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TILING OF CLOSED PLANE CURVES

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Abstract. In this paper, we introduced the tiling, for closed plane curves $\alpha(s)$, and we discussed the properties of tiling. Also if $\alpha(s)$ was arbitrary plane closed curve equipped by tiling $\Im$ then we studied the effect of retraction and tiling retraction on it.

1. Introduction

The notion of the tiling introduced by John M. Lee in [2]. More studies of the tiling of regular curves introduced by M. El-Ghoul and M. Basher in [3]. Tiling are also known as **tessellation**, **paving** or **triangulation**.

Definition 1.1. A regular curve $\alpha : [a, b] \longrightarrow R^2$ is called closed, if there is a (regular curve) $\tilde{\alpha} : R \longrightarrow R^2$ with $\tilde{\alpha}|_{[a,b]} = \alpha$ and $\tilde{\alpha}(t+b-a) = \tilde{\alpha}(t)$ for all $t \in R$, where in particular $\alpha(a) = \alpha(b)$ and $\alpha^{\cdot}(a) = \alpha^{\cdot}(b)$ [4].

The vector $\alpha^{\cdot}$ is the tangent vector to α. A closed curve α is said to be simply closed if $\alpha|_{[a,b)}$ is injective.

Definition 1.2. A regular curve segment is a map $\alpha : [a, b] \longrightarrow R^2$ together with open interval (c, d), with $c < a < b < d$, and a regular curve $\gamma : (c, d) \longrightarrow R^2$ such that $\alpha(t) = \gamma(t)$ for all $t \in [a, b]$ [1].

We know that an arbitrary closed curve in R^2 is homotopic to a simple curve or an eight-shaped curve or a m-self-looped curve. Observe

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that the "eight-shaped" and the "m-self-looped" curves contain branch points.

DEFINITION 1.3. Let X be a space, $A \subset X$. A subset A is called a retract of X if there is a retraction $r : X \to A$, *i.e.*, a continuous map with $r|_A = Id_A$ [5].

2. Tiling of closed plane curves

Now in this section we will discuss the tiling for the regular closed curves in R^2. Let $\alpha : [0, l] \to R^2$ be a regular closed curve given by $\alpha(s) = (x(s), y(s))$, where s is the arc length.

DEFINITION 2.1. A tiling of the regular closed curve $\alpha(s)$ is a collection $\Im = \{T_i, i \in I = \{1, 2, 3, ..., n\}\}$ of segment curves (tiles) which cover the curve α.

The regular segment curves as tiles, $T_i =< v_i, v_{i+1} >$, the interior of the segment curves called edges, α_i, and the boundary of each segment curve called vertices, v_i. See Figure 1.

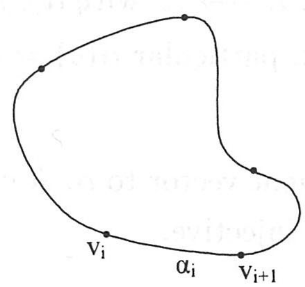


FIGURE 1.

Properties of the tiling

Every tiling of the regular closed curve $\alpha(s)$ must satisfying these properties:

(1) each tiles in not self-intersect,

(2) each vertex lies on exactly two edges,

(3) $\bigcup_{i=1}^{n} T_i = \alpha$.

Also we shall concentrate our attention to tiling that are edge to edge by this we mean that as far as the mutual relation of any two tiles is concerned there are just two possibilities:
(i) They are disjoint (have no point in common),
(ii) They have precisely one common point which is vertex of each of the segment curves.

EXAMPLE 2.1. *Let $\alpha(s)$ be an eight-shaped curve in R^2 equipped by two different covers $\Im_1$ and $\Im_2$. See Figure 2.*

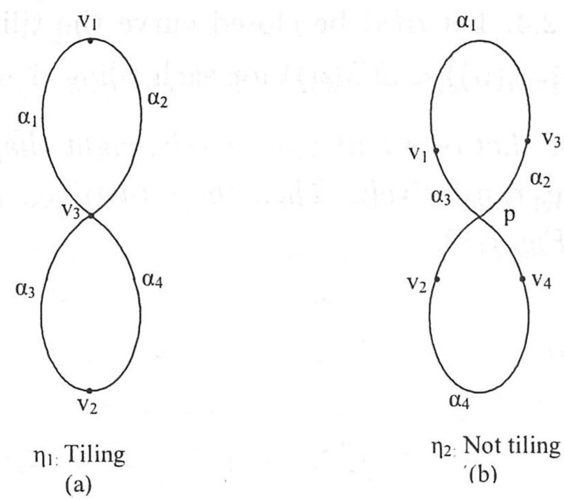


FIGURE 2

In Figure(2-a) the η_1 is tiling since each two tiles $\alpha_i, \alpha_j \in \eta_1$ are intersect in the vertices. But in Figure(2-b) the η_2 is not tiling since the tiles α_2, α_3 intersect in point p which is not vertex.

DEFINITION 2.2. The degree of tiling, $d(\Im)$, is the number of tiles, i.e., $d(\Im) = n$.

Now let α has branch point c, and let p the number of loops at point c.

DEFINITION 2.3. The degree of the vertex, $d(v_i)$, is the number of the loops which connected at the vertex v_i, i.e., $d(v_i) = p$, where p is the number of loops at vertex v_i

LEMMA 2.2. *Let α be closed curve equipped by tiling $\Im$. If α has loop l, then there exist $\Im^* \subset \Im$ such that $\Im^*$ is tiling for l.*

Proof. Obviously from the properties of the tiling $\Im$ there exist $\Im^* \subset \Im$ such that $\Im^*$ is tiling for l. □

EXAMPLE 2.3. *In the Figure (2-a) the degrees of v_3, v_2 are 2,1, respectively, i.e., $d(v_3) = 2$, $d(v_2) = 1$.*

DEFINITION 2.4. Let $\alpha(s)$ be closed curve the tiling $\Im_{\min}$ is called *min* tiling if $d(\Im_{\min}(\alpha)) < d(\Im(\alpha))$ for each tiling $\Im(\alpha)$.

EXAMPLE 2.4. *Let α, β and γ are circle, eight shaped curve and 2-self-looped curve, respectively. Then the min tilings of α, β and γ are given as in the Figure 3.*

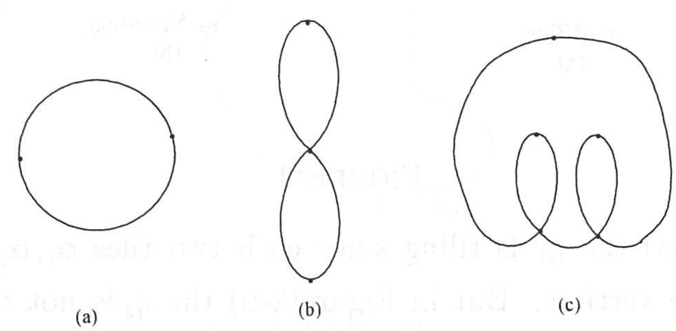


FIGURE 3

3. Tiling retraction

In this section, we will give the definition of the tiling retraction and study the effect of retraction, tiling retraction on the degree of the tiling $d(\Im)$.

Condition A: Let $\alpha(s)$ be closed curve in R^2 equipped by tiling $\Im$ then $d(\Im^*(l)) \geq 3$. for each loop l.

THEOREM 3.1. *Let α be regular closed curve equipped by tiling $\Im$ satisfying Condition A, and let $r : \alpha \setminus \{c\} \to \alpha^*$ be retraction map. Then $r_t(\alpha/c)$ of α are:*
(a) one curve equipped by tiling of degree $n-1$ if $d \in \alpha_i$, where α_i is any edge of the tiling or
(b) p curves equipped by separate tiling, if c is a vertex, where p the degree of the vertex.

Proof. Let α be regular closed curve equipped by tiling $\Im$ satisfying Condition A, then $d(\Im) = n \geq 3$. Now let $r : \alpha \setminus \{c\} \to \alpha^*$ be retraction map, where $c \in \alpha$. Then there exist two cases :

(i) If $d \in \alpha_i$ where α_i is any edge, in this case the retraction map r will map the α_i to α^*. i.e. $r(\alpha_i) = \alpha^*$ and $r \simeq I_{\alpha^*}$ for any points in α^*. See Figure 4. Hence the $r(\alpha \setminus \{c\}) = \alpha^*$ is one curve equipped by tiling of degree $n-1$, i.e., $d(\Im(\alpha^*)) = n-1$.

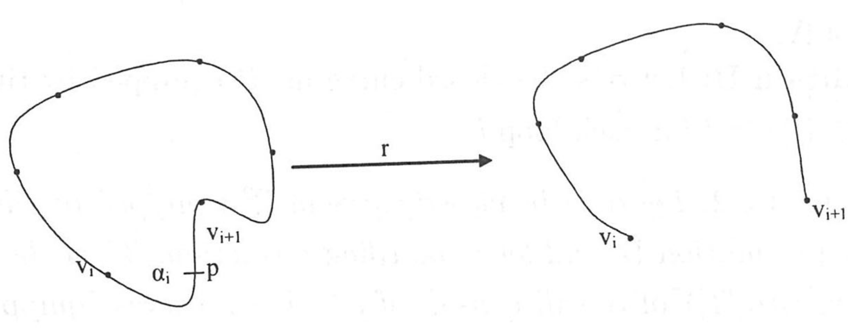


FIGURE 4

(ii) If c is a vertex, then the $d(c) = 0$ or $d(c) = p$. Then in the first case the vertex c lies between two edges so let c lies between T_i, T_{i+1} thus the retraction r will be define as $r(a) = b$ where $b \in \alpha^*$ and $a \in \alpha_i \cup \alpha_j$ and $r \simeq I_{\alpha^*}$ for any points in α^*.

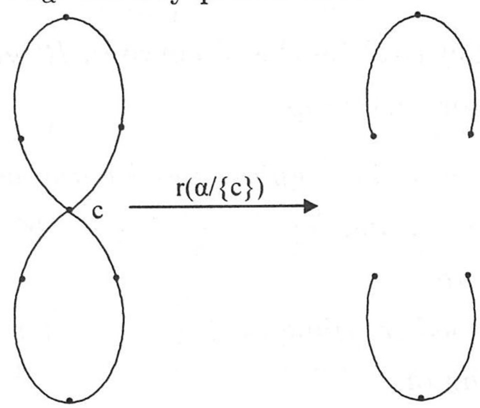


FIGURE 5

Then $d(\Im(\alpha^*)) = n - 2$, in the second case the vertex c will be branch point and all tiles connected to it will called branch tiles, then the retraction r will be define as $r(a) = b$ where $b \in \alpha^*$, a $\in$ branch tiles, $r \simeq I_{\alpha^*}$ for any points in α^*, and since $d(c) = p$ thus the α^* will consist of p curves equipped by separate tiling. See Figure 5. $\square$

DEFINITION 3.1. Let X be space equipped by tiling $\Im$ and $\bar{\Im} \subset \Im$. A continuous map $r_t : X/\{T_i\} \to A$, where $T_i \in \Im$, is called tiling retraction if the tiling retraction of r_t to A is the identity map of A.

If there exists a retraction from X to A, we say that A is tiling retract of X.

Condition B: Let $\alpha(s)$ be closed curve in R^2 equipped by tiling $\Im$ then $d(\Im^*(l)) \geq 4$ for each loop l.

THEOREM 3.2. *Let $\alpha(s)$ be closed curve in R^2 equipped by tiling $\Im$, Satisfying Condition B. and let r_t be tiling retraction. Then the tiling retraction $r_t(\alpha/T_i)$ of α will consist of $p + q - 1$ curves equipped by separate tiling, where p, q are the degree of the boundary vertices of the tile T_i.*

Proof. Let $\alpha(s)$ be closed curve in R^2 equipped by tiling $\Im = \{T_i :_i \in \{1, 2, ..., n\}\}$, and $r_t : \alpha/T_i \to \alpha^*$where $\alpha^* \subset \alpha$. if α be simple closed curve then $d(v_i) = 1$ for any vertex. So $p = q = 1$, then the tiling retraction r_t will map all tiles T_i, T_{i+1} to any tiles in $\Im$ so $r_t(\alpha/T_i) = \alpha^*$ will be one curve equipped by tiling $\Im^*$ of degree $n-2$. Now Let c_i , $i = 1, ..., m$ be m branch point in the curve α and let $r_t : \alpha/T_i \to \alpha^*$ be tiling retraction such that the tile T_i connect between two branch points $c_{1,}c_2$ of degree p, q respectively, then the tiling retraction r_t will map each tile of the branch points to any tiles in $\Im$ so $r_t(\alpha/T_i) = \alpha^*$ will be consist of $p + q - 1$ curves equipped $\Im_i^*$, $i = 1, ..., p + q - 1$. See Figure 6. □

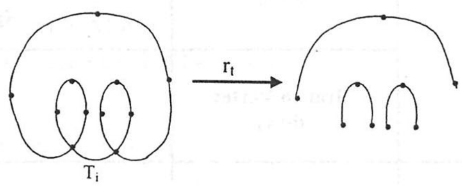


FIGURE 6

EXAMPLE 3.3. *In the following figures we introduce the retraction, tiling retraction for circle, eight shaped curve and 2-self-looped curve respectively which equipped by the min tiling.*

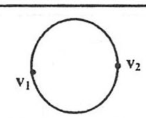
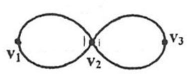
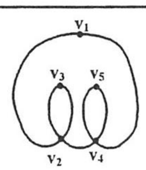
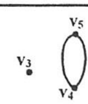
Closed curve	Ti	rt(α/{ Ti})
	Ti=<v1v2> d(v1) = d(v2) =1	∅
	Ti=<v1v2> d(v1) =1, d(v2) =2	•
	Ti=<v1v2> d(v1) =1, d(v2) =2	
	Ti=<v2v4> d(v2) =2, d(v4) =2	v3 • • v1 •v5

FIGURE 8.

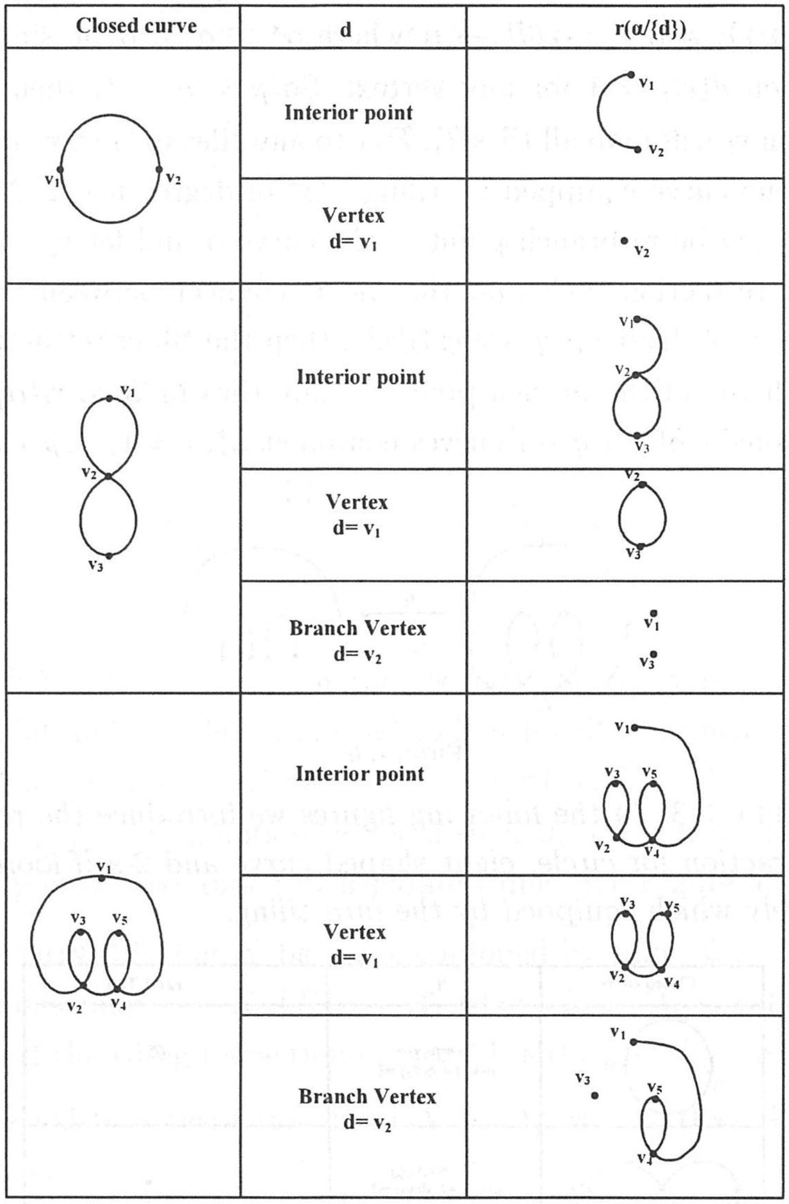

Closed curve
d
r(α/{d})
Interior point
Vertex
d= v1
Interior point
Vertex
d= v1
Branch Vertex
d= v2
Interior point
Vertex
d= v1
Branch Vertex
d= v2
v1
v2
v3
v4
v5

FIGURE 7.

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