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NEIGHBORHOOD SPACES AND P-STACK CONVERGENCE SPACES

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Abstract. We will define p-stack convergence spaces and show that each neighborhood structure is uniquely determined by p-stack convergence structure. Also, we will show that p-stack convergence spaces are a generalization of neighborhood spaces.

1. Introduction and Neighborhood spaces

The study of limits and convergence is an essential part of topology, so finding an appropriate definition of convergence in various structures (supratopology [4], convergence structures [2], neighborhood structures [3], etc.) is a high priority. In generalizing from metric to topological spaces, it was necessary to replace sequences by filters or nets to obtain a convergence theory adequate to characterize topologies and their most basic properties(e.g., closed sets, closures, limit points, continuity, compactness, etc.)

We must likewise replace filters by some more general convergence vehicle "p-stacks" in order to develop a satisfactory convergence theory for neighborhood spaces and p-stack convergence spaces, which are defined in this paper.

Let X be a nonempty set. A nonempty collection $\mathcal{H}$ of subsets of X is called a *stack* on X if it satisfies the following conditions:

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(1) $A \in \mathcal{H}$ and $A \subseteq B \subseteq X$ implies $B \in \mathcal{H}$.

A stack $\mathcal{H}$ on X is called a *p-stack* on X if it satisfies the following conditions:

(2) $A \in \mathcal{H}$ and $B \in \mathcal{H}$ implies $A \cap B \neq \emptyset$

Let $S(X)$ denote the set of all stacks on X, $pS(X)$ the set of all p-stacks on X, $F(X)$ the set of all filters on X, and $P(X)$ the power set of X, partially ordered by inclusion; clearly, $F(X) \subseteq pS(X) \subseteq S(X) \subseteq P(X)$.

Condition (2) is called the *pairwise intersection property* (P.I.P.) which is strictly weaker than the well-known *finite intersection property* (F.I.P.). A collection $\mathcal{B}$ of subsets of X with the P.I.P. is called a *p-stack base*. For any collection $\mathcal{B}$, we denote by $\langle \mathcal{B} \rangle = \{A \subseteq X : \exists B \in \mathcal{B}$ such that $B \subseteq A\}$ the stack generated by $\mathcal{B}$, and if $\mathcal{B}$ is p-stack base, then $\langle \mathcal{B} \rangle$ is a p-stack. If $\mathcal{B}$ is a p-stack base with the F.I.P., then $\mathcal{B}$ is a *filter subbase*, and in this case $\mathcal{B}$ generates the filter $[\mathcal{B}] = \{A \subseteq X : \exists B_1, \cdots, B_n \in \mathcal{B}$ such that $\cap_{i=1}^{n} B_i \subseteq A\}$.

The maximal elements in $pS(X)$ (respectively, $F(X)$) are called *ultrapstacks* (respectively, *ultrafilters*). One may easily verify that every ultrafilter is an ultrapstack, and (via Zorn's Lemma) that every p-stack (respectively, filter) is contained in an ultrapstack (respectively, ultrafilter). Henceforth $upS(X)$ denotes the set of all ultrapstacks on X.

PROPOSITION 1.1. ([3]). *For $\mathcal{H} \in pS(X)$, the following are equivalent.*

(1) $\mathcal{H}$ is an ultrapstack;

(2) If $A \cap H \neq \emptyset$, for all $H \in \mathcal{H}$, then $A \in \mathcal{H}$;

(3) $B \notin \mathcal{H}$ implies $X \setminus B \in \mathcal{H}$.

PROPOSITION 1.2. *If $\mathcal{H}$, $\mathcal{G} \in S(X)$, then so is $\mathcal{H} \cup \mathcal{G}$. If $\mathcal{H}$, $\mathcal{G} \in pS(X)$ and $H \cap G \neq \emptyset$ for every $H \in \mathcal{H}$ and $G \in \mathcal{G}$, then $\mathcal{H} \cup \mathcal{G}$ is a p-stack containing both $\mathcal{H}$ and $\mathcal{G}$.*

Proof. It is obvious. ☐

PROPOSITION 1.3. *If* $\mathcal{H}, \mathcal{G} \in pS(X)$ *and* $\mathcal{G} \not\leq \mathcal{H}$, *then there exists an ultrapstack* $\mathcal{K}$ *such that* $\mathcal{H} \leq \mathcal{K}$ *and* $\mathcal{G} \not\leq \mathcal{K}$.

Proof. Assume that $\mathcal{G} \not\leq \mathcal{H}$. Then there exists $G_0 \in \mathcal{G}$ such that $G_0 \notin \mathcal{H}$. Thus $H \not\subseteq G_0$ for all $H \in \mathcal{H}$, so $(X \setminus G_0) \cap H \neq \emptyset$ for all $H \in \mathcal{H}$. By Proposition 1.2, $\mathcal{H} \cup \langle \{X \setminus G_0\} \rangle \in pS(X)$. By Proposition 1.1, there exists an ultrapstack $\mathcal{K} \geq \mathcal{H} \cup \langle \{X \setminus G_0\} \rangle$, where $\mathcal{H} \leq \mathcal{K}$ and $X \setminus G_0 \in \mathcal{K}$. Therefore $\mathcal{G} \not\leq \mathcal{K}$. □

PROPOSITION 1.4. *If* $\mathcal{H} \in pS(X)$, *then* $\mathcal{H} = \cap\{\mathcal{G} \in upS(X) : \mathcal{H} \leq \mathcal{G}\}$.

Proof. Let $\mathcal{K} = \cap\{\mathcal{G} \in upS(X) : \mathcal{H} \leq \mathcal{G}\}$. Clearly, $\mathcal{H} \leq \mathcal{K}$. If $\mathcal{K} \not\leq \mathcal{H}$, by Proposition 1.3, there exists an ultrapstack $\mathcal{L} \geq \mathcal{H}$ such that $\mathcal{L} \not\geq \mathcal{K}$. Since $\mathcal{L} \geq \mathcal{H}$, $\mathcal{L}$ is one of the ultrapstack's in the set interested to obtain $\mathcal{H}$, and so $\mathcal{K} \leq \mathcal{L}$. This is a contradiction. □

PROPOSITION 1.5. *Let* $\mathcal{H} \in pS(X)$. *If* $A \cup B \in \mathcal{H}$ *implies* $A \in \mathcal{H}$ *or* $B \in \mathcal{H}$, *then* $\mathcal{H}$ *is an ultrapstack*

Proof. It follows from Proposition 1.1. □

EXAMPLE 1.6. Let the condition (*) be the statement: $A \cup B \in \mathcal{H} \implies A \in \mathcal{H}$ or $B \in \mathcal{H}$. Not all members of $upS(X)$ satisfy the condition (*). Let X be any set containing at least 3 distinct elements $\{a, b, c\}$. Then consider the ultrapstack $\mathcal{H}$ generated by p-stack base $\{\{a, b\}, \{a, c\}, \{b, c\}\}$. If $A = \{a\}$ and $B = \{b\}$, then $A \cup B = \{a, b\} \in \mathcal{H}$, but $A \notin \mathcal{H}$ and $B \notin \mathcal{H}$.

REMARK. There seem to be two classes of ultrapstack's: those which satisfy (*) and those which do not.

PROPOSITION 1.7. *Let* $\mathcal{H}$ *be an ultrapstack and* $\{\mathcal{G}_i : i \in I\}$ *a set of ultrapstack's on* X.

Then $\mathcal{H} \geq \cap\{\mathcal{G}_i : i \in I\}$ *iff for every* $H \in \mathcal{H}$, *there exists* $i \in I$ *such that* $H \in \mathcal{G}_i$.

Proof. ($\Longrightarrow$) Suppose that there exists $H_0 \in \mathcal{H}$ such that $H_0 \notin \mathcal{G}_i$ for every $i \in I$. Then by Proposition 1.1, $X \setminus H_0 \in \mathcal{G}_i$ for every $i \in I$. Thus $X \setminus H_0 \in \cap_{i \in I} \mathcal{G}_i \leq \mathcal{H}$. This is contradiction to the fact that $\mathcal{H}$ is a stack.

($\Longleftarrow$) Let $A \in \cap\{\mathcal{G}_i : i \in I\}$. If $A \notin \mathcal{H}$, then by Proposition 1.1, $X \setminus A \in \mathcal{H}$. Thus there exists $j \in I$ such that $X \setminus A \in \mathcal{G}_j$, so $A \notin \mathcal{G}_j$. This is contradiction to the hypothesis that $A \in \mathcal{G}_j$. □

F. Hausdorff [1] gave the first definition of what was later called a "topology" by assigning to each point x in a set X a "family of neighborhoods" subject to certain axioms. We use the same approach for defining a "neighborhood structure".

DEFINITION 1.8. ([3]). Let X be a set, and let $\nu \subseteq pS(X)$ be given by $\nu = \{\nu(x) : x \in X\}$, where $\nu(x) \leq \dot{x}$ for all $x \in X$ and $\dot{x}$ denotes the ultrapstack containing $\{x\}$. Then ν is called a *neighborhood structure* on X, $\nu(x)$ is called the *ν-neighborhood p-stack at x*, and (X, ν) is called a *neighborhood space*. For convenience, "neighborhood" will be henceforth abbreviated by "nbd.".

Let $N(X)$ be the set of all nbd. structures on X, partially ordered as follows: $\nu \leq \mu \Leftrightarrow \nu(x) \leq \mu(x)$, for all $x \in X$ (in which case ν is *coarser* than μ and μ is *finer* than ν). A p-stack $\mathcal{H}$ on X ν-converges to x (written $\mathcal{H} \xrightarrow{\nu} x$) if $\nu(x) \leq \mathcal{H}$.

If (X, ν) is a nbd. space and $A \subseteq X$, let

$$I_\nu(A) = \{x \in A : A \in \nu(x)\};$$
$$Cl_\nu(A) = \{x \in X : A \cap V \neq \emptyset, \text{ for all } V \in \nu(x)\}.$$

PROPOSITION 1.9. ([3]). *If (X, ν) is a nbd. space and $A \subseteq X$, then:*

(1) $I_\nu(A) = \{x \in A : A \in \mathcal{H}$, for every p-stack $\mathcal{H} \xrightarrow{\nu} x\}$;

(2) $Cl_\nu(A) = X \setminus I_\nu(X \setminus A)$;

(3) $Cl_\nu(A) = \{x \in X : \exists \mathcal{H} \in pS(X)$ such that $\mathcal{H} \xrightarrow{\nu} x$ and $A \in \mathcal{H}\}$.

PROPOSITION 1.10. *Let (X,ν) be a nbd. space and $A \subseteq X$.*

(1) $I_\nu(A) = \{x \in A : A \in \mathcal{H}$ for every ultrapstack $\mathcal{H} \xrightarrow{\nu} x\}$.

(2) $Cl_\nu(A) = \{x \in X : \exists \mathcal{H} \in upS(X)$ such that $\mathcal{H} \xrightarrow{\nu} x$ and $A \in \mathcal{H}\}$.

Proof. These follow Proposition 1.9 and Proposition 1.4. □

If (X,ν) is a nbd. space and $\mathcal{H} \in p\mathcal{S}(X)$, the associated *closure p-stack* $Cl_\nu(\mathcal{H})$ and *nbd. p-stack* $\nu(\mathcal{H})$ are generated by the p-stack bases $\{cl_\nu(H) : H \in \mathcal{H}\}$ and $\{A \subseteq X : I_\nu(A) \in \mathcal{H}\}$, respectively. Note that $\nu(\dot{x}) = \nu(x)$ is the ν-nbd p-stack at x.

PROPOSITION 1.11. *The interior (or closure) operator for a nbd space (X,ν) is idempotent iff $\nu(\nu(x)) = \nu(x)$, for all $x \in X$.*

Proof. Let $\nu(\nu(x)) = \nu(x)$, for all $x \in X$. If $A \subseteq X$ and $x \in I_\nu(A)$, then $A \in \nu(\nu(x))$ implies $I_\nu(A) \in \nu(x)$, and so $x \in I_\nu(I_\nu(A))$. Thus $I_\nu = I_\nu \cdot I_\nu$. The converse is obvious. □

DEFINITION 1.12. ([3]). A nbd. space (X,ν) is defined to be:

(1) *pretopological* if $\nu(x)$ is a filter, for all $x \in X$;

(2) *supratopological* if $\nu(\nu(x)) = \nu(x)$, for all $x \in X$;

(3) *topological* if pretopological and supratopological.

The terms *pretopology*, *supratopology*, and *topology* will be used for a nbd structure ν which is pretopological, supratopological, or topological. Also, a pretopological nbd. space (X,ν) will be called a *pretopological space*; likewise for *topological space*. A supratopological nbd. space will be called a *closure space*.

THEOREM 1.13. ([3]). *A nbd. space (X,ν) is:*
(1) pretopological iff $I_\nu(A\cap B)=I_\nu(A)\cap I_\nu(B)$ for all $A,B\subseteq X$;
(2) supratopological iff I_ν is idempotent;

Recall that $A\in\nu(x) \iff x\in I_\nu(A)$. Then $\sigma_\nu=\{A\subseteq X : I_\nu(A)=A\}$ is a supratopology on X, that is, a collection of subsets of a set X which contains X and is closed under arbitrary unions, but, unlike a topology, is not required to be closed under finite intersections.

Define $\nu_\sigma(x)=\{N\subseteq X : \exists G\in\sigma_\nu \text{ s.t. } x\in G\subseteq N\}$, which is the σ-nbd. p-stack at $x\in X$.

Then $\nu_\sigma(x)\leq\nu(x)$, so $\nu_\sigma\leq\nu$.

Also, ν_σ is the finest supratopological nbd. structure on X coarser than ν, which is called the supratopological modification of ν, denoted by $\sigma\nu$ ([3]). Also, $\sigma\nu(x)=\langle\{A\in\sigma_\nu : x\in A\}\rangle$.

If ν is supratopological, then $\nu_\sigma(x)=\nu(x)$ and if ν is pretopological, then σ_ν is a topology on X.

2. p-Stack Convergece Spaces

In this section, we will see why filter convergence space is inadequate for defining convergence in nbd. spaces and supratopological spaces and define p-stack convergence structures extending filter convergence structures.

For a supratopological space (X,σ), define $\mathcal{V}_\sigma(x)=\{A\subseteq X : \exists G\in\sigma$ such that
$x\in G\subseteq A\}$ for each $x\in X$. Then $\mathcal{V}_\sigma(x)$ is a p-stack, so $\mathcal{V}_\sigma$ is a supratopological nbd. structure on X. In particular, if σ is a topology on X, $\mathcal{V}_\sigma(x)$ is a filter on X, so $\mathcal{V}_\sigma$ is a topological nbd. structure on X. A p-stack $\mathcal{H}$ on X σ-converges to x (written $\mathcal{H}\xrightarrow{\sigma}x$) if $\mathcal{V}_\sigma(x)\leq\mathcal{H}$.

For a topological space (X,τ) and $U\subseteq X$, define U is semi-open $\iff \exists G\in\tau$ such that $G\subseteq U\subseteq cl_\tau G$, equivalently, $U\subseteq cl_\tau(Int_\tau(U))$. Then the set of all semi-open sets of X, is a supratopology on X, denoted by $s\tau$, which is called the supratopology relative to a topology τ. Then $\tau\leq s\tau$.

PROPOSITION 2.1. *Let (X, τ_1) and (X, τ_2) be topological spaces. Then $\tau_1 = \tau_2 \iff$ they have the same filter convergence.*

Proof. ($\Longrightarrow$) It is obvious.

($\Longleftarrow$) Suppose that (X, τ_1) and (X, τ_2) have the same filter convergence. Then each $\mathcal{F} \in F(X)$ and $x \in X$,

$$\mathcal{V}_{\tau_1}(x) \leq \mathcal{F} \iff \mathcal{V}_{\tau_2}(x) \leq \mathcal{F}.$$

Since $\mathcal{V}_{\tau_1}(x)$ and $\mathcal{V}_{\tau_2}(x)$ are filters, substituting these in place of $\mathcal{F}$, we obtain $\mathcal{V}_{\tau_1}(x) = \mathcal{V}_{\tau_2}(x)$, so $\mathcal{V}_{\tau_1} = \mathcal{V}_{\tau_2}$, and hence $\tau_1 = \tau_2$. □

PROPOSITION 2.2. *Let (X, σ_1) and (X, σ_2) be supratopological spaces. If $\sigma_1 = \sigma_2$ then they have the same filter convergence, but the converse is not true.*

Proof. ($\Longrightarrow$) It is obvious.

(Counter example.) Let τ be the usual topology on R and let $s\tau$ be the supratopology of semi-open sets relative to τ. Let δ denote the discrete topology on R. Recall that $s\tau$ includes all non-trivial closed intervals $[a, b]$ in R, so any $x \in R$ has $s\tau$-neighborhoods of the form $[x, x+\epsilon]$ and $[x-\epsilon, x]$. Thus, $< \mathcal{V}_{s\tau}(x) >= \dot{x} = \mathcal{V}_\delta(x)$, so $\dot{x} \to x$ and no other filters converge, that is, $s\tau$ and δ have the same filter convergence, but $\mathcal{V}_{s\tau} \neq \mathcal{V}_\delta$ and they are obviously different supratopologies ($s\tau \neq \delta$). □

PROPOSITION 2.3. *Let (X, σ_1) and (X, σ_2) be supratopological spaces. Then $\sigma_1 = \sigma_2 \iff$ they have the same p-stack convergence.*

Proof. ($\Longrightarrow$) It is obvious.

($\Longleftarrow$) Suppose that (X, σ_1) and (X, σ_2) have the same p-stack convergence. Then each $\mathcal{F} \in pS(X)$ and $x \in X$,

$$\mathcal{V}_{\sigma_1}(x) \leq \mathcal{F} \iff \mathcal{V}_{\sigma_2}(x) \leq \mathcal{F}.$$

Since $\mathcal{V}_{\sigma_1}(x)$ and $\mathcal{V}_{\sigma_2}(x)$ are p-stacks, substituting these in place of $\mathcal{F}$, we obtain $\mathcal{V}_{\sigma_1}(x) = \mathcal{V}_{\sigma_2}(x)$, so $\mathcal{V}_{\sigma_1} = \mathcal{V}_{\sigma_2}$, and hence $\sigma_1 = \sigma_2$. □

Now, we define *p-stack convergence structure* by using "p-stacks" in stead of "filters", which are used in (filter) convergence structures. ([2]).

A *p-stack convergence structure* c on a set X is defined to be a function from $pS(X)$ into $P(X)$ satisfying the following conditions :

(1) $x \in c(\dot{x})$ for each $x \in X$;

(2) if $\mathcal{H} \subseteq \mathcal{G}$, then $c(\mathcal{H}) \subseteq c(\mathcal{G})$;

(3) if $x \in c(\mathcal{H})$, then $x \in c(\mathcal{H} \cap \dot{x})$;

where $\dot{x}$ denotes the ultrapstack containing $\{x\}$; $\mathcal{H}$ and $\mathcal{G}$ are in $pS(X)$. Then c is said to be a *a p-stack convergence structure* on X, and the pair (X, c) a *p-stack convergence space.* If $x \in c(\mathcal{H})$, we say that $\mathcal{H}$ c *-converges* to x, which is denoted by $\mathcal{H} \xrightarrow{c} x$. The p-stack $\mathcal{V}_c(x)$ obtained by intersecting all stacks which c*-converge* to x is said to be a c*-neighborhood p-stack* at x. If $\mathcal{V}_c(x)$ c*-converges* to x for each $x \in X$, then c is said to be *weakly topological* (or a *weak topological p-stack convergence structure*) on X, and (X, c) a *weak topological p-stack convergence space.*

PROPOSITION 2.4. *Let (X, c) be a p-stack convergence space, The following statements are equivalent:*

(1) c is weakly topological.

(2) $c(\cap\{\mathcal{H}_\lambda : \lambda \in \Lambda\}) = \cap\{c(\mathcal{H}_\lambda) : \lambda \in \Lambda\}$ for every $\{\mathcal{H}_\lambda : \lambda \in \Lambda\} \subseteq pS(X)$.

Proof. Let $\{\mathcal{H}_\lambda : \lambda \in \Lambda\} \subseteq pS(X)$. Since $\cap\{\mathcal{H}_\lambda : \lambda \in \Lambda\} \subseteq \mathcal{H}_\lambda$, $c(\cap\{\mathcal{H}_\lambda : \lambda \in \Lambda\}) \subseteq c(\mathcal{H}_\lambda)$ for all $\lambda \in \Lambda$ and so $c(\cap\{\mathcal{H}_\lambda : \lambda \in \Lambda\}) \subseteq \cap\{c(\mathcal{H}_\lambda) : \lambda \in \Lambda\}$. Let $x \in \cap\{c(\mathcal{H}_\lambda) : \lambda \in \Lambda\}$. Then $x \in c(\mathcal{H}_\lambda)$ and $\mathcal{V}_c(x) \subset \mathcal{H}_\lambda$ for all $\lambda \in \Lambda$. Since c is a weak topological p-stack convergence structure, $x \in c(\mathcal{V}_c(x)) \subseteq c(\cap\{\mathcal{H}_\lambda : \lambda \in \Lambda\})$. Thus $\cap\{c(\mathcal{H}_\lambda) : \lambda \in \Lambda\} \subseteq c(\cap\{\mathcal{H}_\lambda : \lambda \in \Lambda\})$. Finally, $\cap\{c(\mathcal{H}_\lambda) : \lambda \in \Lambda\} = c(\cap\{\mathcal{H}_\lambda : \lambda \in \Lambda\})$.

Conversely, let $x \in X$. Then $c(\mathcal{V}_c(x)) = c(\cap\{\mathcal{H} : x \in c(\mathcal{H})\})$ $= \cap\{c(\mathcal{H}) : x \in c(\mathcal{H})\} \ni x$. Hence c is weakly topological. □

Let $pSC(X)$ be the set of all p-stack convergence structures on X, partially ordered as follows :

$$c_1 \leq c_2 \text{ iff } c_2(\mathcal{H}) \subseteq c_1(\mathcal{H}) \text{ for all } \mathcal{H} \in pS(X).$$

If $c_1 \leq c_2$, then we say that c_1 is *coarser* than c_2, and c_2 is *finer* than c_1.

Also, we know that if c_1 is weakly topological, then

$$c_1 \leq c_2 \text{ iff } \mathcal{V}_{c_1}(x) \leq \mathcal{V}_{c_2}(x) \text{ for all } x \in X.$$

Let $C(X)$ and $N(X)$ be the collection of all (filter) convergence structures on X and the collection of all nbd. structures on X, respectively. Then we know that $C(X)$ and $N(X)$ are complete lattices. Also, the following Proposition 2.6 shows that $pSC(X)$ is a complete lattice

PROPOSITION 2.6. *For* $\{c_i : i \in I\} \subseteq pSC(X)$, *let* c *and* d *be defined by*

$$c(\mathcal{H}) = \cap_{i \in I} c_i(\mathcal{H}), \quad d(\mathcal{H}) = \cup_{i \in I} c_i(\mathcal{H})$$

for every $\mathcal{H} \in pS(X)$. *Then,*

(1) $c \in pSC(X)$ *and* $c = \sup_{pSC(X)}\{c_i : i \in I\}$.

(2) $d \in pSC(X)$ *and* $d = \inf_{pSC(X)}\{c_i : i \in I\}$.

Proof. It is easy to check $c, d \in pSC(X)$.

(1) Let c' be an upper bound of $\{c_i : i \in I\} \in pSC(X)$. Then $c' \geq c_i$ for all $i \in I$, so $c'(\mathcal{H}) \subseteq c_i(\mathcal{H})$ for all $i \in I$ and $\mathcal{H} \in pSC(X)$, and hence $c'(\mathcal{H}) \subseteq \cap_{i \in I} c_i(\mathcal{H}) = c(\mathcal{H})$. Thus, $c' \geq c$, so c is the least upper bound of $\{c_i : i \in I\}$ in $pSC(X)$. Consequently, $c = \sup_{pSC(X)}\{c_i : i \in I\}$.

(2) It is similar (1). □

3. Relations between Nbd. Spaces and p-Stack Convergence Spaces

Finally, we show that each nbd. structure ν on X is uniquely determined by p-stack convergence structure c_ν on X, and p-stack convergence spaces are a generalization of nbd. spaces.(Theorem 3.4).

DEFINITION 3.1. Let (X, c) be a p-stack convergence space. Then $(X, \mathcal{V}_c)$ is called the *nbd. space related to the p-stack convergence space* (X, c), if $\mathcal{V}_c$ is defined as follows;

$$\mathcal{V}_c(x) = \cap\{\mathcal{H} \in upS(X) : \mathcal{H} \xrightarrow{c} x\}$$

for every $x \in X$. Also, we define the interior and the closure of $A \subseteq X$ respectively, as follows:

$$I_c(A) = \{x \in A : A \in \mathcal{V}_c(x)\},$$

$$Cl_c(A) = \{x \in X : \exists \mathcal{H} \in upS(X) \text{ such that } \mathcal{H} \xrightarrow{c} x \text{ and } A \in \mathcal{H}\}.$$

By the above definition and Proposition 1.4, we know the following:

$$\mathcal{V}_c(x) = \cap\{\mathcal{H} \in pS(X) : \mathcal{H} \xrightarrow{c} x\};$$

$$I_c(A) = I_{\mathcal{V}_c}(A);$$

$$Cl_c(A) = \{x \in X : \exists \mathcal{H} \in pS(X) \text{ such that } \mathcal{H} \xrightarrow{c} x \text{ and } A \in \mathcal{H}\}.$$

DEFINITION 3.2. Let (X, ν) be a nbd. space. Then (X, c_ν) is called the *p-stack convergence space relative to the nbd. space* (X, ν), if c_ν is defined as follows;

$$c_\nu(\mathcal{H}) = \{x \in X : \mathcal{H} \xrightarrow{\nu} x\}$$

for every $\mathcal{H} \in pS(X)$.

It is easy to check that c_ν is a p-stack convergence structure and $\mathcal{V}_c$ is a nbd. structure on X.

By the definitions of 3.1 and 3.2, we know that for a nbd. structure ν on X,

$$\mathcal{H} \xrightarrow{\nu} x \iff x \in c_\nu(\mathcal{H}) \iff \mathcal{H} \xrightarrow{c_\nu} x,$$

and for a p-stack convergence structure c on X,

$$\mathcal{H} \xrightarrow{c} x \implies \mathcal{V}_c(x) \leq \mathcal{H} \iff \mathcal{H} \xrightarrow{\mathcal{V}_c} x,$$

where if c is weakly topological, then the converse is true, so c and $\mathcal{V}_c$ have the same p-stack convergence.

Thus we obtain the following Theorem 3.3.

THEOREM 3.3. *Under the above definitions, the following are true:*

(1) (X, c_ν) *is a weak topological p-stack convergence space;*

(2) $\mathcal{V}_{c_\nu} = \nu$*;*

(3) $c_{\mathcal{V}_c} \leq c$*. If* c *is weakly topological, then the equality holds.*

Proof. (1) and (2) follow from the following:
$\mathcal{V}_{c_\nu}(x) = \cap\{\mathcal{H} : \mathcal{H} \xrightarrow{c_\nu} x\} = \cap\{\mathcal{H} : \mathcal{H} \xrightarrow{\nu} x\} = \nu(x) \xrightarrow{\nu} x$, so $\mathcal{V}_{c_\nu}(x) \xrightarrow{c_\nu} x$.

(3) $c_{\mathcal{V}_c}(\mathcal{H}) = \{x \in X : \mathcal{H} \xrightarrow{\mathcal{V}_c} x\} \supseteq \{x \in X : \mathcal{H} \xrightarrow{c} x\} = c(\mathcal{H})$.

If c is weakly topological, then $x \in c_{\mathcal{V}_c}(\mathcal{H}) \implies \mathcal{H} \xrightarrow{\mathcal{V}_c} x \implies \mathcal{V}_c(x) \leq \mathcal{H} \implies c(\mathcal{V}_c(x)) \leq c(\mathcal{H}) \implies x \in c(\mathcal{H})$, c, and so the equality holds. ☐

THEOREM 3.4. *Let* ν_1 *and* ν_2 *be nbd. structures on* X*, and* c_1 *and* c_2 *p-stack convergence structures on* X*. Then the following are true;*

(1) $\nu_1 \leq \nu_2$ *iff* $c_{\nu_1} \leq c_{\nu_2}$*.*

(2) $c_1 \leq c_2$ *implies* $\mathcal{V}_{c_1} \leq \mathcal{V}_{c_2}$*. If* c_1 *and* c_2 *are weakly topological, then the converse is true.*

Proof. (1) Let $\nu_1 \leq \nu_2$. To show that $c_{\nu_1} \leq c_{\nu_2}$, let $\mathcal{H} \in pS(X)$. Then we will show that $c_{\nu_1}(\mathcal{H}) \supseteq c_{\nu_2}(\mathcal{H})$. Let $x \in c_{\nu_2}(\mathcal{H}) \iff \mathcal{H} \xrightarrow{\nu_2} x \iff \nu_2(x) \leq \mathcal{H} \implies \nu_1(x) \leq \mathcal{H} \iff \mathcal{H} \xrightarrow{\nu_1} x \iff x \in c_{\nu_1}(\mathcal{H})$.

Conversely, let $c_{\nu_1} \leq c_{\nu_2}$. Then $\mathcal{V}_{c_{\nu_1}} \leq \mathcal{V}_{c_{\nu_2}}$, so by Theorem 3.3 (2), $\nu_1 \leq \nu_2$.

(2) Let $c_1 \leq c_2$. Then $\mathcal{H} \xrightarrow{c_2} x \implies \mathcal{H} \xrightarrow{c_1} x$. Thus,

$$\mathcal{V}_{c_1}(x) = \cap\{\mathcal{H} \in pS(X) : \mathcal{H} \xrightarrow{c_1} x\} \leq \cap\{\mathcal{H} \in pS(X) : \mathcal{H} \xrightarrow{c_2} x\} = \mathcal{V}_{c_2}(x).$$

If c_1 and c_2 are weakly topological, then, by Theorem 3.3 (3), the converse is true. ☐

We know that if a p-stack convergence structure c is weakly topological, c and the related nbd. structure $\mathcal{V}_c$ have the same p-stack convergence, so their interior operator and closure operators are equal.

The following Proposition shows that even though a p-stack convergence structure c is not weakly topological and c and $\mathcal{V}_c$ have different p-stack convergences, their interior operator and closure operators are equal.

PROPOSITION 3.5. *Let (X, c) be a p-stack convergence space and $A \subseteq X$. Then*

(1) $X \setminus Cl_c(A) = I_c(X \setminus A)$.

(2) $Cl_c(A) = Cl_{\mathcal{V}_c}(A)$.

Proof. (1) $x \in X \setminus Cl_c(A) \iff x \notin Cl_c(A) \iff \forall \mathcal{H} \in upS(X)$ with $\mathcal{H} \xrightarrow{c} x$, $A \notin \mathcal{H} \iff \forall \mathcal{H} \in upS(X)$ with $\mathcal{H} \xrightarrow{c} x$, $X \setminus A \in \mathcal{H} \iff X \setminus A \in \mathcal{V}_c(x) \iff x \in I_c(X \setminus A)$.

(2) Recall that $Cl_{\mathcal{V}_c}(A) = \{x \in X : A \cap V \neq \emptyset$, for all $V \in \mathcal{V}_c(x)\}$. Let $x \in Cl_c(A)$. Then there exists $\mathcal{H} \in upS(X)$ such that $\mathcal{H} \xrightarrow{c} x$ and $A \in \mathcal{H}$, so $\mathcal{V}_c(x) \leq \mathcal{H}$. Since $\mathcal{H}$ is a p-stack, $A \cap V \neq \emptyset$, for all $V \in \mathcal{V}_c(x)$. Thus $x \in Cl_{\mathcal{V}_c}(A)$. Conversely, let $x \in Cl_{\mathcal{V}_c}(A)$. Then $V \cap A \neq \emptyset$ for all $V \in \mathcal{V}_c(x)$. By Proposition 1.2, $\mathcal{H} = \mathcal{V}_c(x) \cup \dot{A}$ is a p-stack, where $\dot{A} = \langle \{A\} \rangle$. Thus, $A \in \mathcal{H}$ and $\cap\{\mathcal{H} \in upS(X) : \mathcal{H} \xrightarrow{c} x\} = \mathcal{V}_c(x) \leq \mathcal{H}$. By Proposition 1.7, there exists $\mathcal{G}_A \in upS(X)$ such that $A \in \mathcal{G}_A$ and $\mathcal{G}_A \xrightarrow{c} x$. Since $\mathcal{V}_c(x) \leq \mathcal{G}_A$ and $\dot{A} \leq \mathcal{G}_A$, we obtain $\mathcal{H} \leq \mathcal{G}_A$. Therefore $x \in Cl_c(A)$. □

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