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DERIVATIVES OF INNER FUNCTIONS ON EXTENSION WEIGHTED HARDY SPACES

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Abstract. We have extended the H^p space and estabilished the derivative of inner functions, Blaschke product on weighted Hardy spaces for the unit disc in complex plane.

1. Introduction

Much attention has been given to the factorization and boundary properties of functions with derivatives in H^p and B^p. In [4], G. Caughran and L. Shields showed that if the holomorphic function f is in a Hardy space, then f has a factorization $f = BSQ$, where B is Blaschke product, Q is an outer function in H^p. The singular function of $f(z)$ has the form

$$S(z) = \exp\left\{-\int \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(e^{i\theta})\right\},$$

where μ is a positive singular measure on the unit circle. We raised the questions whether there exists a singular inner function $S(z)$ with derivative $S'(z)$ in $H^{\frac{1}{2}}$. They also conjectured that the derivative of non singular inner function lies in $B^{\frac{1}{2}}$. But H. A. Allen and C. L.

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Belna [3] disproved this conjecture by giving an example of singular inner functions with derivatives in B^p for $0 < p < \frac{2}{3}$

P. Ahern and N. Clark [2] gave the condition in which the derivative of Blaschke product is a member of H^p and B^p spaces. N. Linden [7] generalized the previous argument.

P. Ahern [1] constructed A^p_q spaces which are the extension of B^p, and investigated various properties of the space. Especially, he considered derivatives of inner functions and Blaschke products on A^p_q spaces, using modulus of continuity and the moduli of the Taylor coefficients. And then he got results concerned with A^p_q spaces to which the derivative of an inner function can belong.

In this paper, we try to extend the H^p spaces and investigate the derivative of inner functions. Moreover, we fined conditions which the derivative of inner functions and Blaschke product are contained in A^p_q spaces.

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Furthermore, in 1997, K. Shibata [8] generalized the result of the P. Ahern's work and showed that if the derivative of inner function $M(z)$ belongs to A^p_q spaces, then the value of p is $\frac{2}{3} < p < 1$.

Last year, K. Shibata, A. Sakai and Y. M. Nam co-worked to extend the theorem of [8]. We try to generalize properties of the extension of A^p_q spaces and find the value of p and q which satisfies the derivative of inner functions.

Let H^p be Hardy space and B^p denote the spaces of functions $f(z)$ holomorphic in the unit disc D for which

$$\|f\|_{B_p} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |f(re^{i\theta})|(1-r)^{\frac{1}{p-2}}drd\theta$$

is finite.

If the quantity

$$M_p(f,r) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{\frac{1}{p}} \quad (0 < p < \infty)$$

is used, it can be rewritten as follows;

$$\|f\|_{B_p} = \int_0^1 (1-r)^{\frac{1}{p-2}} M_1(f,r)dr.$$

A Blaschke sequence is a (finite or infinite) sequence $\{a_n\}$ of complex numbers satisfying the conditions: $0 < |a_n| < 1$ and

$$\sum (1 - |a_n|) < \infty.$$

A Blaschke product $B(z)$ with zeros $\{a_n\}$ is a function defined by the formula

$$B(z) = \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}$$

where $\{a_n\}$ is a Blaschke sequence. We note that every Blaschke product is an inner function. The set of Blaschke products is uniformly dense in the set of inner function by the Frostman's theorem [6]. An inner function without zeros which is positive at the origin is called a singular inner product. It is well known that a singular inner function is a function $S(z)$ which has the form

$$S(z) = \exp \int_0^{2\pi} \frac{z + e^{it}}{z - e^{it}} d\mu(e^{it}),$$

where μ is a positive measure on $\overline{D}$, and singular with respect to Lebesgue measure.

Now we introduce the definition of A_q^p spaces and develop its some properties. If $f(z)$ is holomorphic in D and $0 < p < 1$ and $q > 0$, we define the weighed L^p norm by

$$\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^q (1-r)^{1/p-2} d\theta dr.$$

If this is finite, we say $f(z)$ belongs to A_q^p. Especially, $A_q^p = B^p$ when $q = 1$.

P. Ahren [1] first considered the problems that determine the derivative of inner function in A_q^p spaces.

2. Derivative of Inner Function on B^p Spaces

Fix p, $0 < p < 1$. Let B^p denote the space of function $f(z)$ holomorphic in D for which

$$||f||_{B^p} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|(1-r)^{1/p-2} M_1(f,r) dr.$$

It turns out H^p is a subspace of B^p, especially $B^p = H^p$ for $p = \frac{1}{2}$. Thus the space B^p is in respect "extended" than H^p space. For typographical reasons we shall frequently omit the superscript p in written norms, $||f||_B$ denote the norm in B^p. The following lemmas are very important to prove the theorem.

LEMMA 2.1. *Let f be in B^p. Then we claim the following:*

$$|f(z)| \leq C_p ||f||_B (1-r)^{-1/p}, \quad z \in D,$$

where C_p is a constant depend on p.

Proof. Let $R < r < 1$. Then we have

$$\begin{aligned} ||f||_B &\geq \int_R^1 (1-r)^{1/p-2} M_1(f,r) dr \\ &\geq M_1(f,R) \left(\frac{1}{p} - 1\right)^{-1} (1-R)^{1/p-1}. \end{aligned}$$

Hence

$$M_1(f,R) \leq \left(\frac{1}{p} - 1\right) ||f||_B (1-R)^{1-1/p}.$$

From this, the estimate follows by writing

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $R = \frac{1}{2}(1 + |z|)$. □

LEMMA 2.2. *Let $f_\rho(z) = f(\rho z)$ be in B^p. Then we have that $f_\rho \to f$ in B^p-norm as $\rho \to 1$.*

Proof. Given $f \in B^p$ and $\varepsilon > 0$, choose $r > 1$ such that

$$\int_R^1 (1-r)^{1/p-2} M_1(f,r)dr < \varepsilon. \tag{2.1}$$

Since $M_1(f,r)$ is an increasing function of r, (2.1) remains valid when f is replaced by f_ρ. Now choose ρ so close to 1 that $|f_\rho(z) - f(z)| < \varepsilon$ on $|z| \leq R$. Then we have

$$\int_0^R (1-r)^{1/p-2} M_1(f_\rho - f, r)dr < \varepsilon ||1||_B,$$

which, upon combining with (2.1), yields

$$||f_\rho - f||_B \leq \varepsilon ||1||_B + 2\varepsilon.$$

We, therefore, have $f_\rho \to f$ in B^p-norm as $\rho \to 1$. □

LEMMA 2.3. *H^p is a dense subset of B^p.*

LEMMA 2.4. *Let f be in H^p spaces then we have the following inequality*

$$||f||_B \leq C_p ||f||_p.$$

The properties from Lemma 2.3 and Lemma 2.4 implies that $H^p \subset B^p$, and given the norm inequality. Also, H^p contains all functions holomorphic in a bigger disc, and such functions are dense in B^p by Lemma 2.2.

If $1 < p < \infty$, it is well known that every bounded linear functional ψ in $(H^p)^*$ has a unique representation.

$$\psi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{-i\theta}) d\theta,$$

where $g \in H^q$, $q = p/(p-1)$. The following may be regarded as an extension of this result to $0 < p < 1$.

THEOREM 2.5. ([5]) Let $\psi \in (H^p)^*$, $0 < p < 1$. Then there is unique function g such that

$$\psi(f) = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(e^{-i\theta}) d\theta, \quad f \in H^p,$$

where $g(z)$ is hololmorphic in D and continuous on $\overline{D}$.

THEOREM 2.6. *B^p and H^p have the same continuous linear functionals; more precisely, Theorem 2.5 remains true if in its statements H^p is everywhere replaced by B^p.*

Proof. Let $\psi \in (B^p)^*$ be given and the associated function $g(z) = \sum b_k z^k$ as in the proof of Theorem 2.5. By Lemma 2.4, ψ is also a bounded linear functionals on H^p and hence g has desired smoothness. Furthermore, if $f(z) = \sum a_k z^k \in B^p$, then by Theorem 3.5 we have

$$\psi(f) = \lim_{\rho \to 1} \sum a_k z^k \rho^k = \lim_{\rho \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{1\theta}) g(e^{-i\theta}) d\theta \tag{2.2}$$

where $f_p \to f$ in norm, by Lemma 2.2.

Conversely let g (holomorphic and continuous) be given and suppose that g has the smoothness described in Theorem 2.5. We must show that the limit in (2.2) exists for every $f \in B^p$ and bounded by $C||f||$. The proof is identical to the proof of Theorem 2.5. □

A Blaschke sequence is a (finite or infinite) sequence $\{a_n\}$ of complex numbers satisfying the conditions: $0 < |a_n| < 1$ and

$$\sum (1 - |a_n|) < \infty.$$

A Blaschke product $B(z)$ with zeros $\{a_n\}$ is a function defined by the formula

$$B(z) = \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a}_n z}$$

where $\{a_n\}$ is a Blaschke sequence. It is well-known if zeros $\{a_n\}$ of a Blaschke product $B(z)$ satisfy the condition

$$\sum (1-|a_n|)\log\frac{1}{1-|a_n|} < \infty,$$

then $B'(z) \in B^p$ for $p=\frac{1}{2}$. The following implies that for each $p<1$ there exist infinite Blaschke products with derivative B^p.

THEOREM 2.7. *Let $B(z)$ be a Blaschke product with zeros $\{a_n\}$ such that*

$$\sum (1-|a_n|)^\alpha < \infty$$

for some α $(0<\alpha<1)$. Then $B'(z) \in B^{1/(1+\alpha)}$.

Proof. It is easily seen that

$$\begin{aligned} B'(z) =& B(z)\sum \frac{1-|a_n|^2}{(z-a_n)(1-\overline{a}_n z)} \\ =&\left(\frac{\overline{a}_1}{|a_1|}\frac{a_1-z}{1-\overline{a}_1 z}\right)\cdot\left(\frac{\overline{a}_2}{|a_2|}\frac{a_2-z}{1-\overline{a}_2 z}\right)\cdots\left(\frac{\overline{a}_n}{|a_n|}\frac{a_n-z}{1-\overline{a}_n z}\right)\cdots \\ &\cdot\left\{\frac{1-|a_1|^2}{(z-a_1)(1-\overline{a}_1 z)}+\frac{1-|a_2|^2}{(z-a_2)(1-\overline{a}_2 z)}+\cdots\right. \\ &\left.+\frac{1-|a_n|^2}{(z-a_n)(1-\overline{a}_n z)}+\cdots\right\} \\ =&\sum\frac{\beta_n(z)(1-|a_n|^2)}{(1-\overline{a}_n z)^2}, \end{aligned}$$

where $\beta_n(z) = B(z)(1-\overline{a}_n z)/(z-a_n)$, and this implies that

$$\begin{aligned} |B'(z)| &\le \sum (1-|a_n|^2)/|a-\overline{a}_n z|^2 \\ &\le 2\sum (1-|a_n|)/|a-\overline{a}_n z|^2 \end{aligned}$$

for all $|z|<1$. Therefore, for $0<r<1$,

$$\begin{aligned} \int_0^{2\pi} |B'(z)(re^{it})|dt &\le 2\sum (1-|a_n|)\int_0^{2\pi}\frac{dt}{|1-\overline{a}_n re^{it}|^2} \\ &= 4\pi\sum\frac{1-|a_n|}{1-r^2|a_n|^2}\ . \end{aligned}$$

The inequalities

$$2(1-r^2|a_n|^2) \geq 2(1-r|a_n|) \geq 2-r^2-|a_n|^2$$
$$\geq 1-r+1-|a_n|$$

implies that

$$\int_0^{2\pi} |B'(re^{it})|dt \leq 8\pi \sum \frac{1-|a_n|}{1-r+1-|a_n|} .$$

If we write $p = 1/(1+\alpha)$, then $1/p-2 = \alpha-1$; setting $1-|a_n| = d_n$, we now obtain the estimate

$$\begin{aligned}\int_0^1 \frac{d_n(1-r)^{\alpha-1}}{1-r+d_n}dr &= \int_0^1 \frac{d_n s^{\alpha-1}}{s+d_n}ds \\ &\leq \int_0^c n_s^{\alpha-1}ds + \int_{d_n}^1 d_n s^{\alpha-2}ds \\ &= \frac{d_n^\alpha}{\alpha} + \frac{d_n^\alpha - d_n}{1-\alpha} \\ &\leq \frac{d_n^\alpha}{\alpha(1-\alpha)}.\end{aligned}$$

It follows immediately that

$$||B'(z)||_B \leq \frac{4}{\alpha(1-\alpha)} \sum (1-|\alpha_n|)^\alpha.$$

□

3. A_q^p-Derivatives of Inner Functions and Blaschke Products

In this section, we will construct more extended Hardy spaces A_q^p and try to find conditions which the derivative of $M(z)$, $B(z)$ are contained in A_q^p spaces.

Now we introduce the definition of A_q^p spaces and develop its some properties. If $f(z)$ is holomorphic in D and $0 < p < 1$ and $q > 0$, we define the weighed L^p norm by

$$\frac{1}{2\pi}\int_0^1\int_0^{2\pi}|f(re^{i\theta})|^q(1-r)^{1/p-2}d\theta dr.$$

If this is finite, we say $f(z)$ belongs to A_q^p. Especially, $A_q^p = B^p$ when $q = 1$.

Here we consider the problem that determine the value of p when $M'(z)$ and $B'(z)$ are in A_q^p spaces.

If $M(z)$ is an inner function, then the following fact holds.

LEMMA 3.1. *If $M(z) = \sum a_n z^n$ is an inner function, then*

$$\int_0^1\int_0^{2\pi}|M'(re^{i\theta})|^2(1-r)^{1/p-1}d\theta dr = \sum |a_n|^2 n^{2-1/p}, \quad 0 < p < 1.$$

If $0 < r < 1$, then we have $r < 1/(1-r)$. Thus the following fact holds.

LEMMA 3.2. *For any $q > 0$, $0 < r < 1$,*

$$r^q < \frac{1}{(1-r)^q}.$$

THEOREM 3.3. *Let $M(z) = \sum_{n>k} a_n z^n$ be an inner function such that $a_n = o(\frac{1}{n})$. Then for $q = \frac{1}{2}$ and $0 < p < \frac{2}{3}$, $M'(z) \in A_q^p$.*

Proof. By Lemma 3.1 and 3.2, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^1\int_0^{2\pi}&|M'(re^{i\theta})|^{\frac{1}{2}}(1-r)^{1/p-2}d\theta dr\\ &\le \sum_{n>k} n^{\frac{1}{2}}|a_n|^{\frac{1}{2}}\int_0^1 r^{(n-1)/2}(1-r)^{-2+1/p}dr\\ &\le \sum_{n>k} n^{\frac{1}{2}}|a_n|^{\frac{1}{2}}\int_0^1 r^{\frac{1}{2}}(1-r)^{-2+1/p}dr\\ &\le \sum_{n>k} n^{\frac{1}{2}}|a_n|^{\frac{1}{2}}\int_0^1 (1-r)^{\frac{1}{2}-2+1/p}dr, \quad k=1,2,\cdots.\end{aligned}$$

Since $\int_0^1(1-r)^t dr$ is finite for any numbers $t > -1$, the proof is complete. □

In view of Theorem 3.3, we have the following restatement.

COROLLARY 3.4. *If $1/(q+1) < p < 1/q$, then $M' \in A_q^p$ if and only if $M' \in B^t$ with $t = p/(1-p(q-1))$.*

The above corollary is false if $p = 1/(q+1)$, for example, if $q = 2$ then $p = 1/3$ and

$$\begin{aligned}\iint |M'(re^{i\theta})|^2(1-r)^{-2+1/p}drd\theta &\le \sum n^2|a_n|^2\int_0^1(r^2-r^3)dr\\ &= \frac{1}{12}\sum n^2|a_n|^2\end{aligned}$$

is finite if $a_n = o\left(\frac{1}{n}\right)$, but if $q = \frac{1}{2}$ then

$$\int_0^1\int_0^{2\pi}|M'(re^{i\theta})|drd\theta$$

dose not always converge.

Next we consider the derivative of Blaschke products.

$\iint |B'(re^{i\theta})|^2 dr d\theta$ is finite if and only if $B(z)$ is a finite Blaschke products.

If $M(z)$ is an inner function and $p > 1/q$ $(1 \le q \le 2)$ then $M' \notin A_q^p$ unless $M(z)$ is a finite Blaschke.

Let us restrict our attention to infinite Blaschke product, then we have the following result.

LEMMA 3.5. ([5]) If we take the value of p $(\frac{1}{2} < p < 1)$, then we have the following:

$$\int_0^{2\pi} \frac{d\theta}{(1 - 2r\cos\theta + r^2)^p} = O\left(\frac{1}{(1-r)^{2p-1}}\right)$$

as $r \to 1$.

LEMMA 3.6. *If we take the value of p $(\frac{1}{2} < p < 1)$, then there exists a constant C such that*

$$\int_0^{2\pi} \frac{d\theta}{|1 - \overline{a}_n re^{i\theta}|^{2p}} < C(1-r)^{1-2p}$$

for $n = 1, 2, \cdots$, and all r $(0 < r < 1)$.

Proof. By Lemma 3.5,

$$\int_0^{2\pi} \frac{d\theta}{|1 - \overline{a}_n re^{i\theta}|^{2p}} = \int_0^{2\pi} \frac{d\theta}{(1 + r^2|a_n|^2 - 2r|a_n|\cos\theta)^p} < C(1-r)^{1-2p}.$$

□

Finally, we prove the following theorem using the above lemmas.

THEOREM 3.7. *Let B(z) be infinite Blaschke product with zeros $\{a_n\}$ such that*

$$\sum_n (1 - |a_n|)^q < \infty$$

for some q $(\frac{1}{2} < q < 1)$. Then for $0 < p < 1/2q$, $B' \in A_q^p$.

Proof. The derivative of $B(z)$ is given by the following formula

$$B'(z) = \sum_n \beta_n(z)(1 - |a_n|^2)/(1 - \overline{a}_n z)^2$$

where $\beta_n(z) = B(z)(1 - \overline{a}_n z)/(z - a_n)$. This implies that

$$|B'(z)| < 2 \sum_n (1 - |a_n|)/(1 - \overline{a}_n z)^2$$

for all $|z| < 1$. Since $\frac{1}{2} < q < 1$,

$$|B'(z)|^q < 2^q \sum_n (1 - |a_n|)^q/(1 - \overline{a}_n z)^{2q},$$

which, upon integrating each side and using Lemma 3.6, yields the inequality

$$\int_0^1 \int_0^{2\pi} |B'(z)(re^{i\theta})|^q (1-r)^{-2+1/p} d\theta dr$$
$$< 2^q C \sum_n (1 - |a_n|)^q \int_0^1 (1-r)^{-1-2q+1/p} dr.$$

Since $0 < p < 1/2q$, it follows that $-1 - 2q + 1/p > -1$. Thus the proof is complete. ❑

COROLLARY 3.8. *Let $B(z)$ be finite Blaschke product with zeros $\{a_n\}$ such that*

$$\sum_n (1 - |a_n|)^q < \infty$$

for some q with $\frac{2}{3} < q < 1$. Then we have, for $0 < p < \frac{1}{2q}$, $B' \in A_q^p$.

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