East Asian Math. J. **21** (2005), No. 1, pp. 105–112

PURITY OF GENERALIZED INVERSE POLYNOMIAL MODULES

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Abstract. In this paper we show that we can extend the purity extension properties of left R-modules to the various generalized inverse polynomial modules.

1. Introduction

Let M be a left R-module, then the inverse polynomial $M[x^{-1}]$ can be defined as a left $R[x]$-module and we know that the polynomial module $M[x]$ and the inverse polynomial module $M[x^{-1}]$ are not isomorphic as left $R[x]$-modules([1], [3], [5]). Let S be a submonoid of the natural number $\mathbb{N}$, then we can generalized the definition of inverse polynomial module and define $M[x^{-s}]$ as a left $R[x^s]$-module([6]). In this paper we prove the purity extension properties of various generalized inverse polynomial modules.

Definition 1.1. ([4]) Let R be a ring and M be a left R-module, then $M[x^{-1}]$ is a left $R[x]$-module by

$$x(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = m_1 + m_2x^{-1} + \cdots + m_nx^{-n+1}$$

and

$$r(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \cdots + rm_nx^{-n}$$

where $r \in R$. We call $M[x^{-1}]$ an **inverse polynomial module** .

Received April 15, 2005.

2000 Mathematics Subject Classification: Primary 16E30; Secondary 13C11, 16D80.

Key words and phrases: module, pure submodule, generalized inverse polynomial module.

This paper was supported by Dong-A University Research Fund, in 2002.

We consider the natural number $\mathbb{N}$ contains 0.

DEFINITION 1.2. ([7]) Let R be a ring and M be a left R-module, and $S = \{0, k_1, k_2, \cdots\}$ be a submonoid of $\mathbb{N}$. Then $M[x^{-s}]$ is a left $R[x^s]$-module defined by

$$\begin{aligned} &x^{k_i}(m_0 + m_1x^{-k_1} + m_2x^{-k_2} + \cdots + m_nx^{-k_n}) \\ &\quad = m_1x^{-k_1+k_i} + m_2x^{-k_2+k_i} + \cdots + m_nx^{-k_n+k_i} \end{aligned}$$

and

$$\begin{aligned} &r(m_0 + m_1x^{-k_1} + m_2x^{-k_2} + \cdots + m_nx^{-k_n}) \\ &\quad = rm_0 + rm_1x^{-k_1} + rm_2x^{-k_2} + \cdots + rm_nx^{-k_n}, \end{aligned}$$

where

$$x^{-k_j+k_i} = \begin{cases} x^{-k_j+k_i} & \text{if} \quad k_j - k_i \in S \\ 0 & \text{if} \quad k_j - k_i \notin S. \end{cases}$$

For example, if $S = \{0, 2, 3, 4, \cdots\}$, then $m_0+m_2x^{-2}+m_3x^{-3}+\cdots+m_ix^{-i} \in M[x^{-s}]$ and if $S = \{0, 1, 2, 3, 4, \cdots\}$, then $M[x^{-s}] = M[x^{-1}]$. Similarly, we can define $M[[x^{-s}]]$, $M[x^s, x^{-s}]$, $M[[x^s, x^{-s}]]$ as a left $R[x^s]$-modules.

DEFINITION 1.3. ([2]) Let S be a submonoid of $\mathbb{N}$ and S contains all n in $\mathbb{N}$ lager than some n_0 in $\mathbb{N}$. Then the **conductor** of S is the largest element of $\mathbb{Z}$ not in S (where $\mathbb{Z}$ is the set of all integers).

EXAMPLE 1.4. *Let $S = \{0, 3, 4, 5, \cdots\}$, then the conductor of S is* 2.

Let $S \subset \mathbb{N}$ be a submonoid where we assume that for some $n_0 \in \mathbb{N}$, all $n \geq n_0$ are in S. S is *symmetric* if and only if it has a conductor c, such that the function $n \mapsto c - n$ from $\mathbb{Z}$ to $\mathbb{Z}$ maps S bijectively to its complement in $\mathbb{Z}$.

EXAMPLE 1.5. *$S = \{0, 2, 3, 4, 5, \cdots\}$ is a symmetric submonoid with the conductor* 1.

EXAMPLE 1.6. *$S = \{0, 3, 4, 5, 6, \cdots\}$ is a nonsymmetric submonoid with the conductor* 2.

THEOREM 1.7. ([6]) Let M be a left R-module and S be a symmetric submonoid. Then there is an exact sequence

$$0 \to M[x^s] \to M[x, x^{-1}] \to M[x^{-s}] \to 0$$

as $R[x^s]$-modules.

An exact sequence of left R-modules

$$0 \to A' \xrightarrow{\lambda} A \to A'' \to 0$$

is **pure exact** if, for every right R-module B, we have exactness of

$$0 \longrightarrow B \otimes A' \xrightarrow{1 \otimes \lambda} B \otimes A \longrightarrow B \otimes A'' \longrightarrow 0.$$

We say that $\lambda A'$ is a **pure submodule** of A in this case([9]).

EXAMPLE 1.8. *A split exact sequence $0 \to A' \xrightarrow{f} A \to A'' \to 0$ is a pure exact.*

Let M be a R-module, then the character module M^+ of M is defined by $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

Let M, N be left R-modules. Then $f : N^+ \longrightarrow M^+$ having a section means that there exist $s : M^+ \longrightarrow N^+$ such that $f \circ s = id_{M^+}$.

THEOREM 1.9. ([8]) $M \subset N$ is pure as left R-modules if and only if $f : N^+ \to M^+$ has a section.

THEOREM 1.10. *If $M \subset N$ is pure as left R-modules, then r divides a in N implies r divides a in M.*

Proof. Let $M \subset N$ be pure and $I = (r)$, $r \in R$, then we have

$$0 \to R/I \otimes_R M \to R/I \otimes_R N.$$

But since $R/I \otimes_R M \cong M/rM$ and $R/I \otimes_R N \cong N/rN$,

$$0 \to M/rM \to N/rN.$$

Therefore, $M \subset N$ is pure implies that if r divides a in N, then r divides a in M. □

EXAMPLE 1.11. *$M[x] \subset M[x, x^{-1}]$ is not pure as a left $R[x]$-module.*

Proof. Let $m \in M$, then since $x(mx^{-1}) = m$. But x does not divide m in $M[x]$. Hence, $M \subset M[x, x^{-1}]$ is not pure as a left $R[x]$-module. □

2. Purity Extensions

THEOREM 2.1. *Let M, N be left R-modules and S be a submonoid of N. Then*

$$Hom_{\mathbb{Z}}(M[x^s], N) \cong Hom_{\mathbb{Z}}(M, N)[[x^{-s}]]$$

as left $R[x^s]$-modules.

Proof. Let $S = \{0, k_1, k_2, \cdots\}$. Let $\phi \in Hom_{\mathbb{Z}}(M[x^s], \mathbb{N})$ and define $d_{Mx^{k_i}} : M \to Mx^{k_i}$ by $d_{Mx^{k_i}}(m) = mx^{k_i}$ and $\phi|_{Mx^{k_i}} : Mx^{k_i} \to \mathbb{N}$. Let $f_{k_i} = \phi|_{Mx^{k_i}} \circ d_{Mx^{k_i}}$ for each $x^{k_i} = 0, k_1, k_2, k_3, \cdots$. Define

$$\psi : Hom_{\mathbb{Z}}(M[x^s], N) \to Hom_{\mathbb{Z}}(M, N)[[x^{-s}]]$$

by $\psi(\phi) = f_0 + f_{k_1}x^{-k_1} + f_{k_2}x^{-k_2} + \cdots$. Then easily ψ is a well-defined group homomorphism. And $ker(\psi) = 0$, so that ψ is injective. Let

$$f_0 + f_{k_1}x^{-k_1} + f_{k_2}x^{-k_2} + \cdots \in Hom_{\mathbb{Z}}(M, N)[[x^{-s}]].$$

Choose $\phi \in Hom_{\mathbb{Z}}(M[x^s], N)$ such that

$$\phi(m_0 + m_{k_1}x^{k_1} + \cdots + m_{k_i}x^{k_i}) = f_0(m_0) + f_{k_1}(m_{k_1}) + \cdots + f_{k_i}(m_{k_i}).$$

Then

$$\psi(\phi) = \sum_{n \in S} f_n x^{-n}.$$

Therefore, ψ is surjective. Hence, $Hom_{\mathbb{Z}}(M[x^s], N)$ and $Hom_{\mathbb{Z}}(M, N)$ $[[x^{-s}]]$ are isomorphic as left $R[x^s]$-modules. □

Similarly, we can get the following two Theorems.

THEOREM 2.2. *Let M be a left R-modules and S be a submonoid of N. Then*

$$Hom_{\mathbb{Z}}(M[x^{-s}], N) \cong Hom_{\mathbb{Z}}(M, N)[[x^s]]$$

as left $R[x^s]$-modules.

THEOREM 2.3. *Let M, N be left R-modules and S be a submonoid of N. Then*

$$Hom_{\mathbb{Z}}(M[x^s, x^{-s}], N) \cong Hom_{\mathbb{Z}}(M, N)[[x^s, x^{-s}]]$$

as left $R[x^s]$-modules.

THEOREM 2.4. *If $M \subset N$ is pure as left R-modules and $S = \{0, k_1, k_2, \cdots\}$, then $M[x^s] \subset N[x^s]$ is pure as left $R[x^s]$-modules.*

Proof. Suppose $M \subset N$ is pure as left R-modules, then $f : N^+ \to M^+$ has a section and by Theorem 2.1, $(M[x^s])^+ \cong M^+[[x^{-s}]]$. Let $f^* : N^+[[x^{-s}]] \to M^+[[x^{-s}]]$ be defined by

$$f^*(\psi_0+\psi_{k_1}x^{-k_1}+\psi_{k_2}x^{-k_2}+\cdots) = f(\psi_0)+f(\psi_{k_1})x^{-k_1}+f(\psi_{k_2})x^{-k_2}+\cdots.$$

Since $f : N^+ \to M^+$ has a section $g : M^+ \to N^+$ such that $g \circ f = id_{M^+}$. Define

$$g^*(\phi_0+\phi_{k_1}x^{-k_1}+\phi_{k_2}x^{-k_2}+\cdots) = g(\phi_0)+g(\phi_{k_1})x^{-k_1}+g(\phi_{k_2})x^{-k_2}+\cdots.$$

Then

$$\begin{aligned}
&(f^*\circ g^*)(\phi_0 + \phi_{k_1}x^{-k_1} + \phi_{k_2}x^{-k_2} + \cdots) \\
&\quad = f^*(g^*(\phi_0 + \phi_{k_1}x^{-k_1} + \phi_{k_2}x^{-k_2} + \cdots)) \\
&\quad = f^*(g(\phi_0) + g(\phi_{k_1})x^{-k_1} + g(\phi_{k_2})x^{-k_2} + \cdots) \\
&\quad = f(g(\phi_0)) + f(g(\phi_{k_1}))x^{-k_1} + f(g(\phi_{k_2}))x^{-k_2} + \cdots \\
&\quad = (f\circ g)(\phi_0) + (f\circ g)(\phi_{k_1})x^{-k_1} + (f\circ g)(\phi_{k_2})x^{-k_2} + \cdots \\
&\quad = \phi_0 + \phi_{k_1}x^{-k_1} + \phi_{k_2}x^{-k_2} + \cdots .
\end{aligned}$$

Therefore, $f^* : N^+[[x^{-s}]] \to M^+[[x^{-s}]]$ has a section g^* such that $f^* \circ g^* = id_{M^+[[x^{-s}]]}$. Hence, $M[x^s] \subset N[x^s]$ is pure as left $R[x^s]$-modules. □

Theorem 2.5. *If $M \subset N$ is pure as left R-modules and S be submonoid $\mathbb{N}$, then $M[x^s, x^{-s}] \subset N[x^s, x^{-s}]$ is pure as left $R[x^s]$-modules.*

Proof. Suppose $M \subset N$ is pure as left R-modules, then $f : N^+ \to M^+$ has a section and by Theorem 2.3, $(N[x^s, x^{-s}])^+ \cong N^+[[x^s, x^{-s}]]$ and $(M[x^s, x^{-s}])^+ \cong M^+[[x^s, x^{-s}]]$. Let $f^* : N^+[[x^s, x^{-s}]] \to M^+[[x^s, x^{-s}]]$ be defined by

$$\begin{aligned}
&f^*(\cdots + n_{-k_1}x^{-k_1} + n_0 + n_{k_1}x^{k_1} + \cdots) \\
&\qquad\qquad = \cdots + f(n_{-k_1})x^{-k_1} + f(n_0) + f(n_{k_1})x^{k_1} + \cdots .
\end{aligned}$$

Define $g^* : M^+[[x^s, x^{-s}]] \to N^+[[x^s, x^{-s}]]$ by

$$\begin{aligned}
&g^*(\cdots + m_{-k_1}x^{-k_1} + m_0 + m_{k_1}x^{k_1} + \cdots) \\
&\qquad\qquad = \cdots + g(n_{-k_1})x^{-k_1} + g(m_0) + g(m_{k_1})x^{k_1} + \cdots .
\end{aligned}$$

Thus

$$\begin{aligned}
&(f^* \circ g^*)(\cdots + m_{-k_1}x^{-k_1} + m_0 + m_{k_1}x^{k_1} + \cdots) \\
&\quad = f^*(g^*(\cdots + m_{-k_1}x^{-k_1} + m_0 + m_{k_1}x^{k_1} + \cdots)) \\
&\quad = f^*(\cdots + g(m_{-k_1})x^{-k_1} + g(m_0) + g(m_{k_1})x^{k_1} + \cdots) \\
&\quad = \cdots + f(g(m_{-k_1}))x^{-k_1} + f(g(m_0)) + f(g(m_{k_1}))x^{k_1} + \cdots \\
&\quad = \cdots + (f \circ g)(m_{-k_1})x^{-k_1} + (f \circ g)(m_0) + (f \circ g)(m_{k_1})x^{k_1} + \cdots \\
&\quad = \cdots + m_{-k_1}x^{-k_1} + m_0 + m_{k_1}x^{k_1} + \cdots .
\end{aligned}$$

Therefore, $f^* : N^+[[x^s, x^{-s}]] \to M^+[[x^s, x^{-s}]]$ has a section g^* such that $f^* \circ g^* = id_{M^+[[x^s,x^{-s}]]}$. Hence, $M[x^s, x^{-s}] \subset N[x^s, x^{-s}]$ is pure as left $R[x^s]$-modules. □

Theorem 2.6. *If $M \subset N$ is pure as left R-modules, then $M[x^{-s}] \subset N[x^{-s}]$ is pure as left $R[x^s]$-modules.*

Proof. Suppose $M \subset N$ is pure as left R-modules, then by $f : N^+ \to M^+$ has a section. By the Theorem 2.2, $Hom_{\mathbb{Z}}(M[x^{-s}], N) \cong Hom_{\mathbb{Z}}(M, N)\ [[x^s]]$. That is, $(M[x^{-s}])^+$ and $M^+[[x^s]]$ are isomorphic as $R[x]$-modules. Let $f^* : N^+[[x^s]] \to M^+[[x^s]]$ be defined by

$$f^*(\psi_0 + \psi_{k_1}x^{k_1} + \psi_{k_2}x^{k_2} + \cdots) = f(\psi_0) + f(\psi_{k_1})x^{k_1} + f(\psi_{k_2})x^{k_2} + \cdots .$$

Since $f : N^+ \to M^+$ has a section there exists $g : M^+ \to N^+$ such that $g \circ f = id_{M^+}$. Define

$$g^*(\phi_0 + \phi_{k_1}x^{k_1} + \phi_{k_2}x^{k_2} + \cdots) = g(\phi_0) + g(\phi_{k_1})x^{k_1} + g(\phi_{k_2})x^{k_2} + \cdots .$$

Then

$$\begin{aligned}
&(f^* \circ g^*)(\phi_0 + \phi_{k_1}x^{k_1} + \phi_{k_2}x^{k_2} + \cdots) \\
&\quad = f^*(g^*(\phi_0 + \phi_{k_1}x^{k_1} + \phi_{k_2}x^{k_2} + \cdots)) \\
&\quad = f^*(g(\phi_0) + g(\phi_{k_1})x^{k_1} + g(\phi_{k_2})x^{k_2} + \cdots) \\
&\quad = f(g(\phi_0)) + f(g(\phi_{k_1}))x^{k_1} + f(g(\phi_{k_2}))x^{k_2} + \cdots \\
&\quad = (f \circ g)(\phi_0) + (f \circ g)(\phi_{k_1})x^{k_1} + (f \circ g)(\phi_{k_2})x^{k_2} + \cdots \\
&\quad = \phi_0 + \phi_{k_1}x^{k_1} + \phi_{k_2}x^{k_2} + \cdots .
\end{aligned}$$

Therefore, $f^* : N^+[[x^s]] \to M^+[[x^s]]$ has a section g^* such that $f^* \circ g^* = id_{M^+[[x^s]]}$. Hence, $M[x^{-s}] \subset N[x^{-s}]$ is pure as left $R[x^s]$-modules. $\square$

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