ON THE GEODESIC SPHERES OF THE 3-DIMENSIONAL HEISENBERG GROUPS

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ABSTRACT. Let $\mathbb{H}^3$ be the 3-dimensional Heisenberg group equipped with a left-invariant metric. In this paper, We characterize the Gaussian curvatures of the geodesic spheres on $\mathbb{H}^3$.

1. Introduction

Let $\mathcal{N}$ be a 2-step nilpotent Lie algebra with an inner product $<,>$ and N be its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $<,>$ on $\mathcal{N}$. The center of $\mathcal{N}$ is depaperd by $\mathcal{Z}$. Then $\mathcal{N}$ can be expressed as the direct sum of $\mathcal{Z}$ and its orthogonal complement $\mathcal{Z}^\perp$.

For $Z \in \mathcal{Z}$, a skew symmetric linear transformation $j(Z) : \mathcal{Z}^\perp \to \mathcal{Z}^\perp$ is defined by $j(Z)X = (\mathrm{ad}X)^*Z$ for $X \in \mathcal{Z}^\perp$. Or, equivalently,

$$\langle j(Z)X, Y \rangle = \langle [X,Y], Z \rangle$$

for $X, Y \in \mathcal{Z}^\perp$. A 2-step nilpotent Lie group N is said to be of Heisenberg type if

$$j(Z)^2 = -|Z|^2 \, \mathrm{id}$$

for all $Z \in \mathcal{Z}$.

The Heisenberg groups are examples of Heisenberg type. That is, let $n \geq 1$ be any integer and let $\{X_1, \cdots, X_n, Y_1, \cdots, Y_n\}$ be any basis

of $R^{2n} = \mathcal{V}$. Let $\mathcal{Z}$ be an one dimensional vector space spanned by $\{Z\}$. Define

$$[X_i, Y_i] = -[Y_i, X_i] = Z$$

for any $i = 1, 2, \cdots, n$ with all other brackets are zero. Give on $\mathcal{N} = \mathcal{V} \oplus \mathcal{Z}$ the inner product such that the vectors $\{X_i, Y_i, Z | i = 1, 2, \cdots, n\}$ form an orthonormal basis. The simply connected 2-step nilpotent group of Heisenberg type, N which is determined by $\mathcal{N}$ and equipped with a left-invariant metric induced by the inner product in $\mathcal{N}$ is called the $(2n + 1)$-dimensional Heisenberg group, which is denoted by $\mathbb{H}^{2n+1}$.

In this paper, we characterize the Gaussian curvature of the geodesic spheres on the 3-dimensional Heisenberg group $\mathbb{H}^3$ as follows:

PROPOSITION 1. *Let e be the identity of $\mathbb{H}^3$ and $0 < r < 2\pi$. And let $S_e(r)$ be the geodesic sphere with center e and radius r on $\mathbb{H}^3$ and $p = \gamma(r) \in S_e(r)$ where $\gamma(t)$ is a unit speed geodesic with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$. Then, the Gaussian curvature $K(p)$ on $S_e(r)$ at p is given by following equation*

$$K(p) = -\frac{1}{4} + \frac{|Z_0|^2(1 - (1 - |Z_0|^2)\cos(|Z_0|r))}{2(1 - \cos(|Z_0|r)) - (1 - |Z_0|^2)|Z_0|r\sin(|Z_0|r)}$$

PROPOSITION 2. *The Gaussian curvature of the geodesic spheres in $\mathbb{H}^3$ is greater than $-\frac{1}{4}$.*

2. Preliminaries

Throughout this section, $\mathcal{N}$ is depaperd by a 2-step nilpotent Lie algebra with an inner product $<,>$ and N its unique simply connected 2-step nilpotent Lie group with the left invariant metric induced by $<,>$ on N.

Let ∇ be unique Riemannian connection on $\mathcal{N}$ If ξ_1, ξ_2 and ξ_3 are left-invariant vector fields, then the formula of the covariant derivative,

$$< \xi_3, \nabla_{\xi_1} \xi_2 >$$

$$= \frac{1}{2}\{\xi_1 < \xi_2, \xi_3 > + < \xi_1, [\xi_3, \xi_2] > + \xi_2 < \xi_1, \xi_3 >$$

$$+ < \xi_2, [\xi_3, \xi_1] > -\xi_3 < \xi_2, \xi_1 > - < \xi_3, [\xi_2, \xi_1] >\}$$

can be reduced to

$$< \xi_3, \nabla_{\xi_1} \xi_2 >= \frac{1}{2} \{ < \xi_1, [\xi_3, \xi_2] > + < \xi_2, [\xi_3, \xi_1] > - < \xi_3, [\xi_2, \xi_1] > \}.$$

Using this, the covariant derivatives on the 2-step nilpotent Lie group N are given as follows:

LEMMA 3. [3] *For a 2-step nilpotent Lie group N with a left invariant metric, the following hold.*
(1) $\nabla_X Y = \frac{1}{2}[X, Y]$ *for* $X, Y \in \mathcal{Z}^\perp$
(2) $\nabla_X Z = \nabla_Z X = -\frac{1}{2}j(Z)X$ *for* $X \in \mathcal{Z}^\perp$ *and* $Z \in \mathcal{Z}$
(3) $\nabla_Z Z^* = 0$ *for* $Z, Z^* \in \mathcal{Z}$.

Let $\gamma(t)$ be a curve in N such that $\gamma(0) = e$ (identity element in N) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Since $\exp : \mathcal{N} \to N$ is a diffeomorphism([7]), the curve $\gamma(t)$ can be expressed uniquely by $\gamma(t) = \exp(X(t) + Z(t))$ with

$$X(t) \in \mathcal{Z}^\perp, \quad X'(0) = X_0, \quad X(0) = 0$$
$$Z(t) \in \mathcal{Z}, \quad Z'(0) = Z_0, \quad Z(0) = 0.$$

A. Kaplan([4],[5]) shows that the curve $\gamma(t)$ is a geodesic in N if and only if

$$X''(t) = j(Z_0)X'(t),$$
$$Z'(t) + \frac{1}{2}[X'(t), X(t)] \equiv Z_0.$$

LEMMA 4. [3] *Let N be a simply connected 2-step nilpotent Lie group with a left invariant metric, and let $\gamma(t)$ be a geodesic of N with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^\perp$ and $Z_0 \in \mathcal{Z}$. Then*

$$\gamma'(t) = dl_{\gamma(t)}(X'(t) + Z_0), t \in R$$

where $X'(t) = e^{tj(Z_0)}X_0$ and $l_{\gamma(t)}$ is the left translation by $\gamma(t)$.

Throughout this paper, different tangent spaces will be identified with $\mathcal{N}$ via left translation. So, by Lemma above, we can consider $\gamma'(t)$ as

$$\gamma'(t) = X'(t) + Z_0 = e^{tj(Z_0)}X_0 + Z_0.$$

3. Main Results

We introduce the concepts of shape operators and Gaussian curvatures in the submanifolds theory. Since the geodesic spheres are the submanifolds with codimension one, we will restrict our attention to the case of the submanifolds with codimension one.

Let $\bar{M}$ be a Riemannian manifold with the metric $<,>$, its induced unique Riemannian connection $\bar{\nabla}$ and M a Riemannian submanifold of codimension one in $\bar{M}$.

For $p \in M$ and a normal vector η to $T_p(M)$, the shape operator

$$S_p : T_p(M) \to T_p(M)$$

is defined by

$$S_p(x) = -(\bar{\nabla}_x N)^T \text{ for any } x \in T_p(M),$$

where N is a local extension of η normal to M and ^{T} denotes the tangential component to $T_p(M)$. It is easy to show that if η and its extension N are unit vector and unit vector field, then the shape operator is given by

$$S_p(x) = -\bar{\nabla}_x N \text{ for any } x \in T_p(M).$$

The shape operator S_p is symmetric, so there exists an orthonormal basis of eigenvectors $\{e_1, e_2, \cdots, e_n\}$ with real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$. We say that the e_i are principal directions and λ_i are principal curvatures of M at p. The determinant of shape operator

$$\det(S_p) = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$$

is called the Gaussian curvature of shape operator S_p.

LEMMA 5. *Let $\bar{M}$ be a Riemannian manifold, $o \in \bar{M}$ and M the geodesic sphere with center o and radius $r > 0$ and $\gamma(t)$ be an unit speed geodesic with $\gamma(0) = o$. Let $J(t)$ be a Jacobi vector fields along γ with $J(0) = 0$, which is normal to γ and S_p the shape operator of M at $p = \gamma(r)$. Then, we have that $S_p(J(r)) = -J'(r)$.*

Proof. Since $J(t)$ is normal to $\gamma(t)$, $J'(t)$ is also normal to $\gamma(t)$. Let $J(t)$ be a Jacobi vector fields along γ with $J(0) = 0$, which is normal to γ. Then, there exists a geodesic variation $H : [0, r] \times (-\varepsilon, \varepsilon) \to \bar{M}$ of γ with the variation field $J(t)$.

By the symmetry Lemma(see p.68 of [2]), we see that

$$\bar{\nabla}_{\frac{\partial H}{\partial s}}\frac{\partial H}{\partial t} = \bar{\nabla}_{\frac{\partial H}{\partial t}}\frac{\partial H}{\partial s}.$$

Hence, we have that

$$S_p(J(r)) = -(\bar{\nabla}_{\frac{\partial H}{\partial s}}\frac{\partial H}{\partial t})_{(r,0)} = -(\bar{\nabla}_{\frac{\partial H}{\partial t}}\frac{\partial H}{\partial s})_{(r,0)} = -J'(r).$$

□

Let $\mathbb{H}^3$ be the 3 dimensional Heisenberg group with a left invariant metric and $\mathcal{N}$ its Lie algebra. Let $\gamma(t)$ be an unit speed geodesic on $\mathbb{H}^3$ with $\gamma(0) = e$(the identity element of $\mathbb{H}^3$) and $\gamma'(0) = X_0 + Z_0$ where $X_0 \in \mathcal{Z}^{\perp}$ and $Z_0 \in \mathcal{Z}$. Assume that $X_0 \neq 0$ and $Z_0 \neq 0$. Since

$$\{X_0 + Z_0, \frac{|Z_0|}{|X_0|}X_0 - \frac{|X_0|}{|Z_0|}Z_0, \frac{1}{|Z_0||X_0|}j(Z_0)X_0\}$$

is an ortonormal set in $\mathcal{N}$, it is easy to show the following Lemma.

LEMMA 6. *Assume that* $X_0 \neq 0$ *and* $Z_0 \neq 0$. *Let*

$$e_1(t) = \frac{|Z_0|}{|X_0|}X'(t) - \frac{|X_0|}{|Z_0|}Z_0,$$

$$e_2(t) = \frac{1}{|Z_0||X_0|}j(Z_0)X'(t).$$

Then, $\{\gamma'(t), e_1(t), e_2(t)\}$ *is an orthonormal frame along* $\gamma(t)$ *on* $\mathbb{H}^3$ *such that*
(1) $\nabla_{\gamma'(t)}e_1(t) = \frac{1}{2}e_2(t)$
(2) $\nabla_{\gamma'(t)}e_2(t) = -\frac{1}{2}e_1(t)$
Or simply,

$$\begin{bmatrix} e_1'(t) \\ e_2'(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

The Jacobi fields on $\mathbb{H}^3$ are completely characterized as you see in the following Proposition. We will use this explicit formula of the Jacobi fields to find the Gaussian curvatures of the geodesic spheres on $\mathbb{H}^3$.

PROPOSITION 7. [1],[6] *Assume that $X_0 \neq 0$ and $Z_0 \neq 0$. If $J(t)$ is a normal Jacobi field along γ in $\mathbb{H}^3$ with $J(0) = 0$, then*

$$J(t) = (c_1(\sin(|Z_0|t) - (1 - |Z_0|^2)|Z_0|t) + c_2(1 - \cos(|Z_0|t)))e_1(t)$$
$$+ (c_1|Z_0|(\cos(|Z_0|t) - 1) + c_2|Z_0|\sin(|Z_0|t))e_2(t)$$

where c_1, c_2 are arbitrary constants.

The following Proposition is a slight modification of Proposition 7, which is useful.

PROPOSITION 8. *For each $k = 1, 2$, let $J_k(t)$ be the Jacobi fields with $J_k(0) = 0, J_k'(0) = e_k(0)$. Then, we have that*

$$\begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix} = B(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

where

$$B(t) = \frac{1}{|Z_0|^3} \begin{bmatrix} \sin(|Z_0|t) - (1 - |Z_0|^2)|Z_0|t & |Z_0|(\cos(|Z_0|t) - 1) \\ |Z_0|(1 - \cos(|Z_0|t)) & |Z_0|^2 \sin(|Z_0|t) \end{bmatrix}$$

Proof. Let $J(t)$ be a normal Jacobi field along $\gamma(t)$ with $J(0) = 0$. Then, by Proposition 6, we can represent $J(t)$ as follow.

$$J(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} B(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

Since $B(0) = 0$ and $B'(0) = I$, we have that

$$J'(0) = c_1 e_1(0) + c_2 e_2(0).$$

Just letting $J'(0) = e_k(0)$ for each $k = 1, 2$, we completes the proof. $\square$

PROPOSITION 9. [8] *In the Heisenberg group $\mathbb{H}^3$, the first conjugate loci and the cut loci are equal. Moreover,*
(1) *If $|Z_0| \neq 0$, then $\gamma(\frac{2\pi}{|Z_0|})$ is the cut point of e along γ.*
(2) *If $|Z_0| = 0$, then e has no cut point along γ.*

Since $|Z_0| \leq 1$, by proposition 9, we see that the geodesic spheres $S_e(r)$ have meaning on $\mathbb{H}^3$ if and only if $r \leq 2\pi$. So, we consider the geodesic spheres $S_e(r)$ with the radius $r \leq 2\pi$. Direct calculation gives the following Lemma.

LEMMA 10.

$$\det(B(t)) = \frac{1}{|Z_0|^4}\{2(1 - \cos(|Z_0|t)) - (1 - |Z_0|^2)|Z_0|t\sin(|Z_0|t)\}$$

Finally, we have the Gaussian curvature on the geodesic spheres of the Heisenberg group $\mathbb{H}^3$ that is the main result of this paper.

PROPOSITION 11. *Let e be the identity of $\mathbb{H}^3$ and $0 < r < 2\pi$. And let $S_e(r)$ be the geodesic sphere with center e and radius r on $\mathbb{H}^3$ and $p = \gamma(r) \in S_e(r)$ where $\gamma(t)$ is a unit speed geodesic with $\gamma(0) = e$ and $\gamma'(0) = X_0 + Z_0$. Then, the Gaussian curvature $K(p)$ on $S_e(r)$ at p is given by following equation*

$$K(p) = -\frac{1}{4} + \frac{|Z_0|^2(1 - (1 - |Z_0|^2)\cos(|Z_0|r))}{2(1 - \cos(|Z_0|r)) - (1 - |Z_0|^2)|Z_0|r\sin(|Z_0|r)}$$

Proof. By Proposition 8, we know that

$$\begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix} = B(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

Since

$$\begin{bmatrix} J_1'(t) \\ J_2'(t) \end{bmatrix} = B'(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} + B(t) \begin{bmatrix} e_1'(t) \\ e_2'(t) \end{bmatrix}.$$

by Lemma 6, we have that

$$\begin{bmatrix} J_1'(t) \\ J_2'(t) \end{bmatrix}$$

$$= B'(t) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} + B(t) \begin{bmatrix} e_1'(t) \\ e_2'(t) \end{bmatrix}$$

$$= (B'(t) + \frac{1}{2}B(t) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$$

Since

$$\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix} = B(t)^{-1} \begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix},$$

we obtain that

$$\begin{bmatrix} J_1'(t) \\ J_2'(t) \end{bmatrix} = (B'(t) + \frac{1}{2}B(t)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})B(t)^{-1}\begin{bmatrix} J_1(t) \\ J_2(t) \end{bmatrix}$$

Since the shape operator S_p of the geodesic sphere $S_e(r)$ is given by

$$S_p(J(r)) = -J'(r),$$

the Gaussian curvature $K(p)$ on $S_e(r)$ at $p = \gamma(r)$ is

$$K(p) = \det(S_p)$$

$$= \frac{\det(B'(r) + \frac{1}{2}B(r)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})}{\det(B(r))}$$

Direct calculations give that

$$\det(B'(r) + \frac{1}{2}B(r)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix})$$

$$= -\frac{1}{4}\det(B(r)) + \frac{1}{|Z_0|^2}(1 - (1 - |Z_0|^2)\cos(|Z_0|r)$$

Hence, the Gaussian curvatue $K(p)$ is given by

$$K(p) = -\frac{1}{4} + \frac{|Z_0|^2(1 - (1 - |Z_0|^2)\cos(|Z_0|r))}{2(1 - \cos(|Z_0|r)) - (1 - |Z_0|^2)|Z_0|r\sin(|Z_0|r)}$$

$\square$

PROPOSITION 12. *The Gaussian curvature of the geodesic spheres in $\mathbb{H}^3$ is greater than $-\frac{1}{4}$.*

Proof. First, assume that $0 < |Z_0| < 1$ and $0 \le r \le 2\pi$. Since $|Z_0|^2(1 - (1 - |Z_0|^2)\cos(|Z_0|r)) > 0$, it is sufficient to show that

$$2(1 - \cos(|Z_0|r)) - (1 - |Z_0|^2)|Z_0|r\sin(|Z_0|r)$$

is positive for $0 < |Z_0| < 1$ and $0 \le r \le 2\pi$.
Since

$$2(1 - \cos(|Z_0|r)) - (1 - |Z_0|^2)|Z_0|r\sin(|Z_0|r)$$

$$= 2\sin\frac{|Z_0|r}{2}(2\sin\frac{|Z_0|r}{2} - (1 - |Z_0|^2)|Z_0|r\cos\frac{|Z_0|r}{2}),$$

it is sufficient to show that

$$f(t) = 2\sin\frac{|Z_0|t}{2} - (1 - |Z_0|^2)|Z_0|t\cos\frac{|Z_0|t}{2}$$

is positive for $0 < |Z_0| < 1$ and $0 \le t \le 2\pi$. If $\pi \le t \le 2\pi$, then it is clear that $f(t) > 0$. Suppose that $0 \le t < \pi$ and let

$$g(t) = \frac{f(t)}{\cos\frac{|Z_0|t}{2}} = 2\tan\frac{|Z_0|t}{2} - (1 - |Z_0|^2)|Z_0|t.$$

Since

$$\frac{d}{dt}\Big|_{t=0}\Big(2\tan\frac{|Z_0|t}{2}\Big) = |Z_0| > (1 - |Z_0|^2)|Z_0| = \frac{d}{dt}\Big|_{t=0}((1 - |Z_0|^2)|Z_0|t),$$

we have that $g(t) > 0$ for $0 < |Z_0| < 1$ and $0 \le t \le \pi$.

Next, if $|Z_0| = 1$, then we have $K = -\frac{1}{4} + \frac{1}{2(1-\cos r)}$, which is clearly greater than $-\frac{1}{4}$. And if $|Z_0| = 0$, then we see that

$$\lim_{|Z_0|\to 0}\Big(-\frac{1}{4} + \frac{|Z_0|^2(1 - (1 - |Z_0|^2)\cos(|Z_0|r))}{2(1 - \cos(|Z_0|r)) - (1 - |Z_0|^2)|Z_0|r\sin(|Z_0|r)}\Big)$$

$$= -\frac{1}{4} + \frac{6(2 + r^2)}{r^2(1 + r^2)},$$

which is also greater than $-\frac{1}{4}$. $\qquad\qquad\square$

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