

# On the goodness of some types of fuzzy paracompactness in Sostak's fuzzy topology

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## Abstract

We introduce in Sostak's fuzzy topological spaces definitions of paracompactness, almost paracompactness, and near paracompactness all of which turn to be good extensions of their classical topological counterparts. Fuzzy semi-paracompact, para S-closed and weakly paracompact spaces are treated to a similar approach.

**Key Words :** Fuzzy (almost, nearly, weakly-) paracompact, good extensions.

## 1. Introduction and Preliminaries

Sostak [16], introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [3], in the sense that not only the objects are fuzzified, but also the axiomatics. In [17,18] Sostak gave some rules and showed how such an extension can be realized. Chattopadhyay et. al have redefined the same concept [4,5]. In [12,7], Ramadan and his colleagues gave a similar definition, namely "smooth topological space" for lattice  $L=[0,1]$ . It has been developed in many directions [8,10,17,18]. In [9], Hoehle and Sostak introduce the concept of an  $L$ -fuzzy topologies and establish their corresponding convergence theory for any lattice  $L$ . Paracompactness is one of the most important notions in topology. Since fuzzy topological spaces were introduced in [16,9], two papers on this problem have been written and a lot of different kinds of fuzzy paracompactness have been introduced and studied [14,15].

The aim of this paper is to introduce some good types of paracompactness in fuzzy topological spaces in view of the definition of Sostak, namely, paracompactness, almost paracompactness, near paracompactness, weak paracompactness and para S-closedness.

## 2. Preliminaries

In this paper, let  $X$  be a nonempty set,  $I=[0,1]$ ,  $I^X$  the family of all fuzzy subsets of  $X$ . For  $\alpha \in I$ ,  $\bar{\alpha}(x) = \alpha$  for all  $x \in X$ . For a subset  $A$  of  $X$ ,  $\chi_A$  is a characteristic function of  $A$ .

A fuzzy topology (in the sense of Sostak) is a map  $\tau: I^X \rightarrow I$  such that

- (O1)  $\tau(\bar{1}) = \tau(\bar{0}) = 1$ .
- (O2)  $\tau(\lambda \wedge \mu) \geq \tau(\lambda) \wedge \tau(\mu)$ .
- (O3)  $\tau(\bigvee_{i \in I} \mu_i) \geq \bigwedge_{i \in I} \tau(\mu_i)$ .

The pair  $(X, \tau)$  is called a fuzzy topological space (fts, for short).

Let  $(X, \tau)$  be a fts. The mapping  $F_\tau: I^X \rightarrow I$  defined by the equality  $F_\tau(\mu) = \tau(\mu^*)$ ,  $\forall \mu \in I^X$  is called a fuzzy cotopology satisfying the following properties:

- (F1)  $F_\tau(\bar{1}) = F_\tau(\bar{0}) = 1$ .
- (F2)  $F_\tau(\lambda \vee \mu) \geq F_\tau(\lambda) \wedge F_\tau(\mu)$ .
- (F3)  $F_\tau(\bigwedge_{i \in I} \mu_i) \geq \bigwedge_{i \in I} F_\tau(\mu_i)$ .

Let  $(X, \tau)$  be a fts. and  $\lambda \in I^X$ . The smooth closure (resp. smooth interior) of  $\lambda \in I^X$ , denoted by  $\text{cl}(\lambda)$  (resp.  $\text{int}(\lambda)$ ), is defined by

$$\begin{aligned} \text{cl}(\lambda) &= \bigwedge \{ \mu \in I^X \mid F_\tau(\mu) > 0, \lambda \leq \mu \} \\ (\text{resp. } \text{int}(\lambda) &= \bigvee \{ \mu \in I^X \mid \tau(\mu) > 0, \mu \leq \lambda \}) [6]. \end{aligned}$$

For a fts  $(X, \tau)$  and  $\lambda, \mu \in I^X$ . Then,

- (i) If  $\lambda \leq \mu$ , then  $\text{cl}(\lambda) \leq \text{cl}(\mu)$ ,
- (ii) If  $\lambda \leq \mu$ , then  $\text{int}(\lambda) \leq \text{int}(\mu)$ ,
- (iii)  $\text{cl}(\lambda)^* = \text{int}(\lambda^*)$  and  $\text{int}(\lambda)^* = \text{cl}(\lambda^*)$
- (iv) If  $\tau(\lambda) > 0$ , then  $\lambda = \text{int}(\lambda)$ ,
- (v) If  $F_\tau(\lambda) > 0$ , then  $\lambda = \text{cl}(\lambda)$  [6].

For a fts  $(X, \tau)$ ,

- (1)  $\lambda \in I^X$  is fuzzy semi-open iff there exists  $\mu \in I^X$  with  $\tau(\mu) > \alpha$ ,  $\forall \alpha \in I_1$  such that  $\mu \leq \lambda \leq \text{cl}(\mu)$
- (2)  $\lambda \in I^X$  is fuzzy regular open iff  $\lambda = \text{int}(\text{cl}(\lambda))$ .
- (3)  $\lambda \in I^X$  is fuzzy regular closed iff  $\lambda = \text{cl}(\text{int}(\lambda))$ .

## 3. Lower semi-continuous

In this section, we introduce the concept of  $\alpha$ -lower semi-continuity ( $\alpha \in I$ ) in order to set up a "goodness of

extension" criterion for fuzzy topological properties.

**Definition 3.1** Let  $(X, \tau)$  be an ordinary topological space and  $\alpha \in I$ . A mapping  $\lambda: (X, T) \rightarrow I$  where  $I$  has its usual topology, is said to be  $\alpha$ -lower semi-continuous if and only if for every  $t \in I_1$  with  $\alpha > t$ ,  $\lambda^{-1}(t, 1] \in T$ . It is clear that if  $\lambda$  is lower semi-continuous, then  $\lambda$  is  $\alpha$ -lower semi-continuous for every  $\alpha \in I$ . Moreover,  $\lambda$  is 1-lower semi-continuous iff  $\lambda$  is lower semi-continuous. Naturally, every mapping from  $(X, \tau)$  to  $I$  is 0-Lower semi-continuous.

**Definition 3.2[1,13].** Let  $(X, \tau)$  be an ordinary topological space. Then

(1) The mapping  $W(T): I^X \rightarrow I$  defined by

$$W(T)(\lambda) = \bigvee \{ \alpha \in I_1 \mid \lambda^{-1}(\alpha, 1] \in T \}$$

for every  $\lambda \in I^X$ , is fuzzy topology on  $X$ .

(2) The mapping  $W(T_c): I^X \rightarrow I$  defined by

$$W(T_c)(\lambda) = \bigvee \{ \alpha \in I_1 \mid \lambda^{-1}(\alpha, 1] \in T_c \}$$

for every  $\lambda \in I^X$ , is fuzzy cotopology on  $X$ .

This provides a "goodness of extension" criterion for fuzzy topological properties. Recall that a fuzzy extension of a topological property of  $(X, \tau)$  is said to be good when it is possessed by  $W(T)$  iff the original property is possessed by  $T$ .

**Proposition 3.3.** For every fuzzy set  $\lambda: (X, T) \rightarrow I$ , and for all  $t \in I_1$ ,

$$\text{cl}(\lambda)^{-1}(t, 1] \subseteq \text{cl}(\lambda^{-1}(t, 1])$$

$$(i) \subseteq \text{cl}(\lambda^{-1}[t, 1]) \subseteq (\text{cl}(\lambda))^{-1}[t, 1]$$

$$\text{int}(\lambda)^{-1}(t, 1] \subseteq \text{int}(\lambda^{-1}(t, 1])$$

$$(ii) \subseteq \text{int}(\lambda^{-1}[t, 1]) \subseteq (\text{int}(\lambda))^{-1}[t, 1].$$

all closures and interiors being taken in  $T$  or  $W(T)$  as appropriate.

**Proof.** (i) We are going to prove that any closed set  $C$  in  $(X, \tau)$  with  $\lambda^{-1}(t, 1] \subseteq C$  satisfies

$(\text{cl}(\lambda))^{-1}(t, 1] \subseteq C$ . Now let  $\lambda^{-1}(t, 1] \subseteq C$ ,  $C$  is closed in  $(X, \tau)$  and let  $\mu: X \rightarrow I$  defined by

$$\mu(x) = \begin{cases} 1, & x \in C, \\ t, & x \notin C. \end{cases}$$

Then,  $\forall s \in I_1$ , we have

$$\mu^{-1}[s, 1] = \begin{cases} X, & s \leq t, \\ C, & s > t. \end{cases}$$

So,  $\mu^{-1}[s, 1]$  is closed set in  $(X, T)$ ,  $\forall s \in I_1$  then  $\mu^*$  is lower semi-continuous. Hence  $W(T)(\mu^*) = 1$  and  $W(T_c)(\mu) = 1$ . We also have that  $\lambda \leq \mu$ .

Hence  $\text{cl}(\lambda) \leq \mu$ . Thus  $(\text{cl}(\lambda))(x) > t$  implies  $\mu(x) > t$  and

$x \in C$  it follows that

$$(\text{cl}(\lambda))^{-1}(t, 1] \subseteq \text{cl}(\lambda^{-1}(t, 1]).$$

Clearly,

$$\lambda^{-1}(t, 1] \subseteq (\text{cl}(\lambda))^{-1}(t, 1] \subseteq (\text{cl}(\lambda))^{-1}[t, 1],$$

since  $\lambda \leq \text{cl}(\lambda)$ . And by lower semi-continuity  $(\text{cl}(\lambda))^{-1}[t, 1]$  is closed in  $T$ . Thus

$$\text{cl}(\lambda^{-1}(t, 1]) \subseteq (\text{cl}(\lambda))^{-1}[t, 1].$$

So,

$$\text{cl}(\lambda^{-1}(t, 1]) \subseteq \text{cl}(\lambda^{-1}[t, 1]) \subseteq (\text{cl}(\lambda))^{-1}[t, 1].$$

(ii) Similarly, for interiors clearly

$$(\text{int}(\lambda))^{-1}(t, 1] \subseteq \lambda^{-1}(t, 1] \subseteq \lambda^{-1}[t, 1].$$

And  $\text{int}(\lambda)^{-1}(t, 1] \in T$ , so

$$(\text{int}(\lambda))^{-1}(t, 1] \subseteq \text{int}(\lambda^{-1}(t, 1]) \subseteq \text{int}(\lambda^{-1}[t, 1]).$$

Secondly, let  $C \subseteq \lambda^{-1}[t, 1]$ , where  $C$  is open in  $T$ , Defining  $\mu \in I^X$  by

$$\mu(x) = \begin{cases} t, & x \in C, \\ 0, & x \notin C, \end{cases}$$

then,  $\forall s \in I_1$

$$\mu^{-1}(s, 1] = \begin{cases} \phi, & s \leq t, \\ C, & s > t. \end{cases}$$

Then,  $\mu^{-1}(s, 1] \in T$ ,  $\forall s \in I_1$  so,  $W(T)(\mu) = 1$  and  $\mu \leq \lambda$ . So,  $\mu \leq \text{int}(\lambda)$ . Hence  $x \in C$  so  $\text{int}(\lambda)(x) \geq t$  and therefore  $\text{int}(\lambda^{-1}[t, 1]) \subseteq (\text{int}(\lambda))^{-1}[t, 1]$ .

**Corollary 3.4.** For a regular open fuzzy set  $\mu \in I^X$  of  $W(T)$  and  $t \in I_1$ ,

$$\text{int}(\text{cl}(\mu^{-1}[t, 1])) = \text{int}(\mu^{-1}[t, 1])$$

This latter set is therefore regular open.

**Proof.** By Proposition 3.3,

$$\text{int}(\text{cl}(\mu^{-1}[t, 1])) \subseteq \text{int}((\text{cl}(\mu))^{-1}[t, 1])$$

$$\subseteq \text{int}(\text{cl}(\mu))^{-1}[t, 1] = \mu^{-1}[t, 1],$$

since  $\text{int}(\text{cl}(\mu)) = \mu$ . But for any set  $A$ ,  $\text{int}(A) \subseteq \text{int}(\text{cl}(A))$ , and if  $\text{int}(\text{cl}(A)) \subseteq A$ , then  $\text{int}(\text{cl}(A)) \subseteq \text{int}(A)$ , so  $\text{int}(\text{cl}(A)) = \text{int}(A)$ . Thus  $\text{int}(\text{cl}(\mu^{-1}[t, 1])) = \text{int}(\mu^{-1}[t, 1])$ . But the interior of the closure of a set is always regular, so  $\text{int}(\mu^{-1}[t, 1])$  is regular open.

**Corollary 3.5.** (i)  $\alpha_{\text{cl}(A)} = \text{cl}(\alpha_A)$

(ii)  $\alpha_{\text{int}(A)} = \text{int}(\alpha_A)$

**Proof.** Since  $A \subseteq \text{cl}(A)$ , we have  $\alpha_A \leq \alpha_{\text{cl}(A)}$ . But

$$\alpha_{\text{cl}(A)} = \begin{cases} \alpha, & x \in \text{cl}(A), \\ 0, & x \notin \text{cl}(A), \end{cases}$$

then  $\forall t \in I_1$ ,

$$(\alpha_{cl(A)})^{-1}[t, 1] = \begin{cases} \phi, & t \geq \alpha, \\ cl(A), & t < \alpha. \end{cases}$$

So,  $(\alpha_{cl(A)})^{-1}[t, 1] \in T_c, \forall t \in I_1$ , so  $\mathcal{W}(T_c)(\alpha_{cl(A)}) = 1 > \alpha$ . Hence  $cl(\alpha_A) \leq \alpha_{cl(A)}$ . Now taking  $\lambda = \alpha_A$  in Proposition 3.3,  $\forall t \in I_1$ ,  $(cl(\alpha_A))^{-1}(t, 1] \subseteq cl(\alpha_A^{-1}(t, 1]) \subseteq (cl(\alpha_A))^{-1}[t, 1]$ . Take  $t < \alpha$ . Then

$$cl(A) = cl(\alpha_A^{-1}(t, 1]) \subseteq (cl(\alpha_A))^{-1}[t, 1],$$

so,  $cl(\alpha_A)(\text{text } cl(A)) \geq t$  for all  $t \in [0, \alpha)$ . Thus

$$cl(\alpha_A)(cl(A)) \geq \alpha$$

and  $\alpha_{cl(A)} \leq cl(\alpha_A)$ . Hence  $\alpha_{cl(A)} = cl(\alpha_A)$  as required. We do not repeat the dual argument for interiors.

**Corollary 3.6.** If  $\lambda$  is fuzzy semiopen in  $\mathcal{W}(T)$  and  $0 \leq t < \alpha$ , then  $\lambda^{-1}(t, 1]$  is semiopen in  $T$ .

**Proof.** For  $\lambda$  semiopen in  $\mathcal{W}(T)$ , there exists  $\mu \in I^X$  with  $\mathcal{W}(T)(\mu) > \alpha \forall \alpha \in I_1, t < \alpha$  such that  $\mu \leq \lambda \leq cl(\mu)$ , Hence  $\mu^{-1}(t, 1] \in T$  and

$$\begin{aligned} \mu^{-1}(t, 1] &\subseteq \lambda^{-1}(t, 1] \subseteq (cl(\mu))^{-1}(t, 1] \\ &\subseteq cl(\mu^{-1}(t, 1]). \end{aligned}$$

So,  $\lambda^{-1}(t, 1]$  is semiopen.

**Proposition 3.7** With the same notation for  $X, \mathcal{W}(T)$  and  $\alpha_A$ :

- (i) If a set  $A$  is semiopen in  $T$ , then  $\alpha_A$  is fuzzy semiopen in  $\mathcal{W}(T)$ .
- (ii) If a set  $A$  is regular open in  $T$ , then  $\alpha_A$  is fuzzy regular open in  $\mathcal{W}(T)$ .

**Proof.** (i) If  $A$  is semiopen, then there exist  $G \in T$  with  $G \subseteq A \subseteq cl(G)$ . Then

$$\alpha_G \leq \alpha_A \leq \alpha_{cl(G)} = cl(\alpha_G)$$

And  $\mathcal{W}(T)(\alpha_G) > \alpha$ , so  $\alpha_A$  is semiopen.

(ii) Let  $A = \text{int}(cl(A))$ . Then  $\alpha_A = \alpha_{\text{int}(cl(A))}$ , so  $\alpha_A = \alpha_{\text{int}(cl(A))} = \text{int}(cl(\alpha_A))$  and  $\alpha_A$  is regular.

#### 4. Proposed definitions and its goodness

**Definition 4.1.** Let  $(X, \tau)$  be a fts. A family of fuzzy sets  $\{\lambda_i \mid i \in I\}$  is said to be locally finite iff for each  $x \in X$ , there exists  $\mu \in I^X$  with  $\tau(\mu) > \alpha$  and  $\mu(x) > \alpha \forall \alpha \in I_1$  and such that  $\lambda_i \wedge \mu = 0$  holds for but at most finitly many  $i \in I$ .

**Definition 4.2.** In an fts with families  $U$  and  $V$  of fuzzy sets,  $U$  is a refinement of  $V$ , written  $U \leq V$ , iff for each  $\mu \in U$  there is an  $\lambda \in V$  such that  $\mu \leq \lambda$ .

**Definition 4.3.** (a) An fts  $(X, \tau)$  is called fuzzy paracompact iff for each  $\beta \subset I^X$  with  $\tau(\lambda) > \alpha, \forall \lambda \in \beta$  and  $\alpha \in I_1$  such that

$$\bigvee_{\lambda \in \beta} \lambda \geq \alpha \text{ and for all } \varepsilon \text{ with } 0 < \varepsilon \leq \alpha,$$

there exists a locally finite refinement  $\beta_0$  of  $\beta$  with  $\tau(\mu) > \alpha, \forall \mu \in \beta_0$  and  $\alpha \in I_1$  such that

$$\bigvee_{\mu \in \beta_0} \mu \geq \alpha - \varepsilon.$$

(b) An fts  $(X, \tau)$  is called fuzzy almost paracompact iff for each  $\beta \subset I^X$  with  $\tau(\lambda) > \alpha, \forall \lambda \in \beta$  and  $\alpha \in I_1$  such that

$$\bigvee_{\lambda \in \beta} \lambda \geq \alpha \text{ and for all } \varepsilon \text{ with } 0 < \varepsilon \leq \alpha,$$

there exists a locally finite refinement  $\beta_0$  of  $\beta$  with  $\tau(\mu) > \alpha, \forall \mu \in \beta_0$  and  $\alpha \in I_1$  such that

$$\bigvee_{\mu \in \beta_0} cl(\mu) \geq \alpha - \varepsilon.$$

(c) An fts  $(X, \tau)$  is called fuzzy near paracompact iff for each  $\beta \subset I^X$  with  $\tau(\lambda) > \alpha, \forall \lambda \in \beta$  and  $\alpha \in I_1$  such that  $\bigvee_{\lambda \in \beta} \lambda \geq \alpha$  and for all  $\varepsilon$  with  $0 < \varepsilon \leq \alpha$ ,

there exists a locally finite refinement  $\beta_0$  of  $\beta$  with  $\tau(\mu) > \alpha, \forall \mu \in \beta_0$  and  $\alpha \in I_1$  such that

$$\bigvee_{\mu \in \beta_0} \text{int } cl(\mu) \geq \alpha - \varepsilon.$$

(d) An fts  $(X, \tau)$  is called fuzzy semi-paracompact iff for each  $\beta \subset I^X$  of fuzzy semiopen sets and each  $\alpha \in I_1$  such that

$$\bigvee_{\lambda \in \beta} \lambda \geq \alpha \text{ and for all } \varepsilon \text{ with } 0 < \varepsilon \leq \alpha,$$

there exists a locally finite semiopen refinement

$$\beta_0 \text{ of } \beta \text{ such that } \bigvee_{\mu \in \beta_0} \mu \geq \alpha - \varepsilon.$$

(e) An fts  $(X, \tau)$  is called fuzzy para S-closed iff

for each  $\beta \subset I^X$  of fuzzy semiopen sets and each  $\alpha \in I_1$  such that  $\bigvee_{\lambda \in \beta} \lambda \geq \alpha$  and for all  $\varepsilon$  with  $0 < \varepsilon \leq \alpha$ ,

there exists a locally finite semiopen refinement

$$\beta_0 \text{ of } \beta \text{ such that } \bigvee_{\mu \in \beta_0} cl(\mu) \geq \alpha - \varepsilon.$$

(f) An fts  $(X, \tau)$  is called fuzzy weakly paracompact iff for each  $\beta \subset I^X$  of fuzzy regular open sets and each  $\alpha \in I_1$  such that

$$\bigvee_{\lambda \in \beta} \lambda \geq \alpha \text{ and for all } \varepsilon \text{ with } 0 < \varepsilon \leq \alpha,$$

there exists a locally finite regular open refinement  $\beta_0$  of  $\beta$

$$\text{such that } \bigvee_{\mu \in \beta_0} cl(\mu) \geq \alpha - \varepsilon.$$

In the crisp case of  $\tau$  this definitions coincides with Definition 3[2].

**Theorem 4.4.**  $(X, \mathcal{W}(T))$  is fuzzy paracompact iff  $(X, T)$  is paracompact. Thus fuzzy paracompactness is a good extension of paracompactness.

**Proof.** Suppose  $(X, T)$  is paracompact. Let  $\beta \subset I^X$  with

$\mathcal{W}(T)(\lambda) > \alpha \forall \lambda \in \beta, \alpha \in I_1$  be such that  $\bigvee_{\lambda \in \beta} \lambda \geq \alpha$  Now let  $0 < \varepsilon \leq \alpha$ . We shall show that there exists a locally finite refinement  $\beta_0$  of  $\beta$  such that

$$\bigvee_{\lambda \in \beta_0} \lambda \geq \alpha - \varepsilon.$$

Take  $\sigma$  such that  $\alpha - \varepsilon < \sigma < \alpha$ . Then from  $\mathcal{W}(T)(\lambda) > \sigma$  we have  $\lambda^{-1}(\sigma, 1] \in T, \forall \lambda \in \beta$ , and so,

$U = \{\lambda^{-1}(\sigma, 1] \mid \lambda \in \beta\}$  is an open cover of  $X$ . Otherwise there exist  $x \in X$  such that, for all  $\lambda \in \beta$ ,  $\lambda(x) \leq \sigma < \alpha$  contradicting the fact that  $\bigvee_{\lambda \in \beta} \lambda \geq \alpha$ . So, there exists a locally finite open refinement  $V$  of  $U$  which is also cover  $X$ . Now consider the family

$$\beta_0 = \{\sigma_v \mid v \in V\} \subset I^X \text{ such that } \mathcal{W}(T)(\sigma_v) > \alpha \quad \forall \sigma_v \in \beta_0, \quad \alpha \in I_1$$

To show that  $\beta_0$  is locally finite, take  $x \in X$ . Since  $V$  is locally finite there exists an open neighbourhood  $u$  of  $x$  in  $T$  such that  $u \cap v = \emptyset$  for all but at most finitely many  $v \in V$ . Then, the characteristic function of every open set is 1-lower semi-continuous, so  $\mathcal{W}(T)(\chi_u) = 1 > \alpha$  with  $\chi_u(x) = 1$  and  $\sigma_v \wedge \chi_u = 0$  holds for all but at most finitely many  $\sigma_v \in \beta_0$ . To show that  $\beta_0$  is a refinement of  $\beta$ , take  $\sigma_v \in \beta_0$ . Then  $v \in V$  and there exists some  $\mu \in \beta$  such that  $v \subseteq \mu^{-1}(\sigma, 1]$ . Hence  $\sigma_v \leq \mu$ . Finally,  $\bigvee_{\sigma \in \beta} \sigma_v \geq \alpha - \varepsilon$  otherwise there exists some  $x \in X$  such that

$$\bigvee_{\sigma \in \beta} \sigma_v(x) < \alpha - \varepsilon < \sigma$$

for all  $\sigma_v \in \beta_0$  i.e.,  $\sigma_v(x) = 0$ . Therefore  $x \notin v$  for all  $v \in V$ , contradicting the covering property of  $V$ .

Conversely, let  $(X, \mathcal{W}(T))$  be fuzzy paracompact, and let  $V$  be an open cover of  $X$  in  $T$ . Then,  $\beta = (\chi_u)_{u \in V} \subset I^X$  with  $\mathcal{W}(T)(\chi_u) = 1 > \alpha$ ,  $\forall \chi_u \in \beta$  and  $\alpha \in I_1$  such that  $\bigvee_{u \in V} \chi_u = 1 \geq \alpha$

Choose  $\varepsilon$  such that  $0 < \varepsilon < \alpha$ . By hypothesis there exists a locally finite refinement  $\beta_0$  of  $\beta$  such that  $\mathcal{W}(T)(\mu) > \alpha$ ,  $\forall \mu \in \beta_0$  with  $\bigvee_{\mu \in \beta_0} \mu \geq \alpha - \varepsilon$

Now take  $t$  such that  $0 < t < \alpha - \varepsilon$ , and consider the family  $V = \{\mu^{-1}(t, 1] \mid \mu \in \beta_0\}$  of open sets in  $T$ .

We show that  $V$  is locally finite open refinement of  $U$  and covers  $X$ . Certainly  $V$  covers  $X$ , since taking  $x \in X$  and observing that  $\bigvee \mu(x) \geq \alpha - \varepsilon$

implies that there exists  $\mu_0 \in \beta_0$  such that  $\mu_0(x) > t$  i.e.,  $x \in \mu_0^{-1}(t, 1] \in V$ . Now let  $x \in X$  and observe that there exists  $\nu \in I^X$  with  $\mathcal{W}(T)(\nu) > \alpha$  such that  $\nu(x) > \alpha$  and  $\mu \wedge \nu = 0$  for all but at most finitely many  $\mu \in \beta_0$ . Then  $\nu^{-1}(t, 1] \in T$  with  $x \in \nu^{-1}(t, 1]$  and  $\mu^{-1}(t, 1] \cap \nu^{-1}(t, 1] = \emptyset$  for all but at most finitely many  $\mu \in \beta_0$ . Finally,  $V$  is a refinement of  $U$ . Let  $\mu^{-1}(t, 1] \in V$ . Then,  $\mu \in \beta_0$  and  $\exists u \in U$  such that  $\mu \leq \chi_u$ . So  $\mu^{-1}(t, 1] \subseteq u$ . Thus  $(X, T)$  is paracompact.

This proof provides the model, and the notation, for the following theorems. We indicate only the details where differences occur.

**Theorem 4.5.**  $(X, \mathcal{W}(T))$  is fuzzy almost paracompact iff  $(X, T)$  is almost paracompact. Thus almost paracompactness

is good.

**Proof.** The locally finite open refinement  $V$  of  $U$  is such that  $\bigcup_{v \in V} \text{cl}(v) = X$ . Then  $\beta_0 = \{\sigma_v \mid v \in V\}$  with  $\mathcal{W}(T)(\sigma_v) > \alpha < \forall \sigma_v \in \beta_0, \alpha \in I_1$  and locally finite refinement of  $\beta$ . Also,  $\bigvee_{v \in V} \text{cl}(\sigma_v) \geq \alpha - \varepsilon$ , since letting  $x \in X$ , there exists  $v \in V$  such that  $x \in \text{cl}(v)$ . And by Corollary 3.5(i),  $\sigma_{\text{cl}(v)}(x) = \text{cl}(\sigma_v)(x) = \sigma > \alpha - \varepsilon$

Conversely, assuming  $(X, \mathcal{W}(T))$  to be almost paracompact we obtain a locally finite refinement  $\beta_0$  of  $\beta$  such that  $\mathcal{W}(T)(\mu) > \alpha, \forall \mu \in \beta_0$  with  $\bigvee_{\mu \in \beta_0} \text{cl}(\mu) \geq \alpha - \varepsilon$ . Again  $V = \{\mu^{-1}(t, 1] \mid \mu \in \beta_0\}$

is an open locally finite refinement of  $U$ . And

$$\bigcup_{\mu \in \beta_0} \text{cl}(\mu^{-1}(t, 1]) = X$$

To see this, let  $x \in X$ . Then since  $\bigvee_{\mu \in \beta_0} \{\text{cl}(\mu)(x)\} \geq \alpha - \varepsilon$  there exists  $\mu_0 \in \beta_0$  such that  $\text{cl}(\mu_0)(x) > t$ , so  $x \in (\text{cl}(\mu_0))^{-1}(t, 1] \subseteq \text{cl}(\mu^{-1}(t, 1])$ . This completes the proof.

**Theorem 4.6.**  $(X, \mathcal{W}(T))$  is fuzzy nearly paracompact iff  $(X, T)$  is nearly paracompact. So near paracompactness is good.

**Proof.** In this case  $\bigcup_{v \in V} \text{int}(\text{cl}(v)) = X$ , and  $\beta_0$  is an open locally finite refinement of  $\beta$ . Also  $\bigvee_{v \in V} \text{int}(\text{cl}(\sigma_v)) \geq \alpha - \varepsilon$

This follows by noting that for any  $x \in X$ , there exists  $v \in V$  such that  $x \in \text{int}(\text{cl}(v))$ . And

$$\sigma_{\text{int}(\text{cl}(v))}(x) = \text{int}(\text{cl}(\sigma_v))(x) = \sigma > \alpha - \varepsilon$$

Conversely,  $\bigvee_{\mu \in \beta_0} \text{int}(\text{cl}(\mu)) \geq \alpha - \varepsilon$ .

And  $\bigcup_{\mu \in \beta_0} \text{int}(\text{cl}(\mu^{-1}(t, 1])) = X$ . Here, taking  $x \in X$ , there exists  $\mu_0 \in \beta_0$  such that  $\text{int}(\text{cl}(\mu_0))(x) > t$ , so

$$\begin{aligned} x \in \text{int}(\text{cl}(\mu_0))^{-1}(t, 1] &\subseteq \text{int}((\text{cl}(\mu_0))^{-1}(t, 1]) \\ &\subseteq \text{int}(\text{cl}(\mu^{-1}(t, 1])) \end{aligned}$$

by Proposition 3.3.

**Theorem 4.7.** Fuzzy semi-paracompactness is good.

**Proof.** An obvious adaptation of the next theorem.

**Theorem 4.8.**  $(X, \mathcal{W}(T))$  is fuzzy para S-closed iff  $(X, T)$  is para S-closed. This property is therefore good.

**Proof.** Supposing  $(X, T)$  to be para S-closed, let  $\beta$  be a family of fuzzy semiopen sets in  $\mathcal{W}(T)$  with  $\bigvee_{\lambda \in \beta} \lambda \geq \alpha$ . This time we get, as in Theorem 4.5, a locally finite refinement of  $U$  with  $\bigcup_{v \in V} \text{cl}(v) = X$ . And  $U, V$  are semiopen, using Corollary 3.6. Then follow verbatim Theorem 4.5, substituting semiopen for  $\mathcal{W}(T)(\lambda) > \alpha, \forall \lambda \in \beta$ .

**Theorem 4.9.**  $(X, \mathcal{W}(T))$  is fuzzy weakly paracompact iff  $(X, T)$  is weakly paracompact. So weak paracompactness is

a good extension.

**Proof.** Assume  $(X, T)$  weakly paracompact, and  $\beta$  a family of fuzzy regular open sets. Then  $\{\text{int}(\lambda^{-1}[t, 1]) \mid \lambda \in \beta\}$  is a regular open cover of  $X$  (Corollary 3.4). The covering property follows from that of  $\lambda^{-1}(t, 1]$  which is an open set contained in  $\lambda^{-1}[t, 1]$  and therefore in its interior. Continue as usual to yield a family  $\beta_0 = \{\sigma_v \mid v \in V\}$  of locally finite fuzzy regular open sets with  $\bigvee_{v \in V} \text{cl}(\sigma_v) \geq \alpha - \varepsilon$ . Also, given  $v \in V$ , there exists  $\lambda \in \beta$  such that

$$v \subseteq \text{int}(\lambda^{-1}[t, 1]) \subseteq \lambda^{-1}[t, 1]$$

So  $\lambda(v) \subseteq [t, 1]$  and  $\sigma_v \leq \lambda$ . Thus  $\beta_0$  is a refinement of  $\beta$ . For the converse take  $V = \{\text{int}(\mu^{-1}[t, 1]) \mid \mu \in \beta_0\}$  a family of regular open sets in  $T$ . Now  $\mu^{-1}(t, 1] \subseteq \text{int}(\mu^{-1}[t, 1])$ , so  $V$  covers  $X$ . And  $V$  refines  $U$ , since given  $\text{int}(\mu^{-1}[t, 1])$  we know  $\mu \in \beta_0$ , so there exists  $u \in U$  with  $\mu \wedge \chi_u = 0$ . Hence

$$\text{int}(\mu^{-1}[t, 1]) \subseteq \mu^{-1}[t, 1] \subseteq u.$$

Local finiteness presents no difficulties.

## References

- [1] H. Aygun, M. W. Warner and S. R. T. Kudri " On smooth L-fuzzy topological spaces" *J. Fuzzy Math.* vol. 5(2) 1997, pp 321-338, 1997.
- [2] A. Bulbul and M. W. Warner, " On the goodness of some types of fuzzy paracompactness" *Fuzzy Sets and Systems*, vol 55, pp.187-191, 1993.
- [3] C. L. Chang, " Fuzzy topological spaces" *J. Math. Anal. Appl.*, vol 24, pp.182-190, 1968.
- [4] K. C. Chattopadhyay and S. K. Samanta," Fuzzy topology: fuzzy closure operator, fuzzy compactness and fuzzy connectedness", *Fuzzy Sets and Systems*, vol 54, pp.207-212, 1993
- [5] K. C. Chattopadhyay, R. N. Hazra, S. K. Samanta, "Gradation of openness: fuzzy topology", *Fuzzy Sets and Systems*, vol 49, pp. 237-242, 1992.
- [6] M. Demirci, " On several types of compactness in smooth topological spaces" *Fuzzy Sets and Systems*, vol 90(1), pp. 83-88, 1997.
- [7] M. K. El Gayyar, E. E. Kerre and A. A. Ramadan, "Almost compactness and near compactness in smooth topological spaces", *Fuzzy Sets and Systems*, vol 62, pp. 193-202, 1994
- [8] U. Hoehle and A. P. Sostak, " A general theory of fuzzy topological spaces", *Fuzzy Sets and Systems*, vol 73, pp. 131-149, 1995
- [9] U. Hoehle and A. P. Sostak,"Axiomatic Foundations of Fixed-Basis fuzzy topology", *The Handbooks of Fuzzy sets series*, Volume 3, Kluwer Academic Publishers, Dordrecht (Chapter 3) 1999.
- [10] T. Kubiak and A. P. Sostak," Lower set-valued fuzzy topologies," *Quaestiones Mathematicae*, vol 20 pp. 423-429, 1997
- [11] R. Lowen, " Fuzzy topological spaces and fuzzy compactness", *J. Math. Anal. Appl.* vol 56 pp. 621-633, 1976.
- [12] A. A. Ramadan, "Smooth topological spaces", *Fuzzy Sets and Systems*, vol 48, pp. 371-375, 1992.
- [13] A. A. Ramadan, "On smooth topological spaces III" *J. Fuzzy Math.* vol 8(1), pp.35-64, 2000
- [14] A. A. Ramadan and S. E. Abbas," On L-smooth paracompactness" *J. Fuzzy Math.*, vol 9(2) 2001.
- [15] A. A. Ramadan, S. E. Abbas and Yong Chan Kim, " Almost paracompactness and near paracompactness in L-smooth topological spaces", *J. Korea Fuzzy Logic and Intelligent Systems*, vol 10(6), pp.564-568, 2000
- [16] A. P. Sostak, " On a fuzzy topological structure," *Suppl. Rend. Circ. Matem. Palerms ser II*, vol 11, pp. 89-103, 1985.
- [17] A. P. Sostak, " On the neighbourhood structure of a fuzzy topological space" *Zb. Rodova Univ. Nis, ser Math*, vol 4, pp. 7-14, 1990.
- [18] A. P. Sostak, "Basic structures of fuzzy topology", *J. of Math. Sciences*, vol 78, No 6, pp. 662-701, 1996.

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