East Asian Math. J. **21** (2005), No. 1, pp. 1–7

ON GENERALIZED LIE IDEALS IN SEMI-PRIME RINGS WITH DERIVATION

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Abstract. The object of this paper is to study (σ,τ)-Lie ideals in semi-prime rings with derivation. Main result is the following theorem: Let R be a semi-prime ring with $2-$torsion free, σ and τ two automorphisms of R such that $\sigma\tau=\tau\sigma$, U be both a non-zero (σ,τ)-Lie ideal and subring of R. If $d^2(U)=0$, then $d(U)=0$ where d a non-zero derivation of R such that $d\sigma=\sigma d,\ d\tau=\tau d$.

1. Introduction

The notion of (σ,τ)-Lie ideals in a ring was introduced by Kaya in [7]. Then, Kaya, Aydin, Kandamar, Soytürk and some authors studied the structure of (σ,τ)-Lie ideals in a prime ring and obtained various generalizations analogous of corresponding parts in Lie ideal(in prime and semi-prime rings).

In [3], Carini proved that if R is a $2-$torsion free semi-prime ring with a derivation d and U is a Lie ideal of R such that $d^2(U)=0$, then $d(U)\subset Z$,the center of R. In [8], Soytürk proved that if R is a prime ring with $char R\neq 2,\ 3$, U is a non-zero (σ,τ)- Lie ideal of R, d is a nonzero derivation of R such that $d(U)\subset U$ and $d^2(U)\subset Z$ then $U\subset Z$, the center of R. Also, in [1], Aydin and Soytürk proved that if U is (σ,τ)-Lie ideal of a prime ring and d is a derivation of R such that $d^2(U)=0$, then $d(U)\subset Z$. In [6], Kaya proved the theorem

Received August 26, 2004.
2000 Mathematics Subject Classification: 16A12, 16A38, 16A66, 16N60, 16Y99.
Key words and phrases: Semi-prime ring, prime ring, derivation, lie ideal, (σ,τ)-Lie ideal.

in [8] when $char\ R = 3$. In this paper, we will generalize the above results.

2. Preliminaries

Let R be a ring, U an additive subgroup of R and σ, τ two mappings of R. We set $[x,y]_{\sigma,\tau} = x\sigma(y) - \tau(y)x$. The definition of (σ,τ)-Lie ideal was given in [7] as follows: (i) U is called a (σ,τ)-right Lie ideal of R if $[U,R]_{\sigma,\tau} \subset U$. (ii) U is called a (σ,τ)-left Lie ideal of R if $[R,U]_{\sigma,\tau} \subset U$. (iii) U is both a (σ,τ)-right Lie ideal and (σ,τ)-left Lie ideal of R, U is called (σ,τ)-Lie ideal of R. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all pairs $x, y \in R$. An additive mapping $d : R \to R$ is called an $(\sigma,\tau)-$inner derivation if there exists $a \in R$ such that $d(x) = [a,x]_{\sigma,\tau}$.

Throughout, R will represent a $2-$torsion free semi-prime ring, σ and τ automorphisms of R such that $\sigma\tau = \tau\sigma$, $\sigma(U) \subset U$ and $\tau(U) \subset U$, d a non-zero derivation of R such that $d\sigma = \sigma d$, $d\tau = \tau d$ and $C_{\sigma,\tau} = \{c \in R | c\sigma(x) = \tau(x)c \text{ for all } x \in R\}$. Further, we shall often use the relations:

$[xy,z]_{\sigma,\tau} = x[y,z]_{\sigma,\tau} + [x,\tau(z)]y = x[y,\sigma(z)] + [x,z]_{\sigma,\tau}y$

and

$[x,yz]_{\sigma,\tau} = \tau(y)[x,z]_{\sigma,\tau} + [x,y]_{\sigma,\tau}\sigma(z)$.

THEOREM 1. **([8, Theorem]).** *Let R be a prime ring $charR \neq 2,3$, U a non-zero (σ,τ)-Lie ideal of R, d a non-zero derivation of R such that $d(U) \subset U$ and $d^2(U) \subset Z$, then $U \subset Z$, the center of R.*

THEOREM 2. **([6, Theorem])** *Let R be a prime ring of characteristic 3, σ and τ two automorphism of R, U a non-zero (σ,τ)-Lie ideal of R. If Z is the center of R and d is a non-zero derivation of R such that $\sigma d = d\sigma$, $\tau d = d\tau$, $d(U) \subset U$ and $d^2(U) \subseteq Z$, then $U \subset Z$.*

3. Results

We remark that let R be a semi-prime ring with $2-torsion\ free$, U a (σ,τ)-Lie ideal of R and d a non-zero derivation of R such that $d^2(U) = 0$. Then, $d(U) + U$ is a (σ,τ)-Lie ideal of R stable under d.

Moreover $d^2(d(U)+U)=0$. Therefore, without lost of generality, we may assume that $d(U)\subset U$.

EXAMPLE 1. *Let* $M := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,\ b,\ c,\ d \in I, \text{ the set of integers} \right\}$, $N := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a,\ d \in I, \text{ the set of integers} \right\} \subset M$, *$I[x]$ be the polynomial ring over the set of integers I. If $R = M \oplus I[x]$, then R is a semi-prime ring. In this case, if $U := \{f \mid f$ is a polynomial in x of degree ≤ 1 $\} \subset I[x]$, $V = N \oplus U$ is a left (σ,τ)-Lie ideal but not Lie ideal of R, where* $\sigma_1 : M \longrightarrow M, \sigma_1(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (\begin{pmatrix} d & -c \\ -b & a \end{pmatrix})$, $\tau_1 : M \longrightarrow M$, $\tau_1(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = (\begin{pmatrix} a & -b \\ -c & d \end{pmatrix})$, $\sigma : M \oplus I[x] \longrightarrow M \oplus I[x]$, $\sigma(m, f(x)) = (\sigma_1(m), f(x))$ *and* $\tau : M \oplus I[x] \longrightarrow M \oplus I[x]$, $\tau(m, f(x)) = (\tau_1(m), f(x))$. *Moreover, let d be the derivation defined on R as follows: d is the usual derivation on the polynomial ring $I[x]$ and $d = 0$ on M. Then, $d(V) \neq 0$, but $d^2(V) = 0$ and $d(V) \subset C_{\sigma,\tau}$.*

LEMMA 1. *Let R be a semi-prime ring and $a \in R$. If $a[a,x]_{\sigma,\tau} = 0$ for all $x \in R$, then $a \in C_{\sigma,\tau}$.*

Proof. Replacing x by xr in the hypothesis and using the hypothesis, we get $a\tau(x)[a,r]_{\sigma,\tau} = 0$ for all $x, r \in R$. Since τ is an automorphism, we have $ax[a,r]_{\sigma,\tau} = 0$ for all $x, r \in R$. But this gives $[a,r]_{\sigma,\tau}R[a,r]_{\sigma,\tau} = 0$ for all $r \in R$. Since R is semi-prime ring, we get $[a,r]_{\sigma,\tau} = 0$ for all $r \in R$, thus $a \in C_{\sigma,\tau}$. □

LEMMA 2. *Let R be a semi-prime ring with $2-$torsion free, U a non-zero (σ,τ)-Lie ideal of R and d a non-zero derivation of R. If $d^2(U) = 0$, then $d([R,U]_{\sigma,\tau}) = 0$.*

Proof. From the hypothesis, $d^2(u) = d^2(v) = 0$ for all $u, v \in U$. Thus, since $[u,v]_{\sigma,\tau} \in U$, $0 = d^2([u,v]_{\sigma,\tau}) = d(d(u\sigma(v) - \tau(v)u)) = d(u)d(\sigma(v)) + ud(d(\sigma(v))) + d(d(u))\sigma(v) + d(u)d(\sigma(v)) - \tau(v)d(d(u)) - d(\tau(v))d(u) - d(d(\tau(v)))u - d(\tau(v))d(u)$. As a consequence, since $d\sigma = \sigma d$, $d\tau = \tau d$ and R is $2-$torsion free, we get, for all $u, v \in U$

$$[d(u), d(v)]_{\sigma,\tau} = 0. \tag{3.1}$$

For $u \in U$ and $x \in R$, since $[\tau(u)x, u]_{\sigma,\tau} \in U$, $\tau(u)[x, u]_{\sigma,\tau} \in U$. Let $v = [x, u]_{\sigma,\tau}$, $x \in R$, then v and $\tau(u)v \in U$. Therefore, $0 = d^2(\tau(u)v) = \tau(u)d^2(v) + d(\tau(u))d(v) + d^2(\tau(u))v + d(\tau(u))d(v)$ and so, from the hypothesis and since $d\tau = \tau d$ and R is $2-torsion\ free$, we get, for all $u, v \in U$

$$d(\tau(u))d(v) = 0. \tag{3.2}$$

Replacing v by $\tau(v)$ and using the fact that τ is an automorphism commuting with d, we get

$$d(u)d(v) = 0. \tag{3.3}$$

for all $u, v \in U$. Also, since U is a nonzero (σ, τ)-Lie ideal of R,we have

$$d(w_1)d(w_2) = 0. \tag{3.4}$$

for all $w_1, w_2 \in [R, U]_{\sigma,\tau} = W$. Let P be a prime ideal of R. If $d(P) \subset P$, then $\overline{R} = R/P$ is a prime ring with induced derivation $\overline{d}$ and $\overline{d^2}(\overline{U}) = \overline{0}$. If $\overline{d} \neq \overline{0}$, then $\overline{U} \subset Z(\overline{R})$ by Theorem 1 and Theorem 2. This implies that $[\overline{R}, \overline{U}]_{\sigma,\tau} = \overline{0}$ and so, $[R, U]_{\sigma,\tau} \subset P$. On the other hand, if $d(P) \not\subset P$ (i.e. $\overline{d} \neq \overline{0}$), then $d(P) + P$ is an ideal of R and so, $\overline{d(P)} + \overline{P} = \overline{d(P)} \neq \overline{0}$ is an ideal of $\overline{R}$. Thus, from (3.4), we get $0 = d(w_1)d([r, d(w_2)]_{\sigma,\tau}) = d(w_1)d(r)d(\sigma(w_2))$ in R. This implies that $\overline{d(w_1)}\,\overline{d(r)}\,\overline{d(\sigma(w_2))} = \overline{0}$ in $\overline{R}$. Therefore since σ is automorphism and $\bar{R}$ is prime ring, we get that $\overline{d(W)} = \overline{0}$ and so $d(W) \subset P$. Thus, $d(W)W \subset P$ for all prime ideals P of R. Since R is semi-prime ring, implies that $d(W)W = 0$. Thus since $d(W) \subset W$, for $w_1, w_2 \in W$, $r \in R$, we get, $0 = d(w_1)[r, d(w_2)]_{\sigma,\tau} = d(w_1)rd(\sigma(w_2)) - d(w_1)d(\tau(w_2))r = d(w_1)rd(\sigma(w_2))$ by (3.4). Thus, $d(w_1)rd(\sigma(w_2)) = 0$ for all $w_1, w_2 \in W$. Since σ is automorphism and R is semi-prime ring, we get that $d(w) = 0$ for all $w \in W$. Therefore $d(W) = 0$, i.e. $d([R, U]_{\sigma,\tau}) = 0$. □

Theorem 3. .*Let R be a semi-prime ring with $2-$torsion free, U be both a non-zero (σ, τ)-Lie ideal and subring of R. If $d^2(U) = 0$, then $d(U) = 0$ where d is a non-zero derivation of R.*

Proof. .Since U is a subring of R, $uv \in U$ for all $u, v \in U$. Then, using $d^2(U) = 0$ and since R is $2-$torsion free, we get, for all $u, v \in U$

$$d(u)d(v) = 0. \tag{3.5}$$

Since $d([x,u]_{\sigma,\tau}) = 0$ for all $u \in U$ and $x \in R$ by Lemma 2, Thus, $0 = d([x,u]_{\sigma,\tau}) = [x, d(u)]_{\sigma,\tau} + [d(x), u]_{\sigma,\tau}$. Replacing x by $\tau(u)x$, $u \in U$ in this last relation,

$$\begin{aligned} 0 &= [\tau(u)x, d(u)]_{\sigma,\tau} + [d(\tau(u)x), u]_{\sigma,\tau} \\ &= \tau(u)[x, d(u)]_{\sigma,\tau} + [\tau(u), d(\tau(u))]x + \tau(u)[d(x), u]_{\sigma,\tau} \\ &\quad +[\tau(u), \tau(u)]d(x) + d(\tau(u))[x, u]_{\sigma,\tau} + [d(\tau(u)), \tau(u)]x \end{aligned}$$

and so, we get, for all $u \in U$ and $x \in R$

$$d(\tau(u))[x, u]_{\sigma,\tau} = 0. \tag{3.6}$$

Replacing u by $u + v$, $v \in U$ in (3.6) and using (3.6) we get, for all $u, v \in U$ and $x \in R$

$$d(\tau(u))[x, v]_{\sigma,\tau} + d(\tau(v))[x, u]_{\sigma,\tau} = 0. \tag{3.7}$$

Replacing v by $d(v)$ in (3.7) and from the hypothesis we get, for all $u, v \in U$ and $x \in R$

$$\begin{aligned} 0 &= d(\tau(u))[x, d(v)]_{\sigma,\tau} \\ &= d(\tau(u))xd(\sigma(v)) - d(\tau(u)d(\tau(v))x \\ &= d(\tau(u))xd(\sigma(v)) - d(\tau(v)d(\sigma(u))x \end{aligned}$$

by (3.1). Now, we replace u and v by $\tau(u)$ and $\sigma(v)$ respectively in this last relation. Since $\sigma\tau = \tau\sigma$, we have $d(\tau^2(u))xd(\sigma^2(v)) - d(\tau(\sigma(v)))d(\tau(\sigma(u)))x = 0$ and so, since σ and τ are automorphisms and using (3.5), this last equality reduces to

$$d(\tau^2(u))xd(\sigma^2(v)) = 0 \tag{3.8}$$

for all $u, v \in U$. Replacing $\tau^2(x)$ by x , $\tau^2(v)$ by v and using τ^2 is an automorphism we get

$$d(u)xd(\sigma^2(v)) = 0 \tag{3.9}$$

for all $x \in R$ and $u, v \in U$. Similarly, replacing $\sigma^2(x)$ by x , $\sigma^2(v)$ by u,we have

$$d(v)xd(v) = 0 \tag{3.10}$$

for all $v \in U$. So, since R is semi-prime ring we get that $d(U) = 0$. □

PROPOSITION 1. *Let R be a semi-prime ring with $2-$torsion free, U be both a non-zero (σ,τ)-Lie ideal and subring of R. If $d^2(U)=0$, then $d(R)[U,R]_{\sigma,\tau}=0$ where d is a non-zero derivation of R.*

Proof. Since $d(u)=0$ for all $u\in U$ by Theorem 3 and since $[u,x]_{\sigma,\tau}\in U$ where $u\in U$ and $x\in R$, $0=d([u,x]_{\sigma,\tau})=d(u)\sigma(x)+ud(\sigma(x))-d(\tau(x))u-\tau(x)d(u)$ and so, since $d\sigma=\sigma d$, $d\tau=\tau d$ we get, for all $u\in U$ and $x\in R$

$$[u,d(x)]_{\sigma,\tau}=0. \tag{3.11}$$

In (3.11), replacing x by xy, $y\in R$ and from (3.11) we get, for all $u\in U$ and $x,y\in R$

$$\tau(d(x))[u,y]_{\sigma,\tau}+[u,x]_{\sigma,\tau}\sigma(d(y))=0. \tag{3.12}$$

But from (3.11), the last equality reduces to

$$\tau(d(x))[u,y]_{\sigma,\tau}+\tau(d(y))[u,x]_{\sigma,\tau}=0 \tag{3.13}$$

for all $u\in U$ and $x,y\in R$. Thus, in (3.13), replacing y by x and since R is $2-$torsion free, we get that $\tau(d(x))[u,x]_{\sigma,\tau}=0$ and so, since $d\tau=\tau d$ and τ is automorphism, we get that $d(R)[U,R]_{\sigma,\tau}=0$. □

COROLLARY 1. *Let R be a semi-prime ring with $2-$torsion free, U a non-zero (σ,τ)-Lie ideal and subring of R. If $d^2(U)=0$, then $d(U)=0$ where d is a non-zero (σ,τ)-inner derivation of R.*

COROLLARY 2. *Let R be a semi-prime ring with $2-$torsion free, U be both a non-zero (σ,τ)-Lie ideal and subring of R. If $[a,[a,U]_{\sigma,\tau}]_{\sigma,\tau}=0$ for some $a\in R$, then $d_a(U)=0$.*

Proof. $d_a(U)=[a,U]_{\sigma,\tau}$ is a (σ,τ)-inner derivation.
Thus, If $d_a(d_a(U))=d_a^2(U)=0$ and so we have $d_a(U)=0$ by Corollary 1. □

ACKNOWLEDGEMENT 1. *The authors are highly grateful to the referees for their valuable comments and suggestions for improving the paper.*

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