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INTUITIONISTIC FUZZY RETRACTS

I. M. Hanafy, F. S. Mahmoud and M. M. Khalaf

Department of Mathematics, Faculty of Education, Suez Canal University, El-Arish, Egypt,
Department of Mathematics, Faculty of Science (82524), Sohag-Egypt.

Abstract

The concept of a intuitionistic fuzzy topology (IFT) was introduced by Coker 1997. The concept of a fuzzy retract was introduced by Rodabaugh in 1981. The aim of this paper is to introduce a new concepts of fuzzy continuity and fuzzy retracts in an intuitionistic fuzzy topological spaces and establish some of their properties. Also, the relations between these new concepts are discussed.

Key words : Intuintionistic fuzzy sets, Intuintionistic fuzzy retracts, Intuintionistic fuzzy semi-retract, Intuintionistic fuzzy almost retract, Intuintionistic fuzzy weakly retract.

1. Introduction

Weaker forms of Intuintionistic fuzzy continuity between Intuintionistic fuzzy topological spaces have been considered by [4,5]. We introduce and study in section 2 a new Intuintionistic fuzzy topological notions called Intuintionistic fuzzy retract, Intuintionistic fuzzy neighborhood retract. In section 3, the notions Intuintionistic fuzzy semi retract, Intuintionistic fuzzy pre retract, Intuintionistic fuzzy strongly semi-retract and Intuintionistic fuzzy semi pre-retract are introduced. In section 4, the notions Intuintionistic fuzzy almost(weakly) retract are introduced. Some of the fundamental properties of these concepts are investigated.

For definitions and results not explained in this paper, we refer to the papers [2,4,5,6,7], assuming them to be will known.. Let X be a non-empty set . A fuzzy set in X is a function with domain X and values in I. The words intuitionistic fuzzy set, intuitionistic fuzzy topological space, will be abbreviated as IF-set, IF-ts, respectively. Also by $int(\upsilon)$, $cl(\upsilon)$ and υ' we will denote respectively the interior, closure, and complement of the IF- set υ of IF- topological space.

First, we give the concept of intuitionistic fuzzy set defined by Atanassov as generalization of the concept of a fuzzy set given by Zadeh [7].

Definition 1.1 [1]. Let X be a nonempty set. An IF-set A is an object of the form $A=\{x,\mu_A(x), \upsilon_A(x): x\in X\}$. Where the functions $\mu_A : X \to I$ and $\upsilon_A : X \to I$ denote respectively, the degree of membership function (namely $\mu_A(x)$) and the degree of non-membership function (namely $\upsilon_A(x)$)of A, $0 \le \mu_A(x)+\upsilon_A(x) \le 1$, for each $x \in X$. An IF-set A= $x,\mu_A(x),\upsilon_A(x): x\in X\}$ can be written in the form A= $\{x,\mu_A,\upsilon_A\}$.

Definition 1.2 [2]. Let $A=\{x,\mu_A,\upsilon_A\}$. B= $\{x,\mu_B,\upsilon_B\}$. , $A_i=\{x,\mu_{A_i},\upsilon_{A_i}\}(i\in J)$. be IF-set on X, and $f: X\to Y$ a function. Then,

(i) $A'=\{x,\mu_A,\upsilon_A\}$.

(ii) $A\le B \Leftrightarrow$ for each $x\in X[\mu_A \le \mu_B$ and $\upsilon_A \ge \mu_B$

(iii) A= $A=B \Leftrightarrow A\le B$ and $B \le A$

(iv) $\wedge A_i=\{x,\wedge\mu_{A_i},\vee\upsilon_{A_i}\}$[7].

(v) $\vee A_i=\{x,\vee\mu_{A_i},\wedge\upsilon_{A_i}\}$[7].

Definition 1.3 [4]. Let A be an IF-set of an IF-ts (X, δ). Then A is called :

(*i*) an IF-regular open (IF-ro, for short) set if $A=int(cl(A))$.

(*ii*) an IF-semiopen (IF-so, for short) set if $A \le cl(int(A))$.

(*iii*) an IF-preopen (IF-po, for short) set if $A \le int(cl(A))$.

(*iv*) an IF-strongly semiopen (IF-so, for short)set if $A \le int(cl(int(A)))$.

(*v*) an IF-semi-preopen (IF-spo, for short) set if $A \le cl(int(cl(A)))$.

Their complements are called IF-semiclosed, IF-preclosed, IF-strongly semiclosed and IF-semi-preclosed sets.

Definition 1.4 [2]. Let X and Y be two nonempty sets and $f: X\to Y$ be a function.

(*i*) If B= $\{y,\mu_B(y),\upsilon_B(y): y\in Y\}$ is an IFS in Y, then the preimage of B under f (denoted by $f^{\leftarrow}(B)$) is defined by

$f^{\leftarrow}(B)=\{x, f^{\leftarrow}(\mu_B(x), f^{\leftarrow}(\upsilon_B(x): x\in X\}$.

(*ii*) If $A=\{x,\lambda_A(y),\lambda_A(y): x\in X\}$ is an IFS in X, then the image of A under f(denoted by $f(A)$) is defined by

$f(A)$= $f(A)=\{x, f(\lambda_A)(y), 1-f(1-\upsilon_A(y): y\in Y\}$.

Definition 1.5 [2]. Let $A, A_i (i\in J)$ be IFS 's in X, $B, B_i(i\in J)$ be IFS 's in Y, and $f: X\to Y$ be a function. Then

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(*i*) $A_1 \le A_2 \Rightarrow f(A_1) \le f(A_2)$

(*ii*) $B_1 \le B_2 \Rightarrow f(B_1) \le f(B_2)$

(*iii*) $A \le f(f^{\leftarrow}(A))$ [If *f* is one to one, then $A = f^{\leftarrow}(f(A))$]

(*iv*) $f(f^{\leftarrow}(B)) \le B$ [If *f* is onto, then $f^{\leftarrow}(f(B))] = B$

(*v*) $f^{\leftarrow}(\cup B_i) = \cup f^{\leftarrow}(B_i), f^{\leftarrow}(\cap B_i) = \cap f^{\leftarrow}(B_i)$

(*vi*) $f(\cup A_i) = \cup f(A_i)$

(*vii*) $f(\cap A_i) \le \cap f(A_i)$ If f is one to one, then, $f(\cap A_i) = \cap f(A_i)$

(*vii*) $f^{\leftarrow}(1) = 1, f^{\leftarrow}(0) = 0, f(1) = 1$ [If *f onto], f*(0)=-0

(*ix*) If *f* is onto, then [If furthermore *f* is one to one, then $cl(f(A)) = f(clA)$], $cl(f^{\leftarrow}(B)) = f^{\leftarrow}(cl(B))$.

Definition 1.6 [2]. Let (*X*,δ) and (*Y*, γ) be IFTSs and Let $f: X \to Y$ be a function. Then *f* is said to be fuzzy continuous iff the preimage of each IFS in γ is an IFS in δ.

Definition 1.7 [5]. Let *X*,*Y* be any IF-sets. If *A* is an IF- set of *X* and *B* is an IF- set of *Y*. Then $A \times B$ is an IF- set of $X \times Y$, defined by $(A \times B) = A(x) \wedge B(y) = \langle \mu_A(x) \wedge \mu_B(y), \mu_A(x) \vee \mu_B(y) \rangle$ for each $(x, y) \in X \times Y$. For a mapping $f: X \to Y$ the graph $g: X \to X \times Y$ of *f* is defined by *g*(*x*)=(*x*, *f*(*x*)) for each $x \in X$.

Definition 1.8 [5]. The product $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ of mappings $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1) \times f_2(x_2))$ for each $(x_1, x_2) \in X_1 \times X_2$.

Definition 1.9. An IF-ts (*X*, δ) is called:

(i) an IF-regular space iff each IF-open set λ is a union of IF-open sets l_α such that $cl(\lambda_\alpha) \le \lambda$ for each α.

(ii) an IF-semi regular space iff the collection of all IF-regular open sets forms a basis of δ.

Lemma 1.1 [5]. For mappings $f_i : X_i \to Y_i$ and IF- sets A_i of $X_i, i = 1,2$ we have $(f_1 \times f_2)(A_1, A_2) = (f_1(A_1) \times f_2(A_2))$

Lemma 1.2 [5]. Let $g: X \to X \times Y$ be the graph of a mapping $f: X \to Y$ .Then, if *A* is an IF- set of *X* and *B* is an IF-set of *Y*, then $g^{\leftarrow}(A \times B) = A \wedge f^{\leftarrow}(B)$.

Lemma 1.3 [5]. For mappings $f_i : X_i \to Y_i$ and IF- sets of $Y_i, (i = 1,2)$, we have $(f_1 \times f_2)^{\leftarrow}(A_1 \times A_2) = (f_1^{\leftarrow}(A_1) \times f_2^{\leftarrow}(A_2))$

Definition 1.10 [3]. If (*X*,δ) is an IF-ts and the induced F-topological subspace (A, δ_A) is defined so that $\delta_A = \{\upsilon \wedge A : \upsilon \in \delta\}$.

2. IF-retracts and IF-neighborhood retracts.

Theorem 2.1 [6]. Let (*X*, δ),(*Y*, γ) and (*Z*, ρ) be IF-ts $'s$ and $f: X \to Y$, $g: Y \to Z$ be mappings. If f and g are IF-continuous, then gf is IF-continuous because $(gf)^{\leftarrow}(\lambda) = f^{\leftarrow}(g^{\leftarrow}(\lambda)) \forall \lambda \in \rho$.

Theorem2.2. Let $(X_1, \delta_1), (X_2, \delta_2), (Y_1, \gamma_1), (Y_2, \gamma_2)$ .and (Y_2, γ_2) b e IF-ts $'s$. Then $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are IF-continuous iff the product. $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is IF-continuous.

Proof. Let $\eta \in (\gamma_1 \times \gamma_2)$ i.e., $\eta = \vee(\lambda_\alpha \times \eta_\beta)$, where λ_α $'s$ and η_β $'s$ are IF- open sets of ((Y_1, γ_1) and (Y_2, γ_2) , respectively, we want to show that $(f_1 \times f_2)^{\leftarrow}(\eta) = (f_1 \times f_2)^{\leftarrow} \vee (\lambda_\alpha \times \mu_\beta)) \in (\delta_1 \times \delta_2)$). Since $f_1 : (X_1, \delta_1) \to (Y_1, \gamma_1)$ is IF-continuous, $\lambda_\alpha \in \gamma_1$ then $f^{\leftarrow}(\lambda_\alpha) \in \delta_1$. Also, since $f_2 : (X_2, \delta_2) \to (Y_2, \gamma_2)$, is IF-continuous, $\mu_\beta \in \gamma_2$ then $f^{\leftarrow}(\mu_\alpha) \in \delta_2$ we get $(f_1^{\leftarrow}(\lambda_\alpha) \times f_2^{\leftarrow}(\eta_\beta) = (f_1 \times f_2)^{\leftarrow}(\lambda_\alpha \times \mu_\beta)) \in \delta_1 \times \delta_2$ hence $(f_1 \times f_2)^{\leftarrow}(\eta_\alpha)) \in \delta_1 \times \delta_2$.

Conversely, Let $\zeta \in \gamma_1$ and $\zeta \times 1 \in \gamma_1 \times \gamma_2$ since $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is an IF-continuous, we have $(f_1 \times f_2)^{\leftarrow}(\zeta \times 1) = (f_1^{\leftarrow}(\zeta) \wedge 1) = f_1^{\leftarrow}(\zeta) \in \delta_1$, i.e., f_1 is an IF-continuous. The proof with respect to f_2 in the same fashion.

Theorem 2.3. Let (*X*, δ),(*Y*, γ) be IF-ts $'s$. and $f: (X, \delta) \to (Y, \gamma)$ be a mapping .Then , the graph $g: (X, \delta) \to (X \times Y, \theta)$ of f is IF-continuous iff f is IF-continuous , where θ is the F- product topology generated by δ and γ.

Proof. Suppose the graph $g: (X, \delta) \to (X \times Y, \theta)$ is IF-continuous . $\eta \in \gamma$, we want to show that $f^{\leftarrow}(\eta) \in \delta$. Since $1 \times \eta \in \theta$ then $g^{\leftarrow}(1 \times \eta) = 1 \wedge f^{\leftarrow}(\eta) = f^{\leftarrow}(\eta) \in \delta$. So f is IF-continuous .

Conversely, Suppose f is IF-continuous , let $\zeta \in \theta$, i.e . $\zeta = \vee(\lambda_\alpha \times \mu_\beta)$, where $\lambda_\alpha 's$ and $\mu_\beta 's$ are IF- open set of δ and γ respectively. Now $g^{\leftarrow}(\zeta) = g^{\leftarrow} \vee (\lambda_\alpha \times \mu_\beta) = \vee(\lambda_\alpha \wedge f^{\leftarrow}(\mu_\beta)) \in \delta$. So *g* is IF- continuous.

Definition 2.1. Let (*X*, δ) be an IF-ts, and $A \subset X$, Then, the F- subspace (A, δ_A) is called an IF-retract (IFR, for short) of (*X*, δ) if there exists an IF-continuous mapping *r* :(*X*, δ) $\to (A, \delta_A)$ such that *r* (*a*)=*a* for all a $\in A$. In this case *r* is called an IF-retraction.

Remark 2.1. Let (*X*, δ) be an IF-ts. Since the identity map $id_X : X \to X$ is IF-continuous, then *X* is an IFR of itself.

Proposition 2.1. Let $Z \subset Y \subset X$, $r_1 : (X, \delta) \to (Y, \delta_Y)$ be IF-retraction , $r_2 : (Y, \delta_Y) \to (Z, (\delta_Y)_Z)$ be IF-retraction. Then $r_2 r_1 : (X, \delta) \to (Z, \delta_Y)_Z)$ is an IF-retraction.

Proof. It follows from Theorem 2.1.

Theorem 2.2. Let (*X*, δ) be an IF-ts, $A \subset X$ and *r* :(*X*, δ) $\to (A, \delta_A)$ be a mapping such that *r*(*a*)=*a* for all a $\in A$. Then the graph $g: (X, \delta) \to (X \times A, \theta)$ of *r* is IF-continuous iff *r* is an IF-retraction , where θ is the product topology generated by δ and δ_A .

Proof. It follows directly from Theorem 2.3.

Proposition 2.2. Let (X, δ),(Y, γ) be IF-ts 's . $A \subset X$, $B \subset Y$ If (A,δ_A) is an IFR of (X, δ) and (B, γ) is an IFR of (Y, γ), then $(A\times B,(\delta\times\gamma))$ is an IF-retract of $(X\times Y,\delta\times\gamma)$.

Definition 2.2. Let (X, δ) be an IF-ts. Then (A,δ_A) is said to be an IF-neighborhood retract (IF-nbd R, for short) of (X, δ) if (A,δ_A) is an IFR of (Y,δ_Y), such that $A \subset Y \subset X$, $1_Y \in \delta$.

Remark 2.3. Every IFR is an IF-nbd R, but the converse is not true.

Example 2.2. Let $X=\{a,b,c\}, A=\{a\} \subset X, \lambda_1$ and λ_2 be IF-sets on X, defined by

$$\lambda_1 = \left\langle x,(\frac{a}{0.2},\frac{b}{0.2},\frac{c}{0.4}),(\frac{a}{0.3},\frac{b}{0.3},\frac{c}{0.4})\right\rangle$$
$$\lambda_2 = \left\langle x,(\frac{a}{1},\frac{b}{1},\frac{c}{0}),(\frac{a}{0},\frac{b}{0},\frac{c}{1})\right\rangle$$

Consider $\delta = \{\underline{0},\underline{1},\lambda_1,\lambda_2,\lambda_1\vee\lambda_2,\lambda_1\wedge\lambda_2\}$ be an IF-ts on X. Then (A,δ_A) is an IF-nbd R of (X, δ), but not an IFR of (X, δ).

Proposition 2.3. Let (X, δ),(Y, γ) be IF-ts 's . $A \subset X$, $B \subset Y$ If (A,δ_A) is an IF-nbd R of (X, δ) and (B,γ_B) is an IF-nbd R of (Y, γ), then $(A\times B,(\delta\times\gamma))$ is an IF-nbd R of $(X\times Y,\delta\times\gamma)$.

Proof. Since (A,δ_A) is an IF-nbd R of (X, δ), then (A,δ_A) is an IFR of (U,δ_U) such that $A\subset U\subset X, 1_U\in\delta$, this implies that, there exists an IF-continuous mapping $r_1:(U,\delta_U)\to(A,(\delta_U)_A)$ such that $r_1(a)=a\forall a\in A$ Also since (B,γ_B) is an IF-nbd R of (Y,γ), then (B,γ_B) is an IFR of (V, γ) such that $B\subset V\subset X, 1_V\in\gamma$, this implies that, there exists an IF-continuous mapping $r_2:(V,\gamma_V)\to(B,(\gamma_V)_B)$ such that $r_2(b)=b\forall b\in B$ By using Theorem 2.2 we have $(r_1\times r_2):(U\times V,(\delta\times\gamma))\to(A\times B,((\delta\times\gamma)))$ is an IF-continuous mapping, $1_U\times 1_V\in\delta\times\gamma$ and $(r_1\times r_2)(a,b)=(r_1(a),r_2(b))=(a,b)\forall(a,b)\in A\times B$. Hence, $A\times B$ is an IF-nbd R of X× Y.

3. Weaker forms of IFR

Definition 3.1. Let (X, δ) be an IF-ts, and $A\subset X$. Then the IF- subspace (A,δ_A) is called an IF-semi retract (IFSR, for short) (resp. IF-pre retract, IF-strongly semi-retract and IF-semi pre retract.) (resp. IFPR, IFSSR, IFSPR, for short)of (X,δ) if there exists an IF-semi continuous (resp. IF-precontinuous, IF-strongly semicontinuous, IF-semi precontinuous.) mapping r :(X, δ) $\to(A,\delta_A)$ such that $r(a)=a\forall a\in A$. In this case r is called an IF-semi-retraction (resp. -IF-pre-retraction, IF-strongly semi-retraction, IF-semi pre-retraction).

The implications between these different concepts are given by the following diagram

$$IFR\Rightarrow IFSSR\Rightarrow \begin{matrix} IFSR\Rightarrow \\ IFPR\Rightarrow \end{matrix} IFSPR$$

But the converse need not be true, in general as shown by the following examples

Example 3.1. Let λ_1 and λ_2 be IF- sets on X={a,b}, defined by

$$\lambda_1 = \left\langle x,(\frac{a}{0.5},\frac{b}{0.3}),(\frac{a}{0.4},\frac{b}{0.5})\right\rangle$$
$$\lambda_2 = \left\langle x,(\frac{a}{0.6},\frac{b}{0.4}),(\frac{a}{0.2},\frac{b}{0.3})\right\rangle$$

Consider $\delta=\{\underline{0},\underline{1},\lambda_1,\lambda_2\}$ }, and A={a}$\subset X$ be an IF-ts on X. Then (A,δ_A) is an IFSS R of (X, δ), but not an IFR of it.

Example 3.2. Let λ be IF- sets on X={a,b}, defined by

$$\lambda = \left\langle x,(\frac{a}{0.1},\frac{b}{0.2}),(\frac{a}{0.3},\frac{b}{0.3})\right\rangle$$

Consider $\delta=\{\underline{0},\underline{1},\lambda\}$ }, and A={a}$\subset X$ be an IF-ts on X. Then (A,δ_A) is an IFP R of (X, δ), but not an IFSSR of it.

Example 3.3. . Let λ_1 and λ_2 be IF- sets on X={a,b}, defined by

$$\lambda_1 = \left\langle x,(\frac{a}{0.3},\frac{b}{0.4}),(\frac{a}{0.2},\frac{b}{0.2})\right\rangle$$
$$\lambda_2 = \left\langle x,(\frac{a}{0.1},\frac{b}{0.1}),(\frac{a}{0.32},\frac{b}{0.31})\right\rangle$$

Consider $\delta=\{\underline{0},\underline{1},\lambda_1,\lambda_2\}$ }, and A={a}$\subset X$ be an IF-ts on X. Then (A,δ_A) is an IFSS R of (X, δ), but not an IFR of it.
IFSR of (X,δ),but not an IFSSR.

Example 3.4. Let λ_1 and λ_2 be IF- sets on X={a,b}, defined by

$$\lambda_1 = \left\langle x,(\frac{a}{0.7},\frac{b}{0.8}),(\frac{a}{0.2},\frac{b}{0.1})\right\rangle$$
$$\lambda_2 = \left\langle x,(\frac{a}{0.2},\frac{b}{0.3}),(\frac{a}{0.4},\frac{b}{0.6})\right\rangle$$

Consider $\delta=\{\underline{0},\underline{1},\lambda_1,\lambda_2\}$ }, and A={a}$\subset X$ be an IF-ts on X. Then (A,δ_A) is an IFSPR of (X, δ), but not an IFSR and IFPR.

Proposition 3.1. Let (X, δ) be an IF-ts, $A\subset X$ and r :(X, δ) $\to(A,\delta_A)$ be a mapping such that $r(a)=a\forall a\in A$ .r is an IF-precontinuous and IF-semicontinuous, then (A,δ_A) is an IFSSR of (X, δ).

Proof . The proof is simple and hence omitted.

Definition 3.2. Let (X, δ) be an IF-ts. Then (A,δ_A) is said to be an IF-neighborhood semi-retract, (IF-nbd SR, for short) (resp. IF-nbd preretract, IF-nbd strongly semi-retract, IF-nbd semi preretract) (resp. IF-nbd PR, IF-nbd SSR, IF-nbd SPR, for short) of (X, δ) if (A,δ_A) is IFSR(resp. IFPR, IFSSR, IFSPR.) of $(Y\delta_Y)$, such that $A \subset Y \subset X, 1_Y \in \delta$.

Remark 3.1. Every IFPR is also an IF-nbd PR, Every IFSSR is also an IF-nbd SSR but the converse is not true in general, as we show in the following example.

Example 3.5. Let λ_1 and λ_2 be IF- sets on *X*={*a*,*b*}, defined by

$$\lambda_1 = \left\langle x, (\frac{a}{1}, \frac{b}{1}, \frac{c}{0}), (\frac{a}{0}, \frac{b}{0}, \frac{c}{1}) \right\rangle$$
$$\lambda_2 = \left\langle x, (\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.2}), (\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.21}) \right\rangle$$

Consider $\delta = \{\underline{0}, \underline{1}, \lambda_1, \lambda_2\}$ }, and A={a} $\subset X$ be an IF-ts on *X*. Then (A,δ_A) is an IF-nbd PR of (*X*, δ), but not an IF-PR and IF-nbdSSR but not IFSSR of it.

Remark 3.2. Every IFSPR is also an IF-nbd SPR, Every IFSR is also an IF-nbd SR but the converse is not true in general, as we show in the following example.

Example 3.6 Let λ_1 and λ_2 be IF- sets on *X*={*a*,*b*}, defined by

$$\lambda_1 = \left\langle x, (\frac{a}{1}, \frac{b}{1}, \frac{c}{0}), (\frac{a}{0}, \frac{b}{0}, \frac{c}{1}) \right\rangle$$
$$\lambda_2 = \left\langle x, (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.3}), (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.5}) \right\rangle$$

Consider $\delta = \{\underline{0}, \underline{1}, \lambda_1, \lambda_2\}$ }, and A={a} $\subset X$ be an IF-ts on *X*. Then (A,δ_A) is an IF-nbd SPR of (*X*, δ), but not an IF-SPR and IF-nbdSR but not IFSR of it.

4. IF-almost (weak) continuity and IF- almost (weakly) retract.

Definition 4.1 [4]. Let (*X*, δ),(*Y*, γ) be IF-ts 's , and $f:(X,\delta) \to (Y,\gamma)$. *f* is called

(*i*) an IF-almost continuous, If for each IF-regular open $\upsilon \in \gamma$, we have $f^{\leftarrow}(\upsilon) \in \delta$. .

(*ii*) an IF-weakly continuous, If for each $\upsilon \in \gamma$ we have $f^{\leftarrow}(\upsilon) \le \text{int}(f^{\leftarrow}(cl\upsilon))$.

Theorem 4.1. Let (*X*, δ),(*Y*, γ) be IF-ts 's , and $f:(X,\delta) \to (Y,\gamma)$. *f* is IF-almost continuous iff $f^{\leftarrow}(\upsilon) \le \text{int}(f^{\leftarrow}(cl\upsilon))$ $\forall \upsilon \in \gamma$

Proof. Let *f* be an IF-almost continuous, $\upsilon \in \gamma$, then $\upsilon = \text{int}(\upsilon) \le \text{int}(cl(\upsilon)) \Rightarrow f^{\leftarrow}(\text{int}(\upsilon)) \le f^{\leftarrow}(\text{int}(cl(\upsilon)))$, then *int*(*cl*(υ)) is IF-regular open, hence $f^{\leftarrow}(\text{int}(cl(\upsilon))) \in \delta$ .Thus, $f^{\leftarrow}(\upsilon) \le f^{\leftarrow}(\text{int}(cl(\upsilon))) = \text{int}(f^{\leftarrow}(\text{int}(cl(\upsilon)))$.

Conversely Let υ be IF-regular open $\in \delta$,then, we have, $f^{\leftarrow}(\upsilon) \le \text{int}(f^{\leftarrow}(\text{int}(cl(\upsilon)) = \text{int}(f^{\leftarrow}(\upsilon))$. Hence $f^{\leftarrow}(\upsilon) = \text{int}(f^{\leftarrow}(\upsilon))$. and $f^{\leftarrow}(\upsilon) \in \delta$.

Proposition 4.1. Let (*X*, δ),(*Y*, γ) be IF-ts 's , and $f:(X,\delta) \to (Y,\gamma)$. If *f* is an IF-almost continuous then it is IF-weakly continuous.

Proof. It follows immediately from Theorem 4.1.

Remark 4.1. The implications between these different concepts are given by the following diagram.

IF-continuous ⇒ IF-almost continuous ⇒ IF-weakly continuous

Example 4.1 Let λ_1 and λ_2 be IF- sets on *X*={*a*,*b*,*c*}, defined by

$$\lambda_1 = \left\langle x, (\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.7}), (\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.2}) \right\rangle$$
$$\lambda_2 = \left\langle x, (\frac{a}{0.1}, \frac{b}{0.4}, \frac{c}{0.6}), (\frac{a}{0.3}, \frac{b}{0.3}, \frac{c}{0.3}) \right\rangle$$

and $\delta = \{\underline{0}, \underline{1}, \lambda_1, \lambda_2\}$ }. Also, ζ_1 and ζ_2 be IF- sets on *X*={*a*,*b*}, defined by

$$\zeta_1 = \left\langle x, (\frac{a}{0.4}, \frac{b}{0.9}), (\frac{a}{0.2}, \frac{b}{0.1}) \right\rangle$$
$$\zeta_2 = \left\langle x, (\frac{a}{0.1}, \frac{b}{0.8}), (\frac{a}{0.2}, \frac{b}{0.1}) \right\rangle$$

and $\gamma = \{\underline{0}, \underline{1}, \zeta_1, \zeta_2\}$ }. Then the function $f:(X,\delta) \to (Y,\gamma)$ defined by *f*(*a*)=*x*, *f*(*b*)=*f*(*c*)=*y* is an IF-almost continuous but not IF-continuous.

Example 4.2. Let λ be IF- sets on *X*={*a*,*b*,*c*}, defined by

$$\lambda = \left\langle x, (\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.2}), (\frac{a}{0.2}, \frac{b}{0.2}, \frac{c}{0.2}) \right\rangle$$

and $\delta = \{\underline{0}, \underline{1}, \lambda\}$ }. Also, ζ be IF- sets on Y={x, y }, defined by

$$\zeta = \left\langle x, (\frac{a}{0.1}, \frac{b}{0.1}), (\frac{a}{0.2}, \frac{b}{0.1}) \right\rangle$$

and $\gamma = \{\underline{0}, \underline{1}, \zeta\}$ }. Then the function $f:(X,\delta) \to (Y,\gamma)$ defined by *f*(*a*)=*x*, *f*(*b*)=*f*(*c*)=*y* is an IF-weakly continuous but not IF-almost continuous.

Theorem 4.2. Let (*X*, δ),(*Y*, γ) be IF-ts 's , and $f:(X,\delta) \to (Y,\gamma)$. is an IF-semi regular space. Then *f* is IF-almost continuous iff *f* is IF-continuous.

Proof. Due to Remark 4.1, it suffices to show that if *f* is IF-almost continuous, then *f* is IF continuous. Let, $\lambda \in \gamma$, then $\lambda = \vee\lambda_\alpha$, where λ_α 's are IF-regular open sets of γ . Now, $f^{\leftarrow}(\lambda) = f^{\leftarrow}(\vee\lambda_\alpha) = \vee f^{\leftarrow}(\lambda_\alpha)$, but $\vee f^{\leftarrow}(\lambda_\alpha) \in \delta \forall \alpha \Rightarrow$ $\vee f^{\leftarrow}(\lambda_\alpha) \in \delta \Rightarrow f$ is IF continuous.

Theorem 4.3. Let (*X*,δ) be an IF-ts and (*Y*, γ) be an IF-regular space. Then *f* is IF-weakly continues iff *f* is IF-continuous.

Proof. Due to Remark 4.1, it is suffices to show that if *f* is IF-weakly continuous, then it is IF-continuous. Let *f* be IF-weakly continuous and $\lambda \in \delta$ Since (*Y*, γ) is IF-regular space, $\lambda = \vee\lambda_\alpha, \lambda_\alpha \in \delta$ and $cl(\lambda_\alpha) \leq \lambda$ for each α ,Now, $f^{\leftarrow}(\lambda) = f^{\leftarrow}(\vee\lambda_\alpha) \leq \vee f^{\leftarrow}(\lambda_\alpha) \leq \vee cl(f^{\leftarrow}(\lambda_\alpha)) \leq \vee \text{int}(f^{\leftarrow}(cl\lambda)) \leq \text{int}(f^{\leftarrow}(\lambda))$ then $f^{\leftarrow}(\lambda) = \text{int}(f^{\leftarrow}(\lambda) \in \delta \Rightarrow f$ is IF continuous .

Definition 4.2. Let (*X*, δ) be an IF-ts, $A \subset X$. Then (A, δ_A) is called an IF-almost R (resp., IF-weakly R) of (*X*, δ) if there exists an IF-almost continuous (resp., IF-weakly continuous) mapping *r* :(*X*, δ) $\rightarrow (A, \delta_A)$ such that $r(a) = a \forall a \in A$. In this case *r* is called an IF-almost retraction (resp., IF-weakly retraction)

Proposition 4.2. Consider the following properties:

(*i*) (A, δ_A) is an IF-R of (*X*, δ);

(*ii*) (A, δ_A) is an IF-almost R of (*X*, δ);

(*iii*) (A, δ_A) is an IF-weakly R of (*X*, δ).

Then (*i*) ⇒ (*ii*) ⇒ (*iii*).

Proof. Obvious.

The inverse implications in Proposition 4.2. are not, in general, true.

Example 4.4. Let λ be IF- sets on *X*={*a*,*b*,*c*}, defined by

$$\lambda = \left\langle x, (\frac{a}{0.2}, \frac{b}{0.71}, \frac{c}{0.8}), (\frac{a}{0.5}, \frac{b}{0.5}, \frac{c}{0.5}) \right\rangle$$

Consider $\delta = \{\underline{0}, \underline{1}, \lambda\}$ be an IF-ts on *X* and *A*={*x*,*y*} ⊂ *X*. Then (A, δ_A) is an IF-almost R of (*X*,δ) but not IF-R of it.

Example 4.5. Let λ be IF- sets on *X*={*a*,*b*,*c*}, defined by

$$\lambda = \left\langle x, (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.3}), (\frac{a}{0.2}, \frac{b}{0.3}, \frac{c}{0.4}) \right\rangle$$

Consider $\delta = \{\underline{0}, \underline{1}, \lambda\}$ be an IF-ts on *X* and *A*={*x*, *y*} ⊂ *X*. Then (A, δ_A) is an IF-weakly R of (*X*,δ) but not IF-almost R of it.

Theorem 4.4. Let (*X*, δ),(*Y*, γ) be IF-ts*'s* , and let $A \subset X$ be an IF-semi regular space. Then (A, δ_A) is an IF-almost R of (*X*, δ) iff (A, δ_A) is an IF-R of (*X*, δ).

Proof. follows directly from Theorem 4.1.

Theorem 4.5. Let (*X*, δ) be an IF-ts, and $A \subset X$ be an IF-regular space. Then (A, δ_A) is an IF-weakly R of (*X*, δ) iff (A, δ_A) is an IF-R of (*X*, δ).

Proof. Follows directly from Theorem 4.2.

Theorem 4.6. Let (*X*, δ) be an IF-ts, and $A \subset X$ be an IF-regular space. Then (A, δ_A) is an IF-almost R of (*X*, δ) iff (A, δ_A) is an IF-weakly R of (*X*, δ).

Proof. The proof can be carried by Theorem 4.3. and Proposition 4.1.

Definition 4.3. Let (*X*, δ) be an IF-ts, and $A \subset X$. Then (A, δ_A) is said to be an IF-neighborhood almost R (IF-nbd almost R, for short), (resp., IF-neighborhood weakly R, (IF-nbd weakly R, for short)) of (*X*, δ) iff (A, δ_A) is an IF-almost R (resp., IF weakly R) of (Y, δ_Y) such that $A \subset Y \subset X, 1_Y \in \delta$.

Remark 4.2. Let (*X*, δ) be an IF-ts, and $A \subset X$. (A, δ_A) is an IF-almost R of (*X*, δ), then (A, δ_A) is an IF-nbd almost R of (*X*, δ), but the converse is not true in general, as shown by the following example.

Example 4.6. Let λ_1 and λ_2 be IF- sets on *X*={*a*,*b*,*c*}, defined by

$$\lambda_1 = \left\langle x, (\frac{a}{1}, \frac{b}{1}, \frac{c}{0}), (\frac{a}{0}, \frac{b}{0}, \frac{c}{1}) \right\rangle$$

$$\lambda_2 = \left\langle x, (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.5}), (\frac{a}{0.4}, \frac{b}{0.4}, \frac{c}{0.2}) \right\rangle$$

Take *A*={*x*, *y*}, and $\delta = \{\underline{0}, \underline{1}, \lambda_1, \lambda_2, \lambda_1 \vee \lambda_2, \lambda_1 \wedge \lambda_2\}$. Then the function $f : (X, \delta) \rightarrow (A, \delta_A)$, defined by *f*(*a*)=*x*, *f*(*b*)=*f*(*c*)=*y* is an IF-nbd almost R of (*X*, δ), but not IF-almost R of it.

Remark 4.3. Let (*X*, δ) be an IF-ts and $A \subset X$. (A, δ_A) is an IF-weakly R of (*X*, δ), then (A, δ_A) is an IF-nbd weakly R of (*X*, δ), but the converse is not true in general, as shown by the following example.

Example 4.7. Example 4.6. show that (A, δ_A) is an IF-nbd weakly R of (*X*, δ), but not IF-weakly R of it.

Remark 4.4. The implications between these different notions of IF-R are given by the following diagram:

IFSR ⇒ IFSPR

⇑ ⇑

IFR ⇒ IFSSR ⇒ IFPR

⇓

IF-almost R ⇒ IF-weakly R.

Remark 4.5. The implications between these different notions of IF-nbd. R are given by the following diagram.

IF-nbd. SR ⇒ IF-nbd SPR

⇑ ⇑

IF-nbd. R ⇒ IF-nbd. SSR ⇒ IF-nbd. PR

⇓

IF-nbd. Almost R ⇒ IF-nbd. Weakly R

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Ibrahim Mohamed Hanafy

Degrees :

- B. SC. in Math. from Fac. Of Science, Assiut Univ. 1976.
- M. SC in Math. (numerical analysis) from Fac. Of Science, Assiut Univ. 1981.
- Ph. D. in Math. (Topology) from Fac. Of Science, Tanta Univ. 1990.

Academic Responsibilities:

- Assistant of the Dept. of Math. Fac. Of Science, Assiut Univ.
- Assistant lecturer of the Dept. of Math. Fac. Of Science, Assiut Univ.
- lecturer of the Dept. of Math. Fac. Of Education, Suez Canal University, El-Arish, Egypt.
- Assistant Prof. of the Dept. of Math. Fac. Of Education, Suez Canal University, El-Arish, Egypt.

E-Mail : ihanafy@hotmail.com

F. S. Mahmoud

Department of Mathematics, Faculty of Science (82524), Sohag-Egypt.

M. M. Khalaf

Department of Mathematics, Faculty of Science (82524), Sohag-Egypt.