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MATHEMATICAL CONSTANTS ASSOCIATED WITH THE MULTIPLE GAMMA FUNCTIONS

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Abstract. The theory of multiple Gamma functions was studied in about 1900 and has, recently, been revived in the study of determinants of Laplacians. There is a class of mathematical constants involved naturally in the multiple Gamma functions. Here we summarize those mathematical constants associated with the Gamma and multiple Gamma functions and will show how they are involved, if possible.

1. Introduction and Preliminaries

The double Gamma function Γ_2 and the multiple Gamma functions Γ_n were defined and studied by Barnes [7, 8, 9, 10] and others in about 1900. Although these functions did not appear in the tables of the most well-known special functions, yet the double Gamma function was cited in the exercises by Whittaker and Watson [60, p. 264] and recorded also by Gradshteyn and Ryzhik [39, p. 661, Entry 6.441(4); p. 937, Entry 8.333]. Recently, these functions were revived in the study of the determinants of the Laplacians on the n–dimensional unit sphere S^n (see Choi [14], Kumagai [47], Osgood *et al.* [49], Quine and Choi [50], Vardi [56], and Voros [58]). Shintani [53] also used the double Gamma function to prove the classical

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Kronecker limit formula. Its p-adic analytic extension appeared in a formula of Cassou-Noguès [13] for the p-adic L-functions at the point 0. More recently, Choi *et al.* [24, 25, 31] used these functions in order to evaluate some families of series involving the Riemann Zeta function as well as to compute the determinants of the Laplacians. Very recently, Choi *et al.* [31] addressed the converse problem and applied various (known or new) formulas for series associated with the Zeta and related functions with a view to developing the corresponding theory of multiple Gamma functions.

Matsumoto [48] proved asymptotic expansions of the Barnes double Zeta function and the double Gamma function, and presented an application to the Hecke L-functions of real quadratic fields. Before their investigation by Barnes, these functions had been introduced in a different form by (for example) Hölder [42], Alexeiewsky [5], and Kinkelin [44].

The main object of this paper is to summarize some known mathematical constants associated with the Gamma and multiple Gamma functions.

2. Euler-Mascheroni constant γ and the Gamma function Γ

2.1. **Its Brief History.** About 270 years ago, Euler introduced an important mathematical constant γ defined by

$$\gamma := \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right). \tag{2.1}$$

The discovery that $1+\frac{1}{2}+\frac{1}{3}+\ldots$ is divergent, is attributed by James Bernoulli to his brother (see Glaisher [37]) but the connection between $1+\frac{1}{2}+\cdots+\frac{1}{x}$ and $\log x$ was first established by Euler [35]. Euler (see Walfisz [59]) gave the formula

$$1+\frac{1}{2}+\cdots+\frac{1}{x} = \gamma + \log x + \frac{1}{2x} - \frac{B_1}{2x^2} + \frac{B_2}{4x^4} - \frac{B_3}{6x^6} + \ldots,$$

$B_1, B_2, \ldots$ being Bernoulli numbers, in which, by putting $x = 10$, he calculated

$$\gamma = 0.57721\,56649\,01532\,5\ldots.$$

The value of Euler's constant was given by Mascheroni in 1790 with 32 figures as follows:

$$\gamma = 0.57721\,56649\,01532\,8606\underline{1}811209008239\ldots$$

In 1809, Soldner computed the value of γ as

$$\gamma = 0.57721\,56649\,01532\,8606\underline{0}6065\ldots$$

which differs from Mascheroni's value in the twentieth place. In fact, Masheroni's value turned out to be not correct. However, maybe since Mascheroni's error has led to eight additional calculations of this constant. So γ is often called the *Euler-Mascheroni* constant. Gauss in 1813 computed the 23 first decimals; in 1860 Adams published the 260 first decimals. For a rather recent computation for γ, we may see Knuth [46].

The true nature of Euler's constant (whether an algebraic or a transcendental number) has not yet been known.

In addition, it is remarked in passing that the Euler-Mascheroni constant γ is the third important mathematical constant next to π and e whose transcendence were shown by Ferdinand Lindemann in 1882 and Charles Hermite in 1873, respectively. The mathematical constants π, e, and γ are often referred to as *holy trinity*.

We can show that the infinite product

$$(A) \qquad \prod_{n=1}^{\infty} (1 + \frac{z}{n}) e^{-\frac{z}{n}}$$

converges in the finite complex plane $\mathbb{C}$ to an entire function which has simple zeros at $z = -1, -2, -3, \ldots$, this argument yields that

$$(B) \qquad \prod_{n=1}^{\infty} (1 + \frac{z}{n})^{-1} e^{\frac{z}{n}}$$

converges on every compact subset in $\mathbb{C} \setminus \{-1, -2, -3, \ldots\}$ to a function with simple poles at $z = -1, -2, \ldots$. Using this fact, Weierstrass defined the Gamma function, $\Gamma(z)$, is a meromorphic function on $\mathbb{C}$ with simple poles at $z = 0, -1, -2, \ldots$, given by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{\frac{z}{n}}, \tag{2.2}$$

where γ *is a constant chosen so that* $\Gamma(1) = 1$. The first thing that must be done is to show that the constant γ exists. Substituting $z = 1$ in (B) yields a finite number

$$c = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{\frac{1}{n}}$$

which is clearly positive. Let $\gamma = \log c$; it follows that with this choice of γ, the equation (2.2) for $z = 1$ gives $\Gamma(1) = 1$. This constant γ is just the Euler's constant and it satisfies

$$e^{\gamma} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{\frac{1}{n}}. \tag{C}$$

Since both sides of (C) involve only real positive numbers and the real logarithm is continuous, we may apply the logarithm function to both sides of (C) and obtain

$$\begin{aligned}
\gamma &= \sum_{k=1}^{\infty} \log \left[\left(1 + \frac{1}{k}\right)^{-1} e^{\frac{1}{k}} \right] \\
&= \sum_{k=1}^{\infty} \left[\frac{1}{k} - \log(k+1) + \log k \right] \\
&= \lim_{n \to \infty} \sum_{k=1}^{n} \left[\frac{1}{k} - \log(k+1) + \log k \right] \\
&= \lim_{n \to \infty} \left[\sum_{k=1}^{n} \frac{1}{k} - \log(n+1) \right].
\end{aligned}$$

Subtracting and adding $\log n$ to each term of this sequence and using the fact that

$$\lim_{n\to\infty} \log\left(\frac{n}{n+1}\right) = 0$$

yields

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right),$$

which corresponds to (2.1).

2.2. **Integral Representations for γ.** Here we just present some known integral representations for the Euler's constant.

$$\gamma = \int_0^1 \frac{1-e{-}t}{t}dt - \int_1^\infty \frac{e^{-t}}{t}dt. \tag{2.3}$$

$$\gamma = \int_0^\infty e^{-t}\left(\frac{1}{1-e^{-t}} - \frac{1}{t}\right)dt. \tag{2.4}$$

$$\gamma = \int_0^1 \left(\frac{1}{\log t} + \frac{1}{1-t}\right)dt. \tag{2.5}$$

$$\gamma = \int_0^1 \frac{1-e^{-t}-e^{-\frac{1}{t}}}{t}dt. \tag{2.6}$$

$$\gamma = \int_0^\infty \left(\frac{1}{1+t} - e^{-t}\right)\frac{dt}{t}. \tag{2.7}$$

$$\gamma = \int_0^1 \left(t - \frac{1}{1-\log t}\right)\frac{1}{t\log t}dt. \tag{2.8}$$

$$\gamma = -\int_0^\infty \left(\cos t - \frac{1}{1+t^2}\right)\frac{dt}{t}. \tag{2.9}$$

$$\gamma = 1 - \int_0^\infty \left\{\frac{\sin t}{t} - \frac{1}{1+t}\right\}\frac{dt}{t}. \tag{2.10}$$

$$\gamma = -2\int_0^\infty \frac{\cos t - e^{-t^2}}{t}dt. \tag{2.11}$$

$$\gamma = \log 2 - \pi \int_0^1 \int_0^{\frac{1}{2}} \tan\frac{\pi t}{2}\left(\frac{\sin \pi t u}{\sin \pi u} - t\right) du dt. \tag{2.12}$$

$$\gamma = \frac{1}{2} + 2\int_0^\infty (1+t^2)^{-\frac{1}{2}}\left(e^{2\pi t} - 1\right)^{-1} \sin(\tan^{-1} t)\, dt. \tag{2.13}$$

$$\gamma = \log 2 - 2\int_0^\infty (1+t^2)^{-\frac{1}{2}}\left(e^{2\pi t} + 1\right)^{-1} \sin(\tan^{-1} t)\, dt. \tag{2.14}$$

$$\gamma = \int_0^\infty \left(\frac{1}{1+x} - \cos x\right)\frac{dx}{x}. \tag{2.15}$$

$$\gamma = -\int_0^\infty e^{-t}\log t\, dt. \tag{2.16}$$

$$\gamma = -\int_0^1 \log\left(\log\frac{1}{t}\right) dt. \tag{2.17}$$

$$\gamma = \frac{1}{2} + 2\int_0^\infty \frac{t}{1+t^2} \cdot \frac{dt}{e^{2\pi t}-1}. \tag{2.18}$$

$$\gamma = \frac{1}{2} + \frac{B_2}{2} + \frac{B_4}{4} + \cdots + \frac{B_{2n}}{2n} - (2n+1)! \int_1^\infty \frac{Q_{2n+1}(x)}{x^{2n+2}} dx, \tag{2.19}$$

where the functions $Q_n(x)$ are defined by

$$Q_n(x) := \begin{cases} x - \dfrac{1}{2} & (n = 1;\ 0 < x < 1), \\ \dfrac{1}{n!} B_n(x - [x]) & (n \in \mathbb{N} \setminus \{1\};\ 0 \leq x < \infty), \end{cases}$$

$B_n := B_n(0)$ and $B_n(x)$ being the Bernoulli numbers and polynomials, respectively (see Srivastava and Choi [55, Section 1.6]). As observed by Knopp [45] by an explicit example with $n = 3$ in (2.19), the approximate value of γ can easily be calculated with much greater accuracy than before (and, theoretically, to any degree of accuracy whatever) by means of the formula (2.19).

2.3. **Series Representations for γ.** The object of this subsection is to summarize some known series representations for γ and to point out that one of those series representations for γ seems to be *incorrectly* recorded (see [16], [21], [41]).

We start by recalling a well-known series representation for γ:

$$\gamma = \sum_{k=2}^\infty (-1)^k \frac{\zeta(k)}{k}, \tag{2.20}$$

where the Riemann Zeta function $\zeta(s)$ defined by

$$\zeta(s) := \sum_{n=1}^\infty \frac{1}{n^s} \quad (\Re(s) > 1) \tag{2.21}$$

is a special case when $a = 1$ of the *Hurwitz* (or *generalized*) *Zeta* function $\zeta(s, a)$ defined by

$$\zeta(s,a) := \sum_{k=0}^{\infty} (k+a)^{-s} \qquad (\Re(s) > 1;\ a \neq 0,\ -1,\ -2, \ldots), \tag{2.22}$$

which can be continued meromorphically to the whole complex s-plane (except for a simple pole at $s = 1$ with its residue 1).

$$\gamma = 1 - \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}. \tag{2.23}$$

$$\gamma = 1 - \log 2 + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k) - 1}{k}. \tag{2.24}$$

$$\gamma = 1 - \frac{1}{2} \log 2 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{2k+1}. \tag{2.25}$$

$$\gamma = \log 2 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{k+1}. \tag{2.26}$$

$$\gamma = \log 2 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(2k+1)\, 2^{2k}}. \tag{2.27}$$

$$\gamma = 1 - \log \frac{3}{2} - \sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{(2k+1)\, 2^{2k}}. \tag{2.28}$$

$$\gamma = 2 - 2\,\log 2 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{(k+1)(2k+1)}. \tag{2.29}$$

$$\gamma = 1 + \frac{1}{3}\,\log\frac{8}{15} - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)-1}{2k+1}\left(\frac{3}{2}\right)^{2k}. \tag{2.30}$$

$$\gamma = 1 - \frac{1}{2}\log 6 + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)-1}{k}\,2^{k-1}. \tag{2.31}$$

$$\gamma = 1 - \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)}, \tag{2.32}$$

which is commented in the work of Ramanujan [51] who gave a general formula containing (2.31) as a very special case.

$$\gamma = \log 2 - 2\sum_{k=1}^{\infty} k \sum_{j=A_{k-1}+1}^{A_k} \frac{1}{(3j)^3 - 3j}, \tag{2.33}$$

where

$$A_k := \frac{1}{2}(3^k - 1) \quad (k = 0,\, 1,\, 2,\, \ldots).$$

$$\gamma = \frac{1}{2} - \frac{1}{3} + \sum_{k=2}^{\infty} k \sum_{i=0}^{2^k-1} \frac{(-1)^i}{2^k+i}. \tag{2.34}$$

$$\gamma = \frac{1}{2} + \sum_{k=1}^{\infty} k \sum_{i=2^{k-1}}^{2^k-1} \frac{1}{2i(2i+1)(2i+2)}. \tag{2.35}$$

$$\gamma = 1 - \sum_{k=1}^{\infty} k \sum_{i=2^{k-1}+1}^{2^k} \frac{1}{(2i-1)(2i)}. \tag{2.36}$$

Campbell [12, p. 200] recorded an interesting series representation for γ which can easily be written in the form:

$$\gamma = 1 - \log \frac{3}{2} - \sum_{k=1}^{\infty} \frac{\zeta(2k+1) - 1}{(2k+1)2^k}. \tag{2.37}$$

Note that the expression of γ in (2.37) *seems to be the most rapidly convergent series* among the ever-known series representations of γ by observing $1 < \zeta(2k+1) < 2$ for each positive integer k and the following rough estimations:

$$0 < \sum_{k=21}^{\infty} \frac{\zeta(2k+1) - 1}{(2k+1)2^k} < 10^{-65}$$

and

$$0 < \sum_{k=41}^{\infty} \frac{\zeta(2k+1) - 1}{(2k+1)2^k} < 10^{-153}.$$

However, when (2.37) is compared with (2.28), it is carefully concluded that the expression (2.37) is *incorrectly* recorded.

3. The double Gamma function

3.1. **Its Origin and Revival.** Barnes [7] defined the double Gamma function $\Gamma_2 = 1/G$ satisfying each of the following properties:

(a) $G(z+1) = \Gamma(z)G(z)$ $(z \in \mathbb{C})$;

(b) $G(1) = 1$;

(c) Asymptotically,
(3.1)

$$\log G(z+n+2) = \frac{n+1+z}{2}\log(2\pi) + \left[\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z\right]$$
$$\log n - \frac{3n^2}{4} - n - nz - \log A + \frac{1}{12} + O\left(n^{-1}\right)$$
$$, \qquad (n \to \infty)$$

where Γ is the familiar Gamma function in (2.2) and A is called the Glaisher-Kinkelin constant defined by

$$\text{(3.2)} \quad \log A = \lim_{n\to\infty}\left\{\sum_{k=1}^{n} k\,\log\,k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right)\log n + \frac{n^2}{4}\right\},$$

the numerical value of A being given by

$$A \cong 1.282427130\cdots.$$

From this definition, Barnes [7] deduced several explicit Weierstrass canonical product forms of the double Gamma function Γ_2, one of which is recalled here in the form:
(3.3)

$$\{\Gamma_2(z+1)\}^{-1} = G(z+1)$$
$$= (2\pi)^{\frac{1}{2}z}\exp\left(-\frac{1}{2}z - \frac{1}{2}(\gamma+1)z^2\right)\prod_{k=1}^{\infty}\left\{\left(1+\frac{z}{k}\right)^k \exp\left(-z+\frac{z^2}{2k}\right)\right\},$$

where γ denotes the Euler-Mascheroni constant given (1.1).

Barnes [7] also gave two more equivalent forms of the double Gamma function Γ_2:

$$\begin{aligned} &\{\Gamma_2(z+1)\}^{-1} = G(z+1) \\ \text{(3.4)} \qquad &= (2\pi)^{\frac{1}{2}z}\exp\left(-\frac{1}{2}z(z+1) - \frac{1}{2}\gamma\, z^2\right) \\ &\prod_{k=1}^{\infty}\frac{\Gamma(k)}{\Gamma(z+k)}\exp\left[z\psi(k) + \frac{1}{2}z^2\,\psi'(k)\right]; \end{aligned}$$

(3.5)

$$\{\Gamma_2(z+1)\}^{-1} = G(z+1) = (2\pi)^{\frac{1}{2}z} \exp\left[\left(\gamma - \frac{1}{2}\right) z - \left(\frac{\pi^2}{6} + 1 + \gamma\right) \frac{z^2}{2}\right]$$
$$\Gamma(z+1) \cdot \prod_{m=0}^{\infty} \prod_{n=0}^{\infty}{}' \left(1 + \frac{z}{m+n}\right) \exp\left(-\frac{z}{m+n} + \frac{z^2}{2(m+n)^2}\right),$$

where the prime denotes the exclusion of the case $n = m = 0$ and the Psi (or Digamma) function ψ is given by $\psi(z) = \Gamma'(z)/\Gamma(z)$. Each form of these products is convergent for all finite values of $|z|$, by the Weierstrass factorization theorem ([see Conway [32, p. 170]).

The double Gamma function satisfies the following relations:

$$G(1) = 1 \quad \text{and} \quad G(z+1) = \Gamma(z)G(z) \quad (z \in \mathbb{C}). \tag{3.6}$$

For sufficiently large real x and $a \in \mathbb{C}$, we have the Stirling formula for the G-function:

(3.7)

$$\begin{aligned}\log G(x+a+1) =& \frac{x+a}{2} \log(2\pi) - \log A + \frac{1}{12} - \frac{3x^2}{4} - ax \\ &+ \left(\frac{x^2}{2} - \frac{1}{12} + \frac{a^2}{2} + ax\right) \log x + O\left(x^{-1}\right) \quad (x \to \infty).\end{aligned}$$

The following special values of G (see Barnes [7]) may be recalled here:

$$G\left(\frac{1}{2}\right) = 2^{\frac{1}{24}} \cdot \pi^{-\frac{1}{4}} \cdot e^{\frac{1}{8}} \cdot A^{-\frac{3}{2}}; \tag{3.8}$$

(3.9)

$$G(n+2) = 1!\, 2! \cdots n! \quad \text{and} \quad G(n+1) = \frac{(n!)^n}{1 \cdot 2 \cdot 3^2 \cdot 4^3 \cdots n^{n-1}} \quad (n \in \mathbb{N}).$$

It should be remarked in passing that the double Gamma function Γ_2 and the multiple Gamma function Γ_n in Section 4 have, recently, been revived in the study of determinants of Laplacians (see, e.g. [14], [47], [56], [58]).

3.2. **Various Representations for A.** We just record some of various known representations for the Glaisher-Kinkelin constant A. We can express $\log \Gamma_2(a)$ as improper integrals in many ways. For example, we give two integral representations for $\log \Gamma_2(a)$ (see [20]):
(3.10)

$$\begin{aligned}\log \Gamma_2(a) = &-\frac{1}{12} + \log A - \frac{a^2}{4} + \left(\frac{1}{2}a^2 - \frac{1}{2}a\right)\log a + (1-a)\log \Gamma(a) \\ &+ 2\int_0^\infty \left\{\frac{1}{2}(a^2+t^2)^{\frac{1}{2}} \sin\left(\arctan\frac{t}{a}\right) \log(a^2+t^2)\right. \\ &+ \left.(a^2+t^2)^{\frac{1}{2}} \cos\left(\arctan\frac{t}{a}\right) \arctan\frac{t}{a}\right\} \frac{dt}{e^{2\pi t}-1} \\ &\qquad (\Re(a)>0);\end{aligned}$$

(3.11)

$$\begin{aligned}\log \Gamma_2(a) = &\log A - \frac{a^2}{4} + \left(\frac{a^2}{2} - \frac{a}{2} + \frac{1}{12}\right)\log a + (1-a)\log \Gamma(a) \\ &- \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right)\frac{e^{-at}}{t^2}\,dt \quad (\Re(a)>0).\end{aligned}$$

Glaisher [38, p. 47] expressed the Glaisher-Kinkelin constant A as an integral:

$$A = 2^{\frac{7}{36}}\pi^{-\frac{1}{6}} \exp\left(\frac{1}{3} + \frac{2}{3}\int_0^{\frac{1}{2}} \log\,\Gamma(t+1)\,dt\right). \tag{3.12}$$

By setting $a=1$ in (3.10) and (3.11), we can also obtain integral representations of $\log A$:

$$\begin{aligned}\log A = &\frac{1}{3} - 2\int_0^\infty \left\{\frac{1}{2}(1+t^2)^{\frac{1}{2}} \sin(\arctan t)\, \log(1+t^2)\right. \\ &+ \left.(1+t^2)^{\frac{1}{2}} \cos(\arctan t)\, \arctan t\right\}\frac{dt}{e^{2\pi t}-1}\end{aligned} \tag{3.13}$$

and

$$\log A = \frac{1}{4} + \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}\right)\frac{e^{-t}}{t^2}\,dt. \tag{3.14}$$

Recall the Euler-Maclaurin summation formula (*cf.* Hardy [40, p. 318]):

$$\sum_{k=1}^{n} f(k) \sim C_0 + \int_a^n f(x)\,dx + \frac{1}{2}\,f(n) + \sum_{r=1}^{\infty} \frac{B_{2r}}{(2r)!} f^{(2r-1)}(n), \tag{3.15}$$

where C_0 is an arbitrary constant to be determined in each special case and

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42},$$
$$B_8 = -\frac{1}{30},\ B_{10} = \frac{5}{66},\ \ldots \text{ and } B_{2n+1} = 0 \quad (n \in \mathbb{N})$$

are the Bernoulli numbers (see Srivastava and Choi [55, Section 1.6]).

By using the Euler-Maclaurin summation formula (2.15), we can obtain a number of analytical representations of $\zeta(s)$, such as [*cf.* Hardy [40, p. 333)]

(3.16)
$$\zeta(s) = \lim_{n\to\infty} \left\{ \sum_{k=1}^{n} k^{-s} - \frac{n^{1-s}}{1-s} - \frac{1}{2}n^{-s} + \frac{1}{12}sn^{-s-1} \right\} \quad (\Re(s) > -3).$$

We may differentiate (3.16) under the limit sign at $s = -1$ to get

(3.17)
$$\zeta'(-1) = \lim_{n\to\infty} \left\{ -\sum_{k=1}^{n} k \log k + \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \log n - \frac{n^2}{4} + \frac{1}{12} \right\}.$$

Comparing (3.2) and (3.17), we find that

$$A = \exp\left(-\zeta'(-1) + \frac{1}{12} \right). \tag{3.18}$$

Choi and Nash [18] obtained a class of integral representations of A:

(3.19)
$$\begin{aligned} \log A =& \frac{1}{8} + \frac{1}{24} \left(3 + \frac{2}{p^2-1} \right) \log p - \frac{1}{p^2-1} \sum_{j=1}^{(p-1)/2} \Bigg[j \log j \\ & + p \int_0^{\frac{j}{p}} \left(t - \frac{j}{p} \right) \{\psi(1+t) + \psi(1-t)\}\,dt \Bigg], \end{aligned}$$

where p is an odd positive integer greater than 1.

We give a relationship among π, γ, A, and $\zeta'(2)$:

$$\gamma = \frac{6\,\zeta'(2)}{\pi^2} + \log\,\frac{A^{12}}{2\pi}. \tag{3.20}$$

4. Multiple Gamma functions

4.1. **Definitions.** There are two known ways to define an n-ple Gamma functions Γ_n: Barnes [10] (see also Vardi [56]) defined Γ_n by using the n-ple Hurwitz Zeta functions (see Choi and Quine [19]); A recurrence formula of the Weierstrass canonical product forms of the n-ple Gamma functions Γ_n was given by Vignéras [57] who used the theorem of Dufresnoy and Pisot [34] which provides the existence, uniqueness, and expansion of the series of Weierstrass satisfying a certain functional equation.

By making use of the aforementioned Dufresnoy-Pisot theorem and starting with

$$f_1(x) = -\gamma\,x + \sum_{n=1}^{\infty}\left\{\frac{x}{n} - \log\left(1+\frac{x}{n}\right)\right\}, \tag{4.1}$$

Vignéras [57] obtained a recurrence formula of Γ_n ($n \in \mathbb{N}$) which is given by

Theorem 1. *The n-ple Gamma functions Γ_n are defined by*

$$\Gamma_n(z) = \{G_n(z)\}^{(-1)^{n-1}} \quad (n \in \mathbb{N}), \tag{4.2}$$

where

$$G_n(z+1) = \exp\left(f_n(z)\right) \tag{4.3}$$

and the functions $f_n(z)$ are given by

$$f_n(z) = -x\,A_n(1) + \sum_{k=1}^{n-1}\frac{p_k(z)}{k!}\left[f_{n-1}^{(k)}(0) - A_n^{(k)}(1)\right] + A_n(z), \tag{4.4}$$

with

(4.5)

$$A_n(z) = \sum_{m \in \mathbb{N}_0{}^{n-1} \times \mathbb{N}} \left[\frac{1}{n} \left(\frac{z}{L(m)} \right)^n - \frac{1}{n-1} \left(\frac{z}{L(m)} \right)^{n-1} + \cdots \right.$$
$$\left. + (-1)^{n-1} \frac{z}{L(m)} + (-1)^n \log \left(1 + \frac{z}{L(m)} \right) \right],$$

where

$$L(m) = m_1 + m_2 + \cdots + m_n \quad if \quad m = (m_1, m_2, \ldots, m_n) \in \mathbb{N}_0{}^{n-1} \times \mathbb{N}$$

and the polynomials

$$p_n(z) = 1^n + 2^n + \cdots + (z-1)^n$$

satisfy the following relations:

$$p_n'(z) = B_n(z) \quad and \quad p_n(0) = 0, \tag{4.6}$$

$B_n(z)$ *being the* n*th Bernoulli polynomials.*

By analogy with the Bohr-Mollerup theorem (see Srivastava and Choi [55, p. 13]), which guarantees the uniqueness of the Gamma function Γ, one can give for the double Gamma function and (more generally) for the multiple Gamma functions of order n ($n \in \mathbb{N}$) a definition of Artin by means of the following theorem (see Vignéras [57, p. 239]).

Theorem 2. *For all* $n \in \mathbb{N}$, *there exists a unique meromorphic function* $G_n(z)$ *satisfying each of the following properties*:

(a) $G_n(z+1) = G_{n-1}(z)\, G_n(z) \quad (z \in \mathbb{C})$;

(b) $G_n(1) = 1$;

(c) *For* $x \geq 1$, $G_n(x)$ *are infinitely differentiable and*

$$\frac{d^{n+1}}{dx^{n+1}} \{\log G_n(x)\} \geq 0;$$

(d) $G_0(x) = x$.

When $n = 3$ in (4.2), we readily obtain an explicit form of the *triple Gamma function* $\mathbf{\Gamma}_3$:

(4.7)

$$\begin{aligned}\mathbf{\Gamma}_3(1+z) =& G_3(1+z)\\ =& e^{R(z)} \prod_{k=1}^{\infty} \left\{ \left(1+\frac{z}{k}\right)^{-\frac{1}{2}k(k+1)} \right.\\ & \left. \cdot \exp\left[\frac{1}{2}(k+1)z - \frac{1}{4}\left(1+\frac{1}{k}\right)z^2 + \frac{1}{6k}\left(1+\frac{1}{k}\right)z^3\right]\right\},\end{aligned}$$

where, for convenience,

$$\begin{aligned}R(z) :=& -\frac{1}{6}\left(\gamma + \frac{\pi^2}{6} + \frac{3}{2}\right)z^3 + \frac{1}{4}\left(\gamma + \log(2\pi) + \frac{1}{2}\right)z^2\\ & + \left(\frac{3}{8} - \frac{1}{4}\log(2\pi) - \log A\right)z.\end{aligned}$$

4.2. **Bendersky-Adamchik Constants.** Letting $f(x) = x^2 \log x$ and $f(x) = x^3 \log x$ in (3.15) with $a = 1$, we obtain

(4.8)

$$\log B = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^2 \log k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right)\log n + \frac{n^3}{9} - \frac{n}{12}\right]$$

and

(4.9)

$$\log C = \lim_{n\to\infty}\left[\sum_{k=1}^{n} k^3 \log k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120}\right)\log n + \frac{n^4}{16} - \frac{n^2}{12}\right],$$

respectively; here B and C are constants whose approximate numerical values are given by

$$B \cong 1.03091\,675\ldots$$

and
$$C \cong 0.97955\,746\ldots.$$

The constants B and C were considered recently by Choi and Srivastava ([23, p. 102], [25]). See also Adamchik [2, p. 199]. Bendersky [11] presented a set of constants including B and C: There exist constants D_k defined by

$$\log D_k := \lim_{n\to\infty}\left(\sum_{m=1}^{n} m^k \log m - p(n,k)\right) \quad (k \in \mathbb{N}_0), \tag{4.10}$$

where the definition of $p(n,k)$ in Adamchik [2, p. 198, Eq.(20)] is *corrected* here as follows:

$$\begin{aligned} p(n,k) :=& \frac{n^k}{2}\log n + \frac{n^{k+1}}{k+1}\left(\log n - \frac{1}{k+1}\right) \\ &+ k!\sum_{j=1}^{k}\frac{n^{k-j}\,B_{j+1}}{(j+1)!\,(k-j)!}\left[\log n + (1-\delta_{kj})\sum_{\ell=1}^{j}\frac{1}{k-\ell+1}\right] \end{aligned}$$

and δ_{kj} is the Kronecker symbol defined by $\delta_{kj} = 0$ $(k \neq j)$ and $\delta_{kj} = 1$ $(k = j)$.

For the constants D_k $(k \in \mathbb{N}_0)$ defined in (4.10), we can show that

$$D_0 = (2\pi)^{\frac{1}{2}}, \quad D_1 = A, \quad D_2 = B, \quad \text{and} \quad D_3 = C,$$

and
$$\log D_k = \frac{B_{k+1}\,H_k}{k+1} - \zeta'(-k) \quad (k \in \mathbb{N}_0),$$

where B_n are the Bernoulli numbers and H_n are the harmonic numbers, and the mathematical constants A, B, and C are given as above (see Adamchik [2, pp. 198-199]).

The constants introduced in this section can be seen to be involved in the theory of multiple Gamma functions. For example, see the following identities (Srivastava and Choi [55, p. 39; p. 247]):

$$\int_0^{\frac{1}{2}} \log G(t+1)dt = \frac{1}{24}(\log 2+1)+\frac{1}{16}\log\pi-\frac{1}{4}\log A-\frac{7}{4}\log B \tag{4.11}$$

in terms of the mathematical constant B defined by (4.8);

(4.12)

$$\begin{aligned}\int_0^{\frac{3}{2}} \log\Gamma_3(t+2)\,dt &= \int_0^1 \log G(t+1)\,dt + \int_0^1 \log\Gamma_3(t+1)\,dt \\ &+ \int_0^{\frac{1}{2}} \log\Gamma(t+1)\,dt + 2\int_0^{\frac{1}{2}} \log G(t+1)\,dt + \int_0^{\frac{1}{2}} \log\Gamma_3(t+1)\,dt \\ &= -\frac{259}{768} - \frac{29}{1920}\log 2 + \frac{9}{16}\log\pi - \frac{15}{16}\log A - \frac{5}{4}\log B + \frac{15}{16}\log C\end{aligned}$$

4.3. **Another version of Bendersky-Adamchik Constants.** We begin by recalling the Euler-Maclaurin summation formula (*cf.* Abramowitz and Stegun [1, p. 806]): Let $K \in \mathbb{N}$ and $a, b \in \mathbb{R}$ with $a < b$. Suppose that f has continuous derivatives through the Kth order on the interval $[a, b]$. Then we have

(4.13)

$$\begin{aligned}\sum_{a<n\leq b} f(n) &= \int_a^b f(x)dx + \sum_{m=1}^{K} \frac{(-1)^m}{m!} B_m(\{b\}) f^{(m-1)}(b) \\ -\sum_{m=1}^{K} &\frac{(-1)^m}{m!} B_m(\{a\}) f^{(m-1)}(a) - \frac{(-1)^K}{K!}\int_a^b B_K(\{x\}) f^{(K)}(x)dx,\end{aligned}$$

where $\{x\}$ denotes the fractional part of any $x \in \mathbb{R}$, *i.e.*, $\{x\} = x - [x]$, $[x]$ being the greatest integer $\leq x$.

Setting $f(x) = \frac{1}{x}$, $a = 1$, $b = N$, $K = 1$ in (4.13), and taking the limit of the resulting equation as $N \to \infty$, we obtain the *Euler-Mascheroni constant* γ given in (1.1).

By appealing to the Leibniz's rule for differentiation, we find that the mth order derivative of $f(x) := x^{k-1}\log x$ $(k \in \mathbb{N})$ is

$$f^{(m)}(x) = \alpha_{k,m}\, x^{k-m-1}\log x + \beta_{k,m}\, x^{k-m-1} \quad (0 \leq m \leq k-1;\ m \in \mathbb{N}_0), \tag{4.14}$$

where the constants $\alpha_{k,m}$ and $\beta_{k,m}$ are given as follows:
(4.15)

$$\alpha_{k,m} = \frac{(k-1)!}{(k-m-1)!} \quad \text{and} \quad \beta_{k,m} = \frac{(k-1)!}{(k-m-1)!} \sum_{j=1}^{m} \frac{1}{k-m-1+j}$$
$$(0 \leq m \leq k-1;\ m \in \mathbb{N}_0).$$

Note also that

$$f^{(k)}(x) = \frac{\alpha_{k,k-1}}{x} \quad \text{and} \quad f^{(k+1)}(x) = -\frac{\alpha_{k,k-1}}{x^2}. \tag{4.16}$$

We prove the following identity:
(4.17)

$$\sum_{m=1}^{k-1} \frac{(-1)^m}{m!} B_m\, \beta_{k,m-1} = \frac{(-1)^k}{k} \sum_{m=1}^{k} \binom{k}{m} \frac{1}{m}\, B_{k-m} + \frac{1}{k^2} \quad (k \in \mathbb{N}),$$

the empty sum being interpreted (as usual, in what follows) to be nil.

Indeed, since (4.17) holds trivially for $k = 1$, we assume $k \in \mathbb{N} \setminus \{1\}$. Let

$$\mathcal{S}_k := (-1)^k\, k \sum_{m=1}^{k-1} \frac{(-1)^m}{m!} B_m\, \beta_{k,m-1} \quad (k \in \mathbb{N} \setminus \{1\}),$$

which, upon setting $n = k - m$ and using (4.15), yields

$$\mathcal{S}_k = \sum_{n=1}^{k-1} (-1)^n \binom{k}{n} B_{k-n} \sum_{j=1}^{k-n-1} \frac{1}{n+j}. \tag{4.18}$$

Define a function $\mathcal{H}(x)$ given by

$$\mathcal{H}(x) := \sum_{n=1}^{k-1} (-1)^n \binom{k}{n} B_{k-n} \sum_{j=1}^{k-n-1} x^{n-1+j}. \tag{4.19}$$

Since the innermost sum is a finite geometric series and

$$\mathcal{S}_k = \int_0^1 \mathcal{H}(x)\,dx,$$

we find that

$$\mathcal{H}(x) = \frac{h(x) - x^{k-1}h(1)}{1-x}, \tag{4.20}$$

where, for convenience,

$$h(x) := \sum_{n=1}^{k-1} \binom{k}{n} B_{k-n}\,(-x)^n.$$

It follows from Equations (3) and (4) in Sri+Choi [..., p. 59] that

$$\begin{aligned} h(x) =& B_k(-x) - B_k - (-x)^k \\ =& B_k(1-x) - B_k - k(-x)^{k-1} - (-x)^k, \end{aligned} \tag{4.21}$$

which, upon setting $x = 1$, yields

$$h(1) = (-1)^k(k-1). \tag{4.22}$$

Putting (4.21) and (4.22) into (4.20), we get

$$\mathcal{H}(x) = \frac{B_k(1-x) - B_k}{1-x} + (-1)^k\,x^{k-1},$$

which, upon integrating from 0 to 1 and changing the variable $t = 1 - x$, gives

$$\mathcal{S}_k = \int_0^1 \frac{B_k(t) - B_k}{t}\,dt + \frac{(-1)^k}{k}. \tag{4.23}$$

Applying Equation (3) in Srivastava and Choi [55, p. 59] to (4.23), we obtain

$$\mathcal{S}_k = \sum_{m=1}^{k} \binom{k}{m} \frac{1}{m} B_{k-m} + (-1)^k \frac{1}{k}, \tag{4.24}$$

which, upon dividing by $(-1)^k k$, leads to the desired identity (4.17).

Setting $f(x) = x^{k-1} \log x$ ($k \in \mathbb{N}$), $a = 1$, $b = N$ ($N \in \mathbb{N} \setminus \{1\}$), and $K = k + 1$ in (4.13), and using (4.17), we can obtain a class of mathematical constants $\{C_k\}_{k \in \mathbb{N}}$ defined, for $k \in \mathbb{N}$, by

$$\begin{aligned} C_k := \lim_{N \to \infty} \Bigg[& \sum_{n=1}^{N} n^{k-1} \log n - \frac{N^k}{k} \log N + \frac{N^k}{k^2} \\ & - \sum_{m=1}^{k} \frac{(-1)^m}{m!} B_m \, \alpha_{k,m-1} \, N^{k-m} \log N \\ & - \sum_{m=1}^{k-1} \frac{(-1)^m}{m!} B_m \, \beta_{k,m-1} \, N^{k-m} \Bigg], \end{aligned} \tag{4.25}$$

where $\alpha_{k,m}$ and $\beta_{k,m}$ are defined by (4.15) (see Choi and Lee [17]).

Remark 1. In the process of getting the constants C_k in (4.25), we find that

$$\begin{aligned} C_k = & - \frac{(-1)^k}{k} \sum_{m=1}^{k} \binom{k}{m} \frac{1}{m} B_{k-m} + \frac{(-1)^k}{k(k+1)} B_{k+1} \\ & - \frac{(-1)^k}{k(k+1)} \int_1^\infty \frac{B_{k+1}(\{x\})}{x^2} \, dx \quad (k \in \mathbb{N}). \end{aligned} \tag{4.26}$$

Remark 2. It follows from the Stirling's formula for $n!$:

$$n! \sim \left(\frac{n}{e} \right)^n \sqrt{2\pi n} \quad (n \to \infty) \tag{4.27}$$

that $C_1 = \log \sqrt{2\pi}$. Note also that $C_2 = \log A$, where A is the *Glaisher-Kinkelin constant* given in (3.2); $C_3 = \log B$, and $C_4 = \log C$, where the constants B and C are given in (4.8) and (4.9), respectively.

Remark 3. The set of constants $\{C_k\}_{k\in\mathbb{N}}$ has already been considered in the works of Bendersky [11] and Adamchik [2] who gave the constants $\{D_k\}_{k\in\mathbb{N}_0}$ in (4.10).

It should be noted in passing that $\log D_{k-1} = C_k$ $(k \in \mathbb{N})$.

Remark 4. As a matter of fact, Choi and Lee [17] introduced the constants $\{C_k\}_{k\in\mathbb{N}}$ in (4.26) in order to evaluate the following family of series associated with the Riemann zeta function:

$$\sum_{m=2}^{\infty} \frac{(-1)^m}{m+k}\,\zeta(m) \quad (k \in \mathbb{N}_0)\,.$$

This interesting and useful research subject, closed-form evaluations of various families of series associated with the Zeta functions, has a long history (see Srivastava and Choi [55, Chapter 3]).

Remark 5. Very recently, Choi and Srivastava [28] also introduced a set of mathematical constants which is more generalized than two families of constants given in (4.10) and (4.26).

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