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A COVERING CONDITION FOR THE PRIME SPECTRUMS

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Abstract. Let R be a commutative ring with identity, and let f, g_i $(i = 1, \ldots, n)$, g_α $(\alpha \in S)$ be elements of R. We show that the following statements are equivalent; (i) $X_f \subseteq \cup_{\alpha \in S} X_{g_\alpha}$ only if $X_f \subseteq X_{g_\alpha}$ for some $\alpha \in S$, (ii) $V(f) \subseteq \cup_{\alpha \in S} V(g_\alpha)$ only if $V(f) \subseteq V(g_\alpha)$ for some $\alpha \in S$, (iii) $V(f) \subseteq \cup_{i=1}^{n} V(g_i)$ only if $V(f) \subseteq V(g_i)$ for some i, (iv) $Spec(R)$ is linearly ordered under inclusion.

Let R be a commutative ring (with identity 1). It is known that every prime ideal of R is the radical of a principal ideal if and only if R satisfies the following property; $(*)$ *for a prime ideal P and a (nonempty) set $\{P_\alpha | \alpha \in A\}$ of prme ideals of R, $P \subseteq \cup_{\alpha \in A} P_\alpha$ implies $P \subseteq P_\alpha$ for some $\alpha \in A$.* This was proved for Noetherian rings by Reis and Viswanathan [6], and then completely generalized by Smith [7]. It is clear that if R satisfies $(*)$, then R has a finite number of minimal prime ideals (cf. [2, Theorem 2.1] or [4, Theorem 2.5]). As a natural dual of $(*)$, Gilmer [3] studied the following condition; (#) *If $P \in Spec(R)$ and if $\{I_\alpha\}_{\alpha \in S}$ is a nonempty family of ideals of R, then P contains $\cap_{\alpha \in S} I_\alpha$ only if P contains some I_α.* He proved that R satisfies condition (#) if and only if R is zero-dimensional and semi-quasilocal [3, Theorem 2].

In [5], we studied similar properties for the prime spectrum of a ring R;

(A) *For any elements f and g_α $(\alpha \in S)$ of R, $X_f \subseteq \cup_{\alpha \in S} X_{g_\alpha}$ implies $X_f \subseteq X_{g_\alpha}$ for some $\alpha \in S$.*

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In particular, we showed that if R satisfies (A), then R has at most two maximal ideals [5, Theorem 6]. We next consider a natural dual of (A); for any elements f and g_α ($\alpha \in S$) of R, $X_f \supseteq \cap_{\alpha\in S} X_{g_\alpha}$ only if $X_f \supseteq X_{g_\alpha}$ for some $\alpha \in S$. Note that $X_f \supseteq \cap_{\alpha\in S} X_{g_\alpha} \Leftrightarrow V(f) = (Spec(R) \setminus X_f) \subseteq (Spec(R) \setminus \cap_{\alpha\in S} X_{g_\alpha}) = \cup_{\alpha\in S}(Spec(R) \setminus X_{g_\alpha}) = \cup_{\alpha\in S} V(g_\alpha)$; hence we can restate the dual of the condition (A) as follows;

(B) *For any elements f and g_λ ($\lambda \in S$) of R, $V(f) \subseteq \cup_\lambda V(g_\lambda)$ implies $V(f) \subseteq V(g_\lambda)$ for some $\lambda \in S$.*

It is well known, and easily verified, that the condition (A) is equivalent to the following condition; *for any elements f and g_i ($i = 1, \ldots, n$) of R, $X_f \subseteq \cup_{i=1}^n X_{g_i}$ implies $X_f \subseteq X_{g_i}$ for some i.* Thus it is natural to ask whether the condition (B) is equivalent to

(C) *For any elements f and g_i ($i = 1, \ldots, n$) of R, $V(f) \subseteq \cup_{i=1}^n V(g_i)$ implies $V(f) \subseteq V(g_i)$ for some i.*

We answer this affirmatively. In fact, the purpose of this paper is to prove that the conditions (A), (B), (C), and that $Spec(R)$ is linearly ordered are equivalent.

All rings R considered in this paper are commutative rings (with identity 1) and $Spec(R)$ (called the *prime spectrum* of R) is the set of prime ideals of R. Clearly, $Spec(R)$ is a partially ordered set under inclusion. For an element $f \in R$, $V(f)$ denotes the set of prime ideal of R containing f and $X_f = Spec(R) \setminus V(f)$. It is clear that $V(0) = Spec(R)$ and $V(1) = \emptyset$. Note that $P \in Spec(R) - X_f \Leftrightarrow P \in V(f) \Leftrightarrow fR \subseteq P$ and that $\sqrt{fR} = \cap\{P \in Spec(R) | f \in P\} = \cap\{P \in V(f)\}$. Hence $X_f = X_g \Leftrightarrow V(f) = V(g) \Leftrightarrow \sqrt{fR} = \sqrt{gR}$ for any $f, g \in R$.

LEMMA 1. *Let f and g be elements of a ring R, then*

1. *$X_{f+g} \subseteq X_f \cup X_g$ and*
2. *$V(fg) = V(f) \cup V(g)$.*

Proof. Let P be a prime ideal of R. Then (1) $P \in X_{f+g} \Leftrightarrow f + g \notin P \Rightarrow f \notin P$ or $g \notin P \Leftrightarrow P \in X_f \cup X_g$ and (2) $P \in V(fg) \Leftrightarrow fg \in P \Leftrightarrow f \in P$ or $g \in P \Leftrightarrow P \in V(f) \cup V(g)$. □

We next give the main result of this paper which gives a complete characterization of rings satisfying the property (A).

THEOREM 2. *Let R be a ring, then the following statements are equivalent:*

1. *R satisfies* (A);
2. *R satisfies* (B);
3. *R satisfies* (C);
4. *$Spec(R)$ is linearly ordered under inclusion.*

Proof. (1) $\Rightarrow$ (4) Let P and Q be prime ideals of R such that $P \nsubseteq Q$ and $Q \nsubseteq P$. Let $f \in P \setminus Q$ and $g \in Q \setminus P$. Then $P \in X_{f+g} \setminus X_f$ and $Q \in X_{f+g} \setminus X_g$. Hence $X_{f+g} \nsubseteq X_f$ and $X_{f+g} \nsubseteq X_g$, but $X_{f+g} \subseteq X_f \cup X_g$ by Lemma 1. Thus if R satisfies (A), then $Spec(R)$ is linearly ordered.

(4) $\Rightarrow$ (1) Recall that $Spec(R)$ is linearly ordered if and only if $\{X_f | f \in R\}$ is linearly ordered [5, Theorem 10]. Thus if $Spec(R)$ is linearly ordered, then R satisfies (A).

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4) Let P and Q be prime ideals of R such that $P \nsubseteq Q$ and $Q \nsubseteq P$. Let $f \in P \setminus Q$ and $g \in Q \setminus P$. Then $Q \in V(fg) \setminus V(f)$ and $P \in V(fg) \setminus V(g)$; hence $V(fg) \nsubseteq V(f)$ and $V(fg) \nsubseteq V(g)$, but $V(fg) = V(f) \cup V(g)$ by Lemma 1. Thus if R satisfies (C), then $Spec(R)$ is linearly ordered.

(4) $\Rightarrow$ (2) Let f, g_λ ($\lambda \in S$) be elements of R such that $V(f) \subseteq \cup_{\lambda \in S} V(g_\lambda)$. If f is a unit in R, then $V(f) = \emptyset$; so we assume that f is not a unit. Let $\sqrt{fR} = P$. Then as $Spec(R)$ is linearly ordered, P is a proper prime ideal of R [5, Theorem 10] and $P \in V(f)$. Since $V(f) \subseteq \cup_{\lambda \in S} V(g_\lambda)$, $P \in V(g_\lambda)$ for some $\lambda \in S$; so $g_\lambda \in P$. Thus $V(f) \subseteq V(g_\lambda)$ since $Q \in V(f) \Leftrightarrow f \in Q \Leftrightarrow P = \sqrt{fR} \subseteq Q \Rightarrow g_\lambda \in Q \Leftrightarrow Q \in V(g_\lambda)$. $\square$

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