d\sigma(\xi)d\sigma(\eta),\end{aligned}$$

$J_\Phi=[\Phi(1),\Phi(2))$, and $J_\Psi=[\Psi(1),\Psi(2))$.

We notice that, to prove $T^{(0)}_{l,s,\Omega}(f)\in L^p(\mathbf{R}^n\times\mathbf{R}^m)$, it suffices to show that $Y_{k,j,l,s,t,r}f(x,y)\in L^p\left\{l^2\left(l^2\left[L^2\left(J_\Phi\times\ J_\Psi,\frac{dtdr}{tr}\right),k\right],j\right),dxdy\right\}$. By a duality argument, there is a function $h=h_{k,j}(x,y,t,r)$ satisfying $\|h\|\le 1$ and

$$h\in L^{p'}\left\{l^2\left(l^2\left[L^2\left(J_\Phi\times\ J_\Psi,\frac{dtdr}{tr}\right),k\right],j\right),dxdy\right\}$$

such that

$$\begin{aligned}&\left\|T^{(0)}_{l,s,\Omega}(f)\right\|_p\\&=\int_{\mathbf{R}^n\times\mathbf{R}^m}\sum_{k,j\in\mathbf{Z}}\int_{J_\Phi\times\ J_\Psi}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\Omega(\xi,\eta)\left|h_{k,j}(x,y,t,r)\right|\\&\quad\times((\Gamma^{(0)}_{k+l,\Phi}\otimes\Gamma^{(0)}_{j+s,\Psi})*f)(x-2^kt\xi,y-2^jr\eta)d\sigma(\xi)d\sigma(\eta)\frac{dtdr}{tr}dxdy\end{aligned}$$

$$= \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,j\in\mathbf{Z}} \int_{J_\Phi\times\ J_\Psi} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} ((\Gamma^{(0)}_{k+l,\Phi}\otimes\Gamma^{(0)}_{j+s,\Psi})*f)(x,y)$$

$$\times\ \Omega(\xi,\eta)h_{k,j}(x+2^k t\xi, y+2^j r\eta, r, t)d\sigma(\xi)d\sigma(\eta)\frac{dtdr}{tr}dxdy$$

$$\leq \left\|(R(h))^{1/2}\right\|_{p'} \left\| \left(\sum_{k,j\in\mathbf{Z}} \left|(\Gamma^{(0)}_{k+l,\Phi}\otimes\Gamma^{(0)}_{j+s,\Psi})*f\right|^2 \right)^{1/2} \right\|_p ,$$

where

$$R(h)(x,y) = \sum_{k,j\in\mathbf{Z}} (R_{k,j}h(x,y))^2 ,$$

and

$$R_{k,j}h(x,y) = \int_{J_\Phi\times\ J_\Psi} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} \Omega(\xi,\eta)$$

$$\times\ h_{k,j}(x+2^k t\xi, y+2^j r\eta, r, t)d\sigma(\xi)d\sigma(\eta)\frac{dtdr}{tr}.$$

By the Littlewood-Paley theory we have

$$\left\|T^{(0)}_{l,s,\Omega}(f)\right\|_p \leq C_p \|f\|_p \left\|(R(h))^{1/2}\right\|_{p'} .$$

Since $\left\|(R(h))^{1/2}\right\|_{p'} = \|R(h)\|^{1/2}_{p'/2}$ and $p' > 2$, there is a function $w \in L^{(p'/2)'}(\mathbf{R}^n\times\mathbf{R}^m)$ such that $\|w\|_{(p'/2)'} \leq 1$ and

$$\|R(h)\|_{p'/2} = \int_{\mathbf{R}^n\times\mathbf{R}^m} \sum_{k,j\in\mathbf{Z}} (R_{k,j}h(x,y))^2 \left|w(x,y)\right| dxdy.$$

By Shwarz inequality and Lemma 2.9, we get

$$(R_{k,j}h(x,y))^2 \leq C \int_{J_\Phi\times\ J_\Psi} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(\xi,\eta)|^{\max\{0,2-q\}} \times$$

$$\left|h_{k,j}(x+2^k t\xi, y+2^j r\eta, t, r)\right|^2 d\sigma(\xi)d\sigma(\eta)\frac{dtdr}{tr}. \tag{5.2}$$

Therefore, by (5.2) and changing variables, Fubini's theorem, Hölder's inequality, and invoking Lemma 2.10 we get

$$\|R(h)\|_{p'/2} \ \leq\ \int_{\mathbf{R}^n\times\mathbf{R}^m} \left(\sum_{k,j\in\mathbf{Z}} \int_{J_\Phi\times\ J_\Psi} |h_{k,j}(x,y,t,r)|^2 \frac{dtdr}{tr} \right)$$

$$
\begin{aligned}
&\times \left(\mathcal{M}_{SP}\left(|\tilde{w}|^{\delta/2}\right)(-x,-y)\right)^{2/\delta} dxdy \\
\leq\ & C\left\|\left(\sum_{k,j\in\mathbf{Z}}\int_{J_\Phi\times\ J_\Psi}|h_{k,j}(x,y,t,r)|^2\frac{dtdr}{tr}\right)^{1/2}\right\|_{p'}^2 \\
&\times \left\|\left(\mathcal{M}_{SP}(|\tilde{\omega}|^{\delta/2})\right)^{2/\delta}\right\|_{(p'/2)'}.
\end{aligned}
$$

By noticing that $(2/\delta)(p'/2)' > \max\{n/(n-1), m/(m-1)\}$, the choice of h and invoking Lemma 2.8 we get

$$\|R(h)\|_{p'/2} \leq C \tag{5.3}$$

which in turn implies that

$$\left\|T^{(0)}_{l,s,\Omega}(f)\right\|_p \leq C_p \|f\|_p \tag{5.4}$$

for $\nu(n,m) < p < 2$.

By interpolation between (4.14) (with $\mu = 0$ and $\tilde{b}_0$ replaced by Ω) and (5.4) we get

$$\left\|T^{(0)}_{l,s,\Omega}(f)\right\|_p \leq C_p\lambda^{-\beta_p|l|}\lambda^{-\beta_p|s|}\|f\|_p \tag{5.5}$$

for $\nu(n,m) < p < 2$. This completes the proof of Step 1. □

Step 2. $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $\max\{q'n', q'm'\} < p < \infty$ and $\gamma = 1$.

Proof. Let

$$Q_{t,r}(x,y) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(t)\xi, y-\Psi(r)\eta)\Omega(\xi,\eta)d\sigma(\xi)d\sigma(\eta).$$

Then, by duality

$$
\begin{aligned}
&\mathcal{S}^{(1)}_{\Omega,\Phi,\Psi}f(x,y) \\
=\ & \|Q_{t,r}(x,y)\|_{L^\infty(\mathbf{R}_+\times\mathbf{R}_+,\frac{dtdr}{tr})} = \|Q_{t,r}(x,y)\|_{L^\infty(\mathbf{R}_+\times\mathbf{R}_+,dtdr)} \\
\leq\ & \sup_{t,r>0}\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}|f(x-\Phi(t)\xi, y-\Psi(r)\eta)\Omega(\xi,\eta)|\, d\sigma(\xi)d\sigma(\eta).
\end{aligned}
$$

Thus, by Hölder's inequality, we have

$$\mathcal{S}^{(1)}_{\Omega,\Phi,\Psi}f(x,y) \leq \|\Omega\|_q\left(\mathcal{M}_{SP}\left(|f|^{q'}\right)(x,y)\right)^{1/q'}.$$

Therefore, by Lemma 2.8 we obtain $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $\max\{q'n',q'm'\}<p<\infty$. This ends the proof of Step 2. □

Step 3. $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}$ is bounded on $L^p(\mathbf{R}^n\times\mathbf{R}^m)$ for $\max\{(\gamma'n\delta)/(\gamma'n+n\delta-\gamma'),(\gamma'm\delta)/(\gamma'm+m\delta-\gamma')\}<p<\infty$ and $1<\gamma<2$.

The idea of the proof of this step will be similar to that used in the nonproduct case in [4, pp. 125-126]. For $f\in C_0^\infty(\mathbf{R}^n\times\mathbf{R}^m)$, choose $\tilde{h}$ such that the L^γ-norm of $\tilde{h}(\cdot,\cdot,x,y)$ is not bigger than 1 for every $(x,y)\in\mathbf{R}^n\times\mathbf{R}^m$ and

$$\begin{aligned}&\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}f(x,y)\\&=\int_0^\infty \tilde{h}(t,r,x,y)\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(t)\xi,y-\Psi(r)\eta)\\&\quad\times\Omega(\xi,\eta)d\sigma(\xi)d\sigma(\eta)\frac{dtdr}{tr}\\&\equiv S_{\tilde{h}}f(x,y).\end{aligned}$$

Now, define the analytic family of operators $S_{\tilde{h}_z}$ by

$$\begin{aligned}&S_{\tilde{h}_z}f(x,y)\\&=\int_0^\infty \tilde{h}_z(t,r,x,y)\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} f(x-\Phi(t)\xi,y-\Psi(r)\eta)\\&\quad\times\Omega(\xi,\eta)d\sigma(\xi)d\sigma(\eta)\frac{dtdr}{tr},\end{aligned}$$

where

$$\tilde{h}_z(t,r,x,y)=\left|\tilde{h}(t,r,x,y)\right|^{(1-\frac{z}{2})\gamma} sgn\left(\tilde{h}(t,r,x,y)\right)$$

and z is a complex number. Notice that $\mathcal{S}^{(\gamma)}_{\Omega,\Phi,\Psi}f=S_{\tilde{h}}f=S_{\tilde{h}_z}f$ if $z=2(1-\frac{1}{\gamma})$.

Now, if $\mathrm{Re}(z)=0$,

$$\left\|\tilde{h}_z(\cdot,\cdot,x,y)\right\|_{L^1(\mathbf{R}_+\times\mathbf{R}_+,\frac{dtdr}{tr})}=\left\|\tilde{h}(\cdot,\cdot,x,y)\right\|^{\gamma}_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{dtdr}{tr})}\le 1$$

and hence by Step 2, we have

$$\left\|S_{\tilde{h}_z}f\right\|_p\le\left\|\mathcal{S}^{(1)}_{\Omega,\Phi,\Psi}f\right\|_p\le C_p\|f\|_p \text{ for } \max\{q'n',q'm'\}<p<\infty. \tag{5.6}$$

Also, if $\mathrm{Re}(z)=1$,

$$\left\|\tilde{h}_z(\cdot,\cdot,x,y)\right\|_{L^2(\mathbf{R}_+\times\mathbf{R}_+,\frac{dtdr}{tr})}=\left\|\tilde{h}(\cdot,\cdot,x,y)\right\|^{\gamma/2}_{L^\gamma(\mathbf{R}_+\times\mathbf{R}_+,\frac{dtdr}{tr})}\le 1$$

and hence by Step 1, we have

$$\left\|S_{\tilde{h}_z} f\right\|_p \leq \left\|S^{(2)}_{\Omega,\Phi,\Psi} f\right\|_p \leq C_p \|f\|_p \tag{5.7}$$

for $\max\{(2n\delta)/(2n+n\delta-2),(2m\delta)/(2m+m\delta-2)\} < p < \infty$.

Now, Step 3 follows by (5.6) and (5.7) and interpolating of analytic family of operators and hence the proof of Theorem 1.2 is completed.

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