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ON THE COMPLETE CONVERGENCE OF WEIGHTED SUMS FOR DEPENDENT RANDOM VARIABLES†

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ABSTRACT

Let $\{X_{ni} \mid 1 \leq i \leq n,\ n \geq 1\}$ be an array of rowwise negatively associated random variables. We in this paper discuss the conditions of $n^{-1/p} \max_{1\leq k\leq n} \sum_{i=1}^{k} X_{ni} \longrightarrow 0$ completely as $n \longrightarrow \infty$ for some $1 \leq p < 2$ under not necessarily identically distributed setting. As application, it is obtained that $n^{-1/p} \max_{1\leq k\leq n} \sum_{i=1}^{k} X_{ni} \longrightarrow 0$ completely as $n \longrightarrow \infty$ if and only if $E|X_{11}|^{2p} < \infty$ and $EX_{ni} = 0$ under identically distributed case such that the corresponding results on *i.i.d.* case are extended and the strong convergence for weighted sums of rowwise negatively associated arrays is also considered.

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1. INTRODUCTION

Let $\{X_n \mid n \geq 1\}$ be a sequence of random variables. Hsu and Robbins(1947) introduced the concept of complete convergence of $\{X_n \mid n \geq 1\}$. A sequence $\{X_n \mid n \geq 1\}$ of random variables converges to a constant c completely if $\sum_{n=1}^{\infty} P(|X_n -$

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$c| > \varepsilon) < \infty$ for all $\varepsilon > 0$. If $X_n \longrightarrow c$ completely, then the Borel-Cantelli lemma implies that $X_n \longrightarrow c$ almost sure, but the converse is not true in general.

Let $\{X_{nk}|\ 1 \le k \le n,\ n \ge 1\}$ be an array of random variables with $EX_{nk} = 0$ for all n and k. Many authors studied the complete convergence of $n^{-1/p}\sum_{k=1}^{n} X_{nk}$, which is defined

$$\sum_{n=1}^{\infty} P(|n^{-1/p}\sum_{k=1}^{n} X_{nk}| > \varepsilon) < \infty \quad \text{for each } \varepsilon > 0, \tag{1.1}$$

where $0 \le p < 2$.

In particular, Erdös(1949) showed that for an array of independent identically distribution (*i.i.d.*) random variables $\{X_{nk} \mid 1 \le k \le n,\ n \ge 1\}$, (1.1) holds if and only if $E|X_{11}|^{2p} < \infty$. Hu et al. (1986) showed that Erdös' result could be obtained by replacing the i.i.d. condition by the uniformly bounded condition. We recall that an array $\{X_{nk} \mid 1 \le k \le n,\ n \ge 1\}$ of random variables is said to be uniformly bounded by a random variable X if for all n and k and every real number $x > 0$,

$$P(|X_{nk}| > x) \le P(|X| > x). \tag{1.2}$$

Hu *et al.*(1989) had obtained the following result on complete convergence and they had established (1.3) for non identically random variables when no assumptions of independence between rows of the array is made.

THEOREM 1.1. *Let $\{X_{nk}|\ 1 \le k \le n,\ n \ge 1\}$ be an array of rowwise independent random variables with $EX_{nk} = 0$. Suppose that $\{X_{nk}|\ 1 \le k \le n,\ n \ge 1\}$ are uniformly bounded by some random variable X. If $E|X|^{2p} < \infty$ for some $1 \le p < 2$, then*

$$n^{-1/p}\sum_{k=1}^{n} X_{nk} \longrightarrow 0 \quad \textit{completely as } n \to \infty \tag{1.3}$$

if and only if $E|X_{11}|2p < \infty$.

In this paper, we discuss the strong law of large numbers for weighted sums of rowwise negatively associated random variables. The main purpose of this paper is to extend and generalize Theorem A to rowwise negatively associated random variables which satisfy suitable conditions, since independent and identically random variables are a special case of negatively associated random variables, and the strong convergence for rowwise negatively associated arrays is also considered.

A finite sequence of random variables $\{X_i \mid 1 \le i \le n\}$ is said to be negatively associated (NA) if for any two disjoint nonempty subsets A_1 and A_2 of $\{1, \cdots, n\}$ and f_1 and f_2 are any two coordinatewise nondecreasing functions,

$$\mathrm{Cov}\Big(f_1(X_i,\ i \in A_1),\ f_2(X_j,\ j \in A_2)\Big) \le 0, \tag{1.4}$$

whenever the covariance is finite. If for every $n \ge 2$, $X_1, \cdots, X_n$ are NA, then the sequence $\{X_i \mid i \in N\}$ is said to be NA. This definition is introduced by Alam and Saxena(1981). Many authors derived several important properties about NA sequences and also discussed some applications in the area of statistics, probability, reliability and multivariate analysis. Compared to positively associated random variables, the study of NA random variables has received less attention in the literature. Readers may refer to Karlin and Rinott(1980b), Ebrahimi and Ghosh(1981), Block *et al.*(1982), Newman and Wright(1982), Joag-Dev(1990), Joag-Dev and Proschan(1983), Matula(1992) and Roussas(1994) among others.

Recently, some authors focused on the problem of limiting behavior of partial sums of NA sequences. Su *et al.*(1996) derived some moment inequalities and weak convergence for NA sequence, Su and Qin(1997) studied some limiting results for NA sequences, Shao and Su (1999) discussed for law of the iterated logarithm, Liang and Su (1999), Liang (2000), and Baek *et al.*(2003) considered some complete convergence for weighted sums of NA sequences, some examples and applications.

2. PRELIMINARIES

This section will contain some background materials which will be used in obtaining the main results in the next section and $a = O(b)$ means that there exists some constant $C > 0$ such that $a \le Cb$, $a^+ = \max(0, a)$, $a^- = \max(0, -a)$.

LEMMA 2.1 (Joag and Proschan(1983)). *Let $A_1, \cdots, A_m$ be disjoint subsets of $\{1, \cdots, k\}$ and $f_1, f_2, \cdots, f_m$ be increasing positive functions. If $\{X_i \mid 1 \le i \le k\}$ is NA, then*

$$E\prod_{i=1}^{m} f_i(X_j, j \in A_i) \le \prod_{i=1}^{m} Ef_i(X_j, j \in A_i).$$

LEMMA 2.2 (Hu *et al.*(1986)). *For any $r \ge 1$, $E|X|^r < \infty$ if and only if*

$$\sum_{n=1}^{\infty} n^{r-1} P(|X| > \varepsilon n) < \infty \quad \textit{for any} \quad \varepsilon > 0.$$

More precisely,

$$r2^{-r}\sum_{n=1}^{\infty} n^{r-1}P(|X|>n) \ \le\ E|X|^r \ \le\ 1+r2^r\sum_{n=1}^{\infty} n^{r-1}P(|X|>n).$$

LEMMA 2.3 (Liang and Su(1999)). *Let $\{X_i \mid i \ge 1\}$ be a sequence of NA random variables and $\{a_{ni} \mid 1 \le i \le n,\ n \ge 1\}$ an array of real numbers. If $P(\max_{1\le j\le n} |a_{nj}X_j| > \varepsilon) < \delta$ for δ small enough and n large enough, then*

$$\sum_{j=1}^{n} P(|a_{nj}X_j|>\varepsilon) = O(1)P(\max_{1\le j\le n}|a_{nj}X_j|>\varepsilon)$$

for sufficient large n.

LEMMA 2.4 (Shao(2000)). *Let $\{X_i \mid i \ge 1\}$ be a sequence of NA random variables with $EX_i = 0$ and $E|X_i|^p < \infty$ for some $p \ge 1$. Then, there exists constant $C_p > 0$ and $D_p > 0$ such that*

$$E\max_{1\le k\le n}|\sum_{i=1}^{k} X_i|^p \le C_p\sum_{i=1}^{n} E|X_i|^p \quad \text{for } 1\le p<2,$$

$$E\max_{1\le k\le n}|\sum_{i=1}^{k} X_i|^p \le D_p\{(\sum_{i=1}^{n} EX_i^2)^{p/2}+\sum_{i=1}^{n} E|X_i|^p\} \quad \text{for } p\ge 2.$$

3. STRONG CONVERGENCE

THEOREM 3.1. *Let $\{X_{ni} \mid 1 \le i \le n,\ n \ge 1\}$ be an array of rowwise NA random variables such that $EX_{ni} = 0$. Suppose that there is a random variable X such that $P(|X_{ni}| > x) = O(1)P(|X| > x)$ for all $x \ge 0$. If $E|X|^{2p} < \infty$ for some $1 \le p < 2$, then*

$$n^{-1/p}\max_{1\le k\le n}\sum_{i=0}^{k} X_{ni} \to 0 \text{ completely as } n\to\infty. \tag{3.1}$$

THEOREM 3.2. *Let $1 \le p < 2$, and let $\{X_{ni} \mid 1 \le i \le n,\ n \ge 1\}$ be an array of rowwise NA random variables. Suppose that there is a random variable X such that $P(|X| > x) = O(1)P(|X_{ni}| > x)$ for all $x \ge 0$. Assume that (3.1) holds, then $E|X|^{2p} < \infty$ and $\lim_{n\to\infty}\left(\max_{1\le k\le n}\sum_{i=1}^{k} EX_{ni}/n^{1/p}\right) = 0$.*

COROLLARY 3.1. *Let $1 \le p < 2$, and let $\{X_{ni} \mid 1 \le i \le n,\ n \ge 1\}$ be an array of rowwise NA random variables with $EX_{ni} = 0$. Suppose that there is a random variable X such that $P(|X_{ni}| > x) = O(1)P(|X| > x)$ and $P(|X| > x) = O(1)P(|X_{ni}| > x)$ for all $x \ge 0$. Then the following two statements are equivalent:*

(1) $E|X|^{2p} < \infty$;

(2) $n^{-1/p} \max_{1 \le k \le n} \sum_{i=1}^{k} X_{ni} \to 0$ *completely as* $n \to \infty$.

COROLLARY 3.2. *Let $1 \le p < 2$, and let $\{X_{ni} \mid 1 \le i \le n,\ n \ge 1\}$ be an array of identically distributed rowwise NA random variables. Then the following two statements are equivalent:*

(1) $E|X_{11}|^{2p} < \infty$ *and* $EX_{ni} = 0$;

(2) $n^{-1/p} \max_{1 \le k \le n} \sum_{i=1}^{k} X_{ni} \to 0$ *completely as* $n \to \infty$.

THEOREM 3.3. *Let $\{X_{ni} \mid 1 \le i \le k_n,\ n \ge 1\}$ be a sequence of rowwise NA random variables such that $EX_{ni} = 0$. Suppose that there is a random variable X such that $P(|X_{ni}| > x) = O(1)P(|X| > x)$ for all $1 \le i \le k_n, n \ge 1$ and $x \ge 0$, which $\{k_n | n \ge 1\}$ is a sequence of positive integers. Assume that $\{a_{ni} | \ 1 \le i \le k_n,\ n \ge 1\}$ is an array of real numbers satisfying*

$$\max_{1 \le i \le k_n} |a_{ni}| = O\Big((\log n)^{-1}\Big) \tag{3.2}$$

$$\sum_{i=1}^{k_n} a_{ni}^2 = o\Big((\log n)^{-1}\Big) \tag{3.3}$$

If $Ee^{t|X|} < \infty$ for all $t > 0$, then

$$\sum_{n=1}^{\infty} n^{\beta} P\Big(|\sum_{i=1}^{k_n} a_{ni} X_{ni}| > \varepsilon\Big) < \infty \quad \textit{for all} \quad \varepsilon > 0 \quad \textit{and} \quad \beta \ge 0.$$

REMARK 3.1. *The following example shows that Theorem 3.3 does not hold if the condition (3.3) is replaced by the weaker condition (3.4)* $\sum_{i=1}^{k_n} a_{ni}^2 = O((\log n)^{-1})$.

EXAMPLE 3.1. Let $\{X,\ X_i,\ i \ge 1\}$ be a sequence of i.i.d. $N(0,1)$ random variables. Set

$$a_{ni} = \begin{cases} 1/[\log n], & \text{if } 1 \le i \le [\log n], \\ 0, & \text{if } [\log n] + 1 \le i \le n, \end{cases}$$

where $[x]$ denotes the integer part of x. Then the condition (3.2) of Theorem 3.3 and the above condition (3.4) are easily satisfied.

Note that $X \sim N(0,1)$, it follows that $Ee^{t|X|} \leq 2e^{t^2/2}$ for all $t > 0$. Since $\sum_{i=1}^{[\log n]} X_i/\sqrt{[\log n]} \sim N(0,1)$, we have by Lemma 5.1.1 in Stout(1974) that

$$P\Big(|\sum_{i=1}^{n} a_{ni}X_i| > 1\Big) = P\Big(|\sum_{i=1}^{[\log n]} X_i/\sqrt{[\log n]} > \sqrt{[\log n]}\Big)$$
$$\geq 2\exp\{-[\log n]\} \geq 2/n$$

for all sufficiently large n, and so

$$\sum_{n=1}^{\infty} n^{\beta} P\Big(|\sum_{i=1}^{n} a_{ni}X_i| > 1\Big) = \infty \quad \text{for all} \quad \beta \geq 0,$$

i.e. Theorem 3.3 does not hold.

COROLLARY 3.3. *Let $\{X_{ni}|\ 1 \leq i \leq k_n,\ n \geq 1\}$ be a sequence of rowwise NA random variables with $EX_{ni} = 0$. Suppose that there is a random variable X such that$P(|X_{ni}| > x) = O(1)P(|X| > x)$ for all $1 \leq i \leq k_n, n \geq 1$ and $x \geq 0$, which $\{k_n|n \geq 1\}$ is a sequence of positive integers. Assume that $\{b_{ni}|\ 1 \leq i \leq k_n,\ n \geq 1\}$ is an array of constants satisfying $\limsup_{n\to\infty} \sum_{i=1}^{k_n} b_{ni}^2 < \infty$. If $Ee^{t|X|} < \infty$ for all $t > 0$, then*

$$\sum_{n=1}^{\infty} n^{\beta} P\Big(\sum_{i=1}^{k_n} b_{ni}X_{ni} > \epsilon \log n\Big) < \infty \quad \textit{for any} \quad \varepsilon > 0.$$

4. PROOF OF THEOREMS

In order to prove Theorem 3.1, choosing $p < b < 2p$ and large integer $N > 0$ (to be specialized later), for all $\varepsilon > 0$, we define that for $1 \leq i \leq n,\ n \geq 1$,

$$Y_{ni}(1) = -n^{1/b}I(X_{ni} < -n^{1/b}) + X_{ni}I(|X_{ni}| \leq n^{1/b}) + n^{1/b}I(X_{ni} > n^{1/b}),$$
$$Y_{ni}(2) = (X_{ni} - n^{1/b})I(n^{1/b} < X_{ni} < \frac{\varepsilon}{N}n^{1/p})$$
$$Y_{ni}(3) = (X_{ni} + n^{1/b})I(-n^{1/b} > X_{ni} > -\frac{\varepsilon}{N}n^{1/p})$$

$$Y_{ni}(4) = (X_{ni} - n^{1/b})I(X_{ni} > \frac{\varepsilon}{N}n^{1/p}) + (X_{ni} + n^{1/b})I(X_{ni} < -\frac{\varepsilon}{N}n^{1/p})$$

$$S_{nk}(l) = \sum_{i=1}^{k} Y_{ni}(l), \ l = 1, 2, 3, 4.$$

Then $\max_{1\le k\le n} \sum_{i=1}^{k} X_{ni} = \max_{1\le k\le n} \sum_{l=1}^{4} S_{nk}(l)$, and we know that $\{Y_{ni}(1),\ 1 \le i \le n,\ n \ge 1\}$ is still a array of rowwise NA random variables by the definition of $Y_{ni}(1)$.

Therefore, to prove Theorem 3.1 it is sufficient to show that the following (4.1), (4.2) and (4.3) hold:

$$\sum_{n=1}^{\infty} P(\max_{1\le k\le n} |S_{nk}(l)| \ge \varepsilon n^{1/p}) < \infty, \quad l = 2, 3, 4 \tag{4.1}$$

$$n^{-1/p} \max_{1\le k\le n} |\sum_{i=1}^{k} EY_{ni}(1)| \longrightarrow 0 \tag{4.2}$$

$$\sum_{n=1}^{\infty} P\left(\max_{1\le k\le n} |\sum_{i=1}^{k}(Y_{ni}(1) - EY_{ni}(1))| \ge \varepsilon n^{1/p}\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{4.3}$$

The proofs of (4.1)-(4.3) can be found in the Lemmas 4.1-4.3 below.

LEMMA 4.1. *If $E|X|^{2p} < \infty$, then (4.1) holds:*

$$\sum_{n=1}^{\infty} P(\max_{1\le k\le n} |S_{nk}(2)| \ge \varepsilon n^{1/p}) < \infty \tag{4.4}$$

$$\sum_{n=1}^{\infty} P(\max_{1\le k\le n} |S_{nk}(3)| \ge \varepsilon n^{1/p}) < \infty \tag{4.5}$$

$$\sum_{n=1}^{\infty} P(\max_{1\le k\le n} |S_{nk}(4)| \ge \varepsilon n^{1/p}) < \infty. \tag{4.6}$$

PROOF. We observe from the definition of $Y_{ni}(2)$ that $Y_{ni}(2) \ge 0$. Hence, by using the NA property, we obtain that

$$\sum_{n=1}^{\infty} P(\max_{1\le k\le n} |S_{nk}(2)| \ge \varepsilon n^{1/p}) = \sum_{n=1}^{\infty} P(S_{nn}(2) \ge \varepsilon n^{1/p})$$

$$\le \sum_{n=1}^{\infty} P(\text{there are at least } N\ i's \text{ such that } Y_{ni}(2) \ne 0)$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{\infty} \sum_{1\leq i_1<i_2<\cdots<i_N\leq n} P(X_{ni_1} > n^{1/b}, \cdots, X_{ni_N} > n^{1/b}) \\
&\leq \sum_{n=1}^{\infty} \left[\sum_{i=1}^{n} P(|X_{ni}| > n^{1/b}) \right]^N \\
&\leq O(1) \sum_{n=1}^{\infty} n^{-(2p/b-1)N} (E|X|^{2p})^N < \infty
\end{aligned}
$$

by choosing large N such that $(2p/b-1)N > 1$. Thus, (4.4) is proved. Note that $Y_{ni}(3) \leq 0$, so similar to the arguments for (4.4), we can verify (4.5).

Finally, we prove (4.6). According to Lemma 2.2, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} P(\max_{1\leq k\leq n} |S_{nk}(4)| \geq \varepsilon n^{1/p}) \\
&\leq \sum_{n=1}^{\infty} \sum_{i=1}^{n} P(|X_{ni}| > \frac{\varepsilon}{N} n^{1/p}) \\
&= \sum_{n=1}^{\infty} O(1) n P(|X| > \frac{\varepsilon}{N} n^{1/p}) \\
&\leq O(1)\ E|X|^{2p} < \infty.
\end{aligned}
$$

□

LEMMA 4.2. *If* $E|X|^{2p} < \infty$ *for some* $1 \leq p < 2$ *and* $EX_{ni} = 0$, *then (4.2) holds.*

PROOF. Note that by $EX_{ni} = 0$, we have

$$
\begin{aligned}
&n^{-1/p} \max_{1\leq k\leq n} |\sum_{i=1}^{k} EY_{ni}(1)| \\
&\leq n^{-1/p} \sum_{i=1}^{n} E|X_{ni}|\ I(|X_{ni}| > n^{1/b}) + \sum_{i=1}^{n} P(|X_{ni}| > n^{1/b}) \\
&= I_1 + I_2 \quad \text{(say)}.
\end{aligned}
$$

By $E|X|^{2p} < \infty$ and noticing $p > 1,\ 2p/b > 1$, we have

$$
I_1 = n^{-1/p} \sum_{i=1}^{n} \left[n^{1/b} P(|X_{ni}| > n^{1/b}) + \int_{n^{1/b}}^{\infty} P(|X_{ni}| > x) dx \right]
$$

$$\begin{aligned}
&= O(1)n^{-(1/p-1/b)+1}P(|X|>n^{1/b})+O(1)\frac{n}{n^{1/p}}\int_{n^{1/b}}^{\infty}P(|X|>x)dx \\
&\le O(1)n^{-(1/p-1/b)-(2p/b-1)}E|X|^{2p}+O(1)\frac{n}{n^{1/p}}E|X|^{2p}\int_{n^{1/b}}^{\infty}x^{-2p}dx \\
&= O(1)n^{-(1/p-1/b)-(2p/b-1)}E|X|^{2p}+O(1)n^{-(1/p-1/b)-(2p/b-1)}E|X|^{2p} \\
&\longrightarrow 0.
\end{aligned}$$

As to I_2, we have

$$\begin{aligned}
I_2 &= \sum_{i=1}^{n}P(|X_{ni}|>n^{1/b}) \\
&= O(1)\ nP(|X|^{2p}>n^{2p/b}) \\
&\le O(1)n^{-(2p/b-1)}E|X|^{2p}\longrightarrow 0.
\end{aligned}$$

□

LEMMA 4.3. *If $E|X|^{2p}<\infty$, then (4.3) holds.*

PROOF. Let $q>2$, according to Lemma 2.4, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}P\Big(\max_{1\le k\le n}|\sum_{i=1}^{k}(Y_{ni}(1)-EY_{ni}(1))|\ge\varepsilon n^{1/p}\Big) \\
&\le O(1)\sum_{n=1}^{\infty}n^{-q/p}E\Big(\max_{1\le k\le n}|\sum_{i=1}^{k}(Y_{ni}(1)-EY_{ni}(1))|\Big)^{q} \\
&\le O(1)\sum_{n=1}^{\infty}n^{-q/p}\{\sum_{i=1}^{n}E|Y_{ni}|^{q}+(\sum_{i=1}^{n}E|Y_{ni}|^{2})^{q/2}\} \\
&= I_3+I_4\ \text{(say)}.
\end{aligned}$$

First, we prove that $I_3<\infty$. By assumptions, and noticing $1/p-1/b>0$, we obtain that

$$\begin{aligned}
I_3 &= O(1)\sum_{n=1}^{\infty}n^{-q/p}\sum_{i=1}^{n}\{E|X_{ni}|^{q}I(|X_{ni}|\le n^{1/b})+n^{q/b}P(|X_{ni}|\ge n^{1/b})\} \\
&\le O(1)\sum_{n=1}^{\infty}n^{1-q/p}\Big\{\int_0^{n^{1/b}}x^{q-1}P(|X|\ge x)dx+n^{q/b}P(|X|\ge n^{1/b})\Big\} \\
&\le O(1)\sum_{n=1}^{\infty}n^{1-q/p}\Big\{\int_0^{n^{1/b}}x^{q-1-2p}E|X|^{2p}dx+n^{q/b-2p/b}E|X|^{2p}\Big\} \\
&\le O(1)\sum_{n=1}^{\infty}n^{-(1/p-1/b)q-(2p/b-1)}E|X|^{2p}<\infty
\end{aligned}$$

by choosing large q.

Finally, note that $E|X|^{2p} < \infty$ implies $\sup_{n,i} E|X_{ni}|^2 < \infty$, further $\sup_{n,i} E|Y_{ni}(1)|^2 < \infty$. Hence, by choosing large q, we have

$$I_4 \leq O(1) \sum_{n=1}^{\infty} n^{-(1/p-1/2)q} < \infty.$$

□

PROOF OF THEOREM 3.2. Note that (3.1) implies that for any $\varepsilon > 0$,

$$P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}| \geq \varepsilon n^{1/p}) \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty$$

and

$$\sum_{n=1}^{\infty} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}| \geq \varepsilon n^{1/p}) < \infty. \tag{4.7}$$

Since $\max_{1\leq k\leq n} |X_{nk}| \leq 2\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}|$, according to Lemma 2.3, we obtain that

$$\sum_{i=1}^{n} P(|X_{ni}| \geq \varepsilon n^{1/p}) = O(1)P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_{ni}| \geq \varepsilon n^{1/p}),$$

which, together with (4.7) and assumptions, we have

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(|X| \geq \varepsilon n^{1/p}) < \infty,$$

i.e.

$$\sum_{n=1}^{\infty} nP(|X| \geq \varepsilon n^{1/p}) < \infty,$$

which is equivalent to $E|X|^{2p} < \infty$ by Lemma 2.2.

Now, under $E|X|^{2p} < \infty$, we obtain from Theorem 3.1 that

$$\sum_{n=1}^{\infty} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} (X_{ni} - EX_{ni})| \geq \varepsilon n^{1/p}) < \infty \quad \text{for any} \quad \varepsilon > 0. \tag{4.8}$$

(4.7) and (4.8) yield $\lim_{n\to\infty} \left(\max_{1\leq k\leq n} \sum_{i=1}^{k} EX_{ni}/n^{1/p} \right) = 0.$ □

PROOF OF THEOREM 3.3. Since $a_{ni} = a_{ni}^+ - a_{ni}^-$, it suffices to show that

$$\sum_{n=1}^{\infty} n^\beta P\Big(|\sum_{i=1}^{k_n} a_{ni}^+ X_{ni}| > \varepsilon\Big) < \infty \quad \text{for any} \quad \varepsilon > 0, \tag{4.9}$$

$$\sum_{n=1}^{\infty} n^\beta P\Big(|\sum_{i=1}^{k_n} a_{ni}^- X_{ni}| > \varepsilon\Big) < \infty \quad \text{for any} \quad \varepsilon > 0. \tag{4.10}$$

We prove only (4.9), the proof of (4.10) is analogous. To prove (4.9), we need only to prove that

$$\sum_{n=1}^{\infty} n^\beta P\Big(\sum_{i=1}^{k_n} a_{ni}^+ X_{ni} > \varepsilon\Big) < \infty \quad \text{for any} \quad \varepsilon > 0, \tag{4.11}$$

$$\sum_{n=1}^{\infty} n^\beta P\Big(\sum_{i=1}^{k_n} a_{ni}^- X_{ni} < -\varepsilon\Big) < \infty \quad \text{for any} \quad \varepsilon > 0. \tag{4.12}$$

We first prove (4.11). By the definition of NA random variables, we know that $\{a_{ni}^+ X_{ni}| \ 1 \le i \le k_n, \ n \ge 1\}$ is still an array of rowwise NA random variables. From an inequality $e^x \le 1 + x + \frac{1}{2}x^2 e^{|x|}$ for all $x \in R$, using Lemma 2.1, we obtain for $t = M \log n/\varepsilon$, where M is a large constant and will be specified later on,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^\beta P\Big(\sum_{i=1}^{k_n} a_{ni}^+ X_{ni} > \varepsilon\Big) \\
&\le \sum_{n=1}^{\infty} n^\beta e^{-\varepsilon t} E e^{t \sum_{i=1}^{k_n} a_{ni}^+ X_{ni}} \\
&\le \sum_{n=1}^{\infty} n^{\beta - M} \prod_{i=1}^{k_n} E e^{t a_{ni}^+ X_{ni}} \\
&\le \sum_{n=1}^{\infty} n^{\beta - M} \prod_{i=1}^{k_n} E\Big(1 + t a_{ni}^+ X_{ni} + \frac{1}{2} t^2 (a_{ni}^+)^2 X_{ni}^2 e^{t a_{ni}^+ |X_{ni}|}\Big) \\
&\ll \sum_{n=1}^{\infty} n^{\beta - M} \prod_{i=1}^{k_n} \Big(1 + C(\log n)^2 (a_{ni}^+)^2 E e^{(1+C)|X|}\Big) \\
&\le \sum_{n=1}^{\infty} n^{\beta - M + \varepsilon} < \infty
\end{aligned}$$

provided $M > (\beta + \varepsilon) + 1$, where C denote positive constant whose values are unimportant and may vary at different place. Thus (4.11) is proved.

By replacing X_{ni} by $-X_{ni}$ from the above statement and noticing $\{a_{ni}^{+}(-X_{ni})|$ $1 \le i \le k_n,\ n \ge 1\}$ is still an array of rowwise NA random variables, we know that

$$\sum_{n=1}^{\infty} n^{\beta} P\Big(\sum_{i=1}^{k_n} a_{ni}^{-} X_{ni} < -\varepsilon\Big) < \infty \quad \text{for any} \quad \varepsilon > 0.$$

□

PROOF OF COROLLARY 3.3. Let $a_{ni} = b_{ni}/\log n$. Then, we can obtain the result by Theorem 3.3. □

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