PARALLEL ALGORITHMS FOR INTEGRATION OF NAVIER–STOKES EQUATIONS BASED ON THE ITERATIVE SPACE-MARCHING METHOD

Leonid I. Skurin[1]

This research is based on the iterative space-marching method for incompressible and compressible Navier–Stokes equations[1-4]. A principle of parallel computational schemes construction for steady and unsteady problems is suggested. It is analytically proven that convergence of these schemes is unconditional for incompressible case. When the parallel scheme is used the total volume of computations is the sum of a large number of independent and equal parts. Estimation of the speed-up K shows that K>1000 in ideal case. First results of using the parallel schemes are presented.

1. INTRDUCTION

Now there exist a number of methods for solving Navier-Stokes equations that use advantages of supercomputers. Methods that are based on computational schemes that can be realized using a parallel algorithm are very effective. However existing computational schemes of this type are applicable only to a limited range of problems. This can be seen from a large number of papers presented in recent years (see proceedings of the conferences in[3,5,6]).

The iterative space-marching method (IMM) for solving compressible and incompressible Navier-Stokes equations is highly universal and is based on a simple scheme[1-4] (see also the following section). Therefore it is important to build a parallel algorithm based on this method and substantiate its qualities analytically and numerically. This problem is considered in the present paper.

Here we construct parallel computational schemes in the IMM framework for steady and unsteady problems. We perform an analytical research of convergence of these schemes for 3D problems. Based on these schemes (for any problem) computations of the single algebraic procedure of the IMM (see following section) in each marching station on any GI can be performed simultaneously on different processors. Thus, realizing these schemes on a supercomputer or a cluster can significantly decrease computational time for any problem.

2. SUMMARY OF THE IMM

The iterative space-marching method for incompressible and compressible full Navier-Stokes equations has been developed in recent years (in works [1-4] and others).

First, we give a review of the mathematical formalization of the IMM. In 2D case one of the coordinate axes (let it be x) is declared a marching axis. The pressure p gradient in the projection of the motion equation on this axis is expressed in the following form:

$$\partial p/\partial x \Rightarrow (1+\varepsilon)\partial \bar{p}/\partial x - \varepsilon \partial p/\partial x, \qquad (1)$$

where ε is a given value (as a rule $\varepsilon = 1$), $\bar{p}$ is a given function being refined through the computational process. The Navier-Stokes equations are presented in finite-difference forms and an iterative loop based on marching sweeps - a global iteration (GI), is constructed. Function $\bar{p}$ for the s-th GI is calculated using the following relation:

$$\bar{p}^{s} = (1-\tau)\bar{p}^{s-1} + \tau p^{s-1}, \qquad (2)$$

where τ is a parameter of the scheme (as a rule $\tau = 1$). The objective of the GI is to achieve the

Received: July 15, 2004, Accepted: January 20, 2005
1 St. Petersburg State University, St. Petersburg, Russia

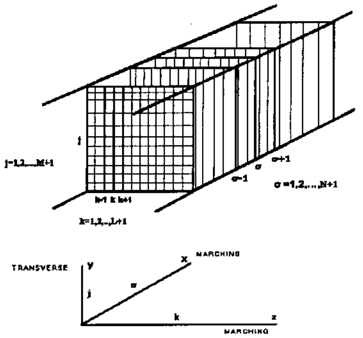

Fig.1

equality between $\bar{p}$ and p (with a given level of accuracy):

$$\bar{p}^s - p^s < \varepsilon_p << 1, \qquad (3)$$

When this condition is met the transformation (1) becomes trivial for all ε and thus solving the steady problem is completed, or in the case of an unsteady problem, one time step is completed.

In case of 3D problems (see sketch of the grid in Fig. 1) two coordinate axes are declared marching axes. Therefore, 3D problems are solved by making multiple successive computations of 2D domains (global iteration) and within each 2D domain the computation is performed using marching sweeps[1,3,4].

1D problems are solved by making multiple successive computations of grid functions at each point of a single space axis.

Thus, we see that a single algebraic procedure is used for solving all problems. This is the procedure of finding the vector of unknown grid functions along the transverse lines.

It is important to emphasize that the original computational schemes of the IMM are a marching schemes. No splitting is used. Any computational scheme for incompressible fluid is a special case of compressible one at zero Mach number. The computational schemes do not contain any varied artificial constants or functions.

Analytical research of the marching sweeps stability and convergence in GI was performed for incompressible fluid using first order finite -difference approximations for first marching derivatives. It was proven (using a linear approach) that stability and convergence are unconditional[4] (see also references in[3]).

Numerical solutions of various problems were obtained. We considered motions of homogeneous and non-homogeneous medias (stratified, two-phase, turbulent), swirled flows and flows with large separated and recirculation regions (wake for a body, vortex tube, vortex pair, Taylor's vortexes and others). The computations were based on conservative form of equations and second order approximations of the marching derivatives. This computational research showed that the IMM is robust at a wide ranges of Mach number (from zero to hypersonic values) and Reynolds number (from values much less than 1 to "turbulent values") as well as for problems without main flow direction where second order marching derivatives play a significant role.

Thus the IMM is universal and simple. A modified scheme[2 - 4] of the IMM is efficient for steady problems. It is based on the principle of convergence in time to a steady state. It does not require meeting condition (3) at each temporal layer and values ε and τ in (1) and (2) are equal one for this scheme. As a result each time step is realized with one GI.

For solving unsteady problems the general scheme[2 - 4] of the IMM is used. In this case the solution at each temporal layer is found using GI up to convergence (until (3) is met).

3. STEADY PROBLEM

We construct parallel schemes for steady problems in framework of the modified scheme. In order to formulate a parallel scheme we change finite difference approximations of derivatives for velocity vector projections. Namely, approximations of first derivatives for the each projection f with respect to each marching coordinate x we write as follows

$$\frac{\partial f}{\partial x} = \frac{f^{\sigma}_{jkm} - f^{\sigma-1}_{jkm-1}}{\Delta x} = \frac{f^{\sigma}_{jkm} - f^{\sigma-1}_{jkm}}{\Delta x} + \frac{f^{\sigma-1}_{jkm} - f^{\sigma-1}_{jkm-1}}{\Delta x}, \qquad (4)$$

where Δx - step along axis x, σ, k - numbers of nodes along marching axes x and z respectively, j - a number of a node along transverse line, m - a number of a temporal layer.

It can be seen that first equality in (4) is an approximation of the derivative given time convergence. The presence of the second term in the right-hand side of (4) makes this approximation different from a similar one for the modified scheme. Approximations of the second marching derivatives are changed in a similar way.

These changes lead to following schemes for incompressible 3D Navier-Stokes equations written in Cartesian coordinates and with "frozen" coefficients (see Fig. 1):

$$\frac{u^{\sigma}_{jkm}-u^{\sigma}_{jkm-1}}{\Delta t}+u_0\frac{u^{\sigma}_{jkm}-S_1u^{\sigma-1}_{jkm-1}-S_2u^{\sigma+1}_{jkm-1}}{\Delta x}+v_0\frac{u^{\sigma}_{j+1km}-u^{\sigma}_{j-1km}}{2\Delta y}+w_0\frac{u^{\sigma}_{jkm}-T_1u^{\sigma}_{jk-1m-1}-T_2u^{\sigma}_{jk+1m-1}}{\Delta z}+\frac{p^{\sigma+1}_{jkm-1}-p^{\sigma}_{jkm}}{\Delta x}$$
$$=\frac{u^{\sigma+1}_{jkm-1}-2u^{\sigma}_{jkm}+u^{\sigma-1}_{jkm-1}}{\mathrm{Re}\Delta x^2}+\frac{u^{\sigma}_{j+1km}-2u^{\sigma}_{jkm}+u^{\sigma}_{j-1km}}{\mathrm{Re}\Delta y^2}+\frac{u^{\sigma}_{jk+1m-1}-2u^{\sigma}_{jkm}+u^{\sigma}_{jk-1m-1}}{\mathrm{Re}\Delta z^2},$$

$$\frac{v^{\sigma}_{jkm}-v^{\sigma}_{jkm-1}}{\Delta t}+u_0\frac{v^{\sigma}_{jkm}-S_1v^{\sigma-1}_{jkm-1}-S_2v^{\sigma+1}_{jkm-1}}{\Delta x}+v_0\frac{v^{\sigma}_{j+1km}-v^{\sigma}_{j-1km}}{2\Delta y}+w_0\frac{v^{\sigma}_{jkm}-T_1v^{\sigma}_{jk-1m-1}-T_2v^{\sigma}_{jk+1m-1}}{\Delta z}+\frac{p^{\sigma}_{j+1km}-p^{\sigma}_{j-1km}}{2\Delta y}$$
$$=\frac{v^{\sigma+1}_{jkm-1}-2v^{\sigma}_{jkm}+v^{\sigma-1}_{jkm-1}}{\mathrm{Re}\Delta x^2}+\frac{v^{\sigma}_{j+1km}-2v^{\sigma}_{jkm}+v^{\sigma}_{j-1km}}{\mathrm{Re}\Delta y^2}+\frac{v^{\sigma}_{jk+1m-1}-2v^{\sigma}_{jkm}+v^{\sigma}_{jk-1m-1}}{\mathrm{Re}\Delta z^2},$$

$$\frac{w^{\sigma}_{jkm}-w^{\sigma}_{jkm-1}}{\Delta t}+u_0\frac{w^{\sigma}_{jkm}-S_1w^{\sigma-1}_{jkm}-S_2w^{\sigma+1}_{jkm-1}}{\Delta x}+v_0\frac{w^{\sigma}_{j+1km}-w^{\sigma}_{j-1km}}{2\Delta y}+w_0\frac{w^{\sigma}_{jkm}-T_1w^{\sigma}_{jk-1m-1}-T_2w^{\sigma}_{jk+1m-1}}{\Delta z}+\frac{p^{\sigma+1}_{jk+1m-1}-p^{\sigma}_{jkm}}{\Delta z}$$
$$=\frac{w^{\sigma+1}_{jkm-1}-2w^{\sigma}_{jkm}+w^{\sigma-1}_{jkm-1}}{\mathrm{Re}\Delta x^2}+\frac{w^{\sigma}_{j+1km}-2w^{\sigma}_{jkm}+w^{\sigma}_{j-1km}}{\mathrm{Re}\Delta y^2}+\frac{w^{\sigma}_{jk+1m-1}-2w^{\sigma}_{jkm}+w^{\sigma}_{jk-1m-1}}{\mathrm{Re}\Delta z^2},$$

$$\frac{u^{\sigma}_{jkm}-u^{\sigma-1}_{jkm-1}}{\Delta x}+\frac{v^{\sigma}_{j+1km}-v^{\sigma}_{j-1km}}{2\Delta y}+\frac{w^{\sigma}_{jkm}-w^{\sigma}_{jk-1m-1}}{\Delta z}=0, \quad (5)$$

where

$$S_1=\frac{1}{2}(1+\frac{u_0}{|u_0|}),\ S_2=1-S_1,$$
$$T_1=\frac{1}{2}(1+\frac{w_0}{|w_0|}),\ T_2=1-T_1,$$

u,v,w - the projection of the velocity vector on axes x,y,z, Re - Reynolds number.

This scheme differs from the modified one in that the values of the grid functions with indexes $(\sigma-1)$ or $(k-1)$ have index $(m-1)$ (not m, as in the modified scheme). The modified scheme is realized with a marching sweep at each temporal layer, in other words by means of finding solutions successively at lines $k=2,3,...,L-1$ of planes $\sigma=2,3,...,N-1$ respectively, where $N+1$ - the number of computational planes in 3D domain, $L+1$ - the number of the transverse lines in each plane. As opposed to that, finding solutions at all transverse lines (σ,k) using system (5) does not require a specific order of steps and therefore may be achieved with a parallel algorithm. However it is important to investigate whether this scheme has time stability.

Consider this question using Fourie method (or von Neyman analysis). Determine the vector of unknown grid functions in (5) as follows

$$\vartheta^{\sigma}_{jkm}=(u_*,v_*,w_*,p_*)^T\lambda^m\exp[i(\alpha j+\gamma\sigma+\beta k)], \quad (6)$$

where α,γ,β - arbitrary constants, u_*,v_*,w_*,p_* and λ - unknown constants. The matrix of equations system for vector $(u_*,v_*,w_*,p_*)^T$ is

$$\begin{pmatrix}\lambda-e^{-i\gamma} & \lambda r_y i\sin\alpha & r_z(\lambda-e^{-i\beta}) & 0\\ \lambda P_1-P_2 & 0 & 0 & e^{i\gamma}-1\\ 0 & \lambda P_1-P_2 & 0 & r_y\lambda i\sin\alpha\\ 0 & 0 & \lambda P_1-P_2 & r_z(e^{i\beta}-\lambda)\end{pmatrix} \quad (7)$$

where

$$P_1=r_t+|u_0|+|w_0|r_z+\frac{2}{\mathrm{Re}\Delta x}+\frac{2r_z}{\mathrm{Re}\Delta z}+\xi+iv_0r_y\sin\alpha,$$
$$P_2=r_t+|u_0|(S_1e^{-i\gamma}+S_2e^{-i\gamma})+|w_0|r_z(T_1e^{-i\beta}+T_2e^{-i\beta})+\frac{2\cos\gamma}{\mathrm{Re}\Delta x}+\frac{2r_z\cos\beta}{\mathrm{Re}\Delta z},$$

$$r_t=\frac{\Delta x}{\Delta t},\quad r_y=\frac{\Delta x}{\Delta y},\quad r_z=\frac{\Delta x}{\Delta z},\quad \xi=\frac{r_y(1-\cos\alpha)}{\mathrm{Re}\Delta y}$$

The characteristic equation of this matrix has three different roots. Two of them are defined by the equation

$$\lambda^2-\frac{2(\cos\gamma+r_z^2\cos\beta)}{1+r_z^2+A^2}\lambda+\frac{1+r_z^2}{1+r_z^2+A^2}=0, \qquad (8)$$

$$A^2=r_y^2\sin^2\alpha$$

and the third one - by the formula $\lambda_3=P_2/P_1$.

The solution of (8) is

$$\lambda_{1,2}=\frac{\cos\gamma+r_z^2\cos\beta}{1+r_z^2+A^2}\pm\sqrt{\left(\frac{\cos\gamma+r_z^2\cos\beta}{1+r_z^2+A^2}\right)^2+\frac{1+r_z^2}{1+r_z^2+A^2}}$$

$$=\frac{\cos\gamma+r_z^2\cos\beta}{1+r_z^2+A^2}\pm\frac{i\sqrt{1-\cos^2\gamma+2r_z^2(1-\cos\gamma\cos\beta)+r_z^4(1-\cos\beta)+(1+r_z^2)A^2}}{1+r_z^2+A^2}$$

Algebraic transformations lead to the formula

$$|\lambda_{1,2}|=\sqrt{\frac{1+r_z^2}{1+r_z^2+A^2}}.$$

It can be seen that these roots do not exceed one. Consider the third root. Taking into account that

$$S_1+S_2=T_1+T_2=1,\ S_2-S_1=\pm1,\ T_2-T_1=\pm1$$

it can be found

$$\begin{aligned}|P_2|^2&=(C+|u_0|\cos\gamma+|w_0|r_z\cos\beta)^2\\&+[|u_0|(S_2-S_1)\sin\gamma+|w_0|r_z(T_2-T_1)\cos\beta]^2\\&=C^2+2C(|u_0|\cos\gamma+|w_0|r_z\cos\beta)+|u_0|^2\cos^2\gamma\\&+2|u_0||w_0|r_z\cos\gamma\cos\beta+|w_0|^2r_z^2\cos^2\beta\\&+|u_0|^2\sin^2\gamma+2|u_0||w_0|r_z(S_2-S_1)(T_2-T_1)\sin\gamma\sin\beta\\&+|w_0|^2r_z^2\sin^2\beta\\&=C^2+2C(|u_0|\cos\gamma+|w_0|r_z\cos\beta)\\&+|u_0|^2+|w_0|^2r_z^2+2|u_0||w_0|r_z\cos(\gamma\pm\beta),\end{aligned}$$

$$\begin{aligned}|P_1|^2&=(B+|u_0|+|w_0|r_z)^2+v_0^2A^2\\&=B^2+2B(|u_0|+|w_0|r_z)+|u_0|^2\\&+2|u_0||w_0|r_z+|w_0|^2r_z^2+v_0^2A^2,\end{aligned}$$

where

$$C=r_t+\frac{2\cos\gamma}{\mathrm{Re}\Delta x}+\frac{2r_z\cos\beta}{\mathrm{Re}\Delta z},$$

$$B=r_t+\frac{2}{\mathrm{Re}\Delta x}+\frac{2r_z}{\mathrm{Re}\Delta z}+\xi.$$

Thus

$$|\lambda_3|^2=\frac{C^2+2C(|u_0|\cos\gamma+|w_0|r_z\cos\beta)+2|u_0||w_0|r_z\cos(\gamma\pm\beta)+|u_0|^2+|w_0|^2r_z^2}{B^2+2B(|u_0|+|w_0|r_z)+2|u_0||w_0|r_z+|u_0|^2+|w_0|^2r_z^2+v_0^2A^2}$$

It can be seen that the difference between the denominator and the numerator is always positive.

Thus all roots do not exceed one so that scheme (5) converges unconditionally.

4. UNSTEADY PROBLEM

Unsteady problem is normally solved using the general scheme of the IMM. In this case the solution at each temporal layer is found by means of GI. Let us make similar changes in the general scheme as the ones we made above (4) to the modified scheme. Then we have the following scheme (see Fig. 1):

$$\begin{aligned}&\frac{u_{jkm}^{\sigma}-u_{jkm-1}^{\sigma}}{\Delta t}+u_0\frac{u_{jkm}^{\sigma}-(S_1u_{jkm}^{\sigma-1})^{s-1}-(S_2u_{jkm}^{\sigma+1})^{s-1}}{\Delta x}\\&+v_0\frac{u_{j+1km}^{\sigma}-u_{j-1km}^{\sigma}}{2\Delta y}+w_0\frac{u_{jkm}^{\sigma}-(T_1u_{jk-1m}^{\sigma})^{s-1}-(T_2u_{jk+1m}^{\sigma})^{s-1}}{\Delta z}\\&+\frac{(p_{jkm-1}^{\sigma+1})^{s-1}-p_{jkm}^{\sigma}}{\Delta x}\\&=\frac{(u_{jkm}^{\sigma+1})^{s-1}-2u_{jkm}^{\sigma}+(u_{jkm}^{\sigma-1})^{s-1}}{\mathrm{Re}\Delta x^2}+\frac{u_{j+1km}^{\sigma}-2u_{jkm}^{\sigma}+u_{j-1km}^{\sigma}}{\mathrm{Re}\Delta y^2}\\&+\frac{(u_{jk+1m}^{\sigma})^{s-1}-2u_{jkm}^{\sigma}+(u_{jk-1m}^{\sigma})^{s-1}}{\mathrm{Re}\Delta z^2},\end{aligned}$$

$$\begin{aligned}&\frac{v_{jkm}^{\sigma}-v_{jkm-1}^{\sigma}}{\Delta t}+u_0\frac{v_{jkm}^{\sigma}-(S_1v_{jkm}^{\sigma-1})^{s-1}-(S_2v_{jkm}^{\sigma+1})^{s-1}}{\Delta x}\\&+v_0\frac{v_{j+1km}^{\sigma}-v_{j-1km}^{\sigma}}{2\Delta y}+w_0\frac{v_{jkm}^{\sigma}-(T_1v_{jk-1m}^{\sigma})^{s-1}-(T_2v_{jk+1m-1}^{\sigma})^{s-1}}{\Delta z}\\&+\frac{p_{j+1km}^{\sigma}-p_{j-1km}^{\sigma}}{2\Delta y}\\&=\frac{(v_{jkm}^{\sigma+1})^{s-1}-2v_{jkm}^{\sigma}+(v_{jkm}^{\sigma-1})^{s-1}}{\mathrm{Re}\Delta x^2}+\frac{v_{j+1km}^{\sigma}-2v_{jkm}^{\sigma}+v_{j-1km}^{\sigma}}{\mathrm{Re}\Delta y^2}\\&+\frac{(v_{jk+1m}^{\sigma})^{s-1}-2v_{jkm}^{\sigma}+(v_{jk-1m}^{\sigma})^{s-1}}{\mathrm{Re}\Delta z^2},\end{aligned}$$

$$\frac{w^{\sigma}_{jkm}-w^{\sigma}_{jkm-1}}{\Delta t}+u_0\frac{w^{\sigma}_{jkm}-(S_1 w^{\sigma-1}_{jkm})^{s-1}-(S_2 w^{\sigma+1}_{jkm})^{s-1}}{\Delta x}$$
$$+v_0\frac{w^{\sigma}_{j+1km}-w^{\sigma}_{j-1km}}{2\Delta y}+w_0\frac{w^{\sigma}_{jkm}-(T_1 w^{\sigma}_{jk-1m})^{s-1}-(T_2 w^{\sigma}_{jk+1m})^{s-1}}{\Delta z}$$
$$+\frac{(p^{\sigma}_{jk+1m})^{s-1}-p^{\sigma}_{jkm}}{\Delta z}$$
$$=\frac{(w^{\sigma+1}_{jkm})^{s-1}-2w^{\sigma}_{jkm}+(w^{\sigma-1}_{jkm})^{s-1}}{\mathrm{Re}\Delta x^2}+\frac{w^{\sigma}_{j+1km}-2w^{\sigma}_{jkm}+w^{\sigma}_{j-1km}}{\mathrm{Re}\Delta y^2}$$
$$+\frac{(w^{\sigma}_{jk+1m})^{s-1}-2w^{\sigma}_{jkm}+(w^{\sigma}_{jk-1m})^{s-1}}{\mathrm{Re}\Delta z^2},$$
$$\frac{u^{\sigma}_{jkm}-(u^{\sigma-1}_{jkm})^{s-1}}{\Delta x}+\frac{v^{\sigma}_{j+1km}-v^{\sigma}_{j-1km}}{2\Delta y}+\frac{w^{\sigma}_{jkm}-(w^{\sigma}_{jk-1m})^{s-1}}{\Delta z}=0, \quad (9)$$

where s - a GI number and it is assumed that all unknown functions are related to the current s-th GI with the exception of those that have index $(s-1)$.

The scheme (9) differs from the general scheme in that the values of grid function with indexes $(\sigma-1)$ or $(k-1)$ have index $(s-1)$ (not s as in the general scheme). Therefore obtaining solution at transverse lines may be achieved for each GI by means of a parallel algorithm. However it is important to investigate whether this scheme converges in GI. Consider this question using Fourie method. Determine the vector of unknown grid functions in (9) (where all unknown values that have index $m-1$ are fixed) as follows

$$(\varphi^{\sigma}_{jkm})^s=(u_*,v_*,w_*,p_*)^T\lambda^s\exp(i(\alpha j+\gamma\sigma+\beta k)).$$

The matrix of the equations system for vector $(u_*,v_*,w_*,p_*)^T$ is (7), where the expression for P_2 is different. Namely in this case it does not contain the positive term r_t. Thus it is clear that we have the same three roots and each of them does not exceed one. It means that scheme (9) converges in GI unconditionally at each temporal layer.

5. FIRST COMPUTED RESULTS

It can be shown that results obtained above for a 3D case hold for a 2D case[4-6]. It is clear that parallel schemes for a case of the second order approximations of the marching derivatives and for the compressible Navier-Stokes equations can be formulated in a similar way.

The fact of convergence of the parallel schemes has been confirmed on solving test problems using conservative form of equations and the second order approximations of the marching derivatives (the incompressible flow over a back facing step and sub- and supersonic longitudinal flow over a circular cylinder of a limited size). These computations showed that the parallel scheme requires 5-20 per cent more computational time (to reach the problem solution) than the modified scheme (when computations are performed using one processor). This is consequence of using formula (4).

Estimation of speed-up shows following. The ratio of the time needed for parallel computations to the total computing time is more than 0.999 for the codes using suggested parallel schemes. This means that the speed-up achievable in an ideal case is more than 1000. In reality achievable speed-up depends on the quality of the code and on architecture of a supercomputer or cluster as well as on the number of the processor.

The test problem (2D Laval nozzle flow) is solved at supercomputer NEC SX-5. According to computations using 6 processors (that have shared memory) the speed-up differs from 6 by a fraction of a percent.

6. CONCLUSIONS

Efficiency of the parallel algorithm depends on the structure of the computational scheme. The structure of the schemes (5), (9) is such that the total volume of computations equals practically the sum of a large number (which is equal to the number of the inner transverse lines) of independent and equal parts. Therefore we can expect achieving high values of the speed-up by using large number of processors in shared memory architecture.

Parallel version of the IMM is universal and simple like the marching one. Therefore it can be used as a method for high performance computing in fluid mechanics.

ACKNOWLEDGEMENTS

This research has been supported by the President of Russia (grant SS-2259.2003.1) and the Ministry for Education (grant E02-4.0-133). We would like to acknowledge the European Representation of the firm NEC, Netherlands that kindly gave us the opportunity to use the supercomputer.

REFERENCES

[1] Polyansky, A.P. and Skurin, L.I., 1999, "Numerical Investigation of 3D Hydro-dynamical Problems Using Iterative Space-Marching Method," *St. Petersburg Univ. Mech. Bulletin*. No.2, pp.90-96.

[2] Skurin, L.I., 2000, "Iterative Space-Marching Method for Solving Fluid Mechanics Problems. Mathematical Modeling," *RAS*, Vol.12, No.6, pp.88 - 94 (in Russian).

[3] Skurin, L.I., 2000, "Iterative Space- Marching Method for Incompressible and Compressible Full Navier-Stokes Equations," N. Satofuka (Ed.), *Proceedings of the First International Conference on Computational Fluid Dynamics*, ICCFD, Kyoto, Japan, pp.319 - 324.

[4] Skurin, L.I., 2004, "Marching and Parallel Algorithms for Integration of Compressible and Incompressible Navier–Stokes Equations," St. Petersburg, St. Petersb. Univ., Russian, p.168. (in Russian)

[5] Skurin, L.I., 2003, "Parallel Version of the Iterative Space-Marching Method for Navier-Stokes Equations. Parallel CFD 2003," *Book of Abstracts*. Parallel Computational Fluid Dynamics, Moskow, Russia, pp.76-79.

[6] Skurin, L.I., 2003, "Parallel Technique for Navier-Stokes Equations Based on the Iterative Space-Marching Method," CD-ROM *Proceedings of the 7th U.S. National Congress on Computational Mechanics*, Albuquerque, New Mexico, USA, p.593.