Journal of the Korean Statistical Society (2005), **34***: 1, pp 11–20*

ON THE RATIO $X/(X+Y)$ FOR WEIBULL AND LEVY DISTRIBUTIONS

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ABSTRACT

The distributional properties of $R = X/(X+Y)$ and related estimation procedures are derived when X and Y are independent and identically distributed according to the Weibull or Levy distribution. The work is of interest in biological and physical sciences, econometrics, engineering and ranking and selection.

AMS 2000 subject classifications. Primary 33C90; Secondary 62E99.
Keywords. Levy distribution, ratio of random variables, Weibull distribution.

1. INTRODUCTION

For given random variables X and Y, the distribution of the ratio $R = X/(X+Y)$ is of interest in biological and physical sciences, econometrics, engineering and ranking and selection. For example, ratios of normal random variables appear as sampling distributions in single equation models, in simultaneous equations models, as posterior distributions for parameters of regression models and as modeling distributions, especially in economics when demand models involve the indirect utility function. Other areas of applications include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, ionic current fluctuations in biological membranes, target to control precipitation in meteorology, and inventory ratios in economics. Another important area is the stress–strength model in the context of reliability. It describes the life of a component which has a random strength Y and is subjected to random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus,

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$\Pr(X < Y)$ is a measure of component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels.

The distribution of the ratio $R = X/(X+Y)$ has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Marsaglia (1965) and Korhonen and Narula (1989) for normal family, Press (1969) for Student's t family, Shcolnick (1985) for stable family, Hawkins and Han (1986) for non-central chi-squared family, Provost (1989) for gamma family, and Pham-Gia (2000) for beta family.

It seems that the distribution of $R = X/(X+Y)$ has not been studied for the Weibull case except for McCool (1991) where inference for $P(X < Y)$ is discussed. The Weibull distribution has popular applications in reliability and quality control. The distribution is often suitable where conditions of 'strict randomness' of the exponential distribution are not satisfied. It is also used as a tolerance distribution in the analysis of quantal response data. The aim of this paper is to derive distributional properties as well as estimation procedures relating to R for Weibull and Levy distributions. Sections 2 and 3 of the paper consider the Weibull case while the Levy case is considered in Sections 4 and 5.

2. DISTRIBUTION OF R FOR WEIBULL CASE

The Weibull distribution is widely used to model breaking strength of materials. The probability density function of a Weibull distribution (see Johnson *et al* (1995)) is given by

$$f(x) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} \exp\left\{-\left(\frac{x}{\beta}\right)^\alpha\right\} \tag{2.1}$$

for $0 < x < \infty$, where the shape parameter $\alpha > 0$ and the scale parameter $\beta > 0$. Let X and Y be independent Weibull random variables with scale parameters β_x and β_y, respectively, and common shape parameter α. To find the distribution of the ratio $R = X/(X+Y)$, let $S = X+Y$. Then the joint pdf of R and S can be obtained as

$$f_{R,S}(r,s) = \frac{\alpha^2}{\beta_x^\alpha \beta_y^\alpha} r^{\alpha-1}(1-r)^{\alpha-1} s^{2\alpha-1} \exp\left\{-\left(\frac{r^\alpha}{\beta_x^\alpha} + \frac{(1-r)^\alpha}{\beta_y^\alpha}\right) s^\alpha\right\}$$

for $0 < r < 1$ and $s > 0$. By standard integration, the marginal pdf of R can be obtained as

$$f_R(r) = \alpha\rho \frac{1}{r^2} \left(\frac{1-r}{r}\right)^{\alpha-1} \left\{1 + \rho \left(\frac{1-r}{r}\right)^{\alpha}\right\}^{-2} \tag{2.2}$$

for $0 < r < 1$, where $\rho = (\beta_x/\beta_y)^\alpha$. The cdf of R corresponding to (2.2) can be obtained as

$$F_R(r) = \left\{1 + \rho \left(\frac{1-r}{r}\right)^{\alpha}\right\}^{-1} \tag{2.3}$$

for $0 < r < 1$. By using the binomial expansion

$$(1+u)^{-k} = \sum_{l=0}^{\infty} \binom{-k}{l} u^l,$$

the kth moment of R can be expressed as

$$\begin{aligned}
E\left(R^k\right) &= \alpha\rho \int_0^1 r^{k-2} \left(\frac{1-r}{r}\right)^{\alpha-1} \left\{1 + \rho \left(\frac{1-r}{r}\right)^{\alpha}\right\}^{-2} dr \\
&= \alpha\rho \int_0^\infty \frac{y^{\alpha-1}}{(1+y)^k (1+\rho y^\alpha)^2} dy \\
&= \alpha\rho \left\{ \int_0^1 \frac{y^{\alpha-1}}{(1+y)^k (1+\rho y^\alpha)^2} dy \right. \\
&\qquad \left. + \int_1^\infty \frac{y^{\alpha-1}}{(1+y)^k (1+\rho y^\alpha)^2} dy \right\} \\
&= \alpha\rho \left\{ \int_0^1 \sum_{l=0}^{\infty} \binom{-k}{l} \frac{y^{\alpha+l-1}}{(1+\rho y^\alpha)^2} dy \right. \\
&\qquad \left. + \int_1^\infty \sum_{l=0}^{\infty} \binom{-k}{l} \frac{y^{\alpha-k-l-1}}{(1+\rho y^\alpha)^2} dy \right\} \\
&= \alpha\rho \sum_{l=0}^{\infty} \binom{-k}{l} \left\{ \int_0^1 \frac{y^{\alpha+l-1}}{(1+\rho y^\alpha)^2} dy + \int_1^\infty \frac{y^{\alpha-k-l-1}}{(1+\rho y^\alpha)^2} dy \right\} \\
&= \rho \sum_{l=0}^{\infty} \binom{-k}{l} \left\{ \int_0^1 \frac{v^{l/\alpha}}{(1+\rho v)^2} dv + \frac{1}{p^2} \int_0^1 \frac{w^{(k+l)/\alpha}}{(1+w/p)^2} dw \right\}
\end{aligned}$$

$$= \alpha\rho \sum_{l=0}^{\infty} \binom{-k}{l} \left\{ \frac{F\left(2, l/\alpha+1; l/\alpha+2; -p\right)}{l+\alpha} + \frac{F\left(2, (k+l)/\alpha+1; (k+l)/\alpha+2; -1/p\right)}{p^2\left(k+l+\alpha\right)} \right\}, \tag{2.4}$$

where the last step follows by using the integration identity

$$\int_0^u \frac{x^{\mu-1}}{(1+\beta x)^\nu} dx = \frac{u^\mu}{\mu} F\left(\nu, \mu; 1+\mu; -\beta u\right)$$

(see equation (3.194.1), Gradshteyn and Ryzhik, 1965) and, where

$$F\left(a_1, a_2; b; x\right) = \sum_{k=0}^{\infty} \frac{\left(a_1\right)_k \left(a_2\right)_k}{(b)_k} \frac{x^k}{k!}$$

denotes the Gauss hypergeometric function with $(c)_k = c(c+1)\cdots(c+k-1)$ denoting the ascending factorial. Because (2.4) is an infinite sum of Gauss hypergeometric functions (which are themselves infinite sums), further inferences for the Weibull case will be mathematically intractable. Because of this difficulty, we consider a particular case of the model given by (2.2) for $\alpha = 2$. The problem is that of the distribution of the ratio of two independent Rayleigh random variables. For $\alpha = 2$, the pdf and cdf in (2.2)–(2.3) reduce to

$$f_R(r) = 2\rho \frac{1}{r^2} \left(\frac{1-r}{r}\right) \left\{1 + \rho\left(\frac{1-r}{r}\right)^2\right\}^{-2} \tag{2.5}$$

and

$$F_R(r) = \frac{r^2}{r^2 + \rho(1-r)^2}, \tag{2.6}$$

respectively, for $0 < r < 1$. By standard integration, the first four moments can be obtained as

$$E\left(R\right) = \left[\sqrt{\rho}\left\{2\rho^{3/2} - \pi\rho + 2\sqrt{\rho}\ln\left(\rho\right) + \pi + 2\sqrt{\rho}\right\}\right] \Big/ \left[2(1+\rho)^2\right],$$

$$E\left(R^2\right) = \left[\rho\left\{\rho^2 - \ln\left(\rho\right) + 3\sqrt{\rho}\pi - \rho^{3/2}\pi - 3 - 2\rho + 3\rho\ln\left(\rho\right)\right\}\right] \Big/ \left[(1+\rho)^3\right],$$

$$E\left(R^3\right) = -\Big[\rho\Big\{18\rho^2 - 12\rho^2\ln(\rho) + 3\sqrt{\rho}\pi - 2\rho^3 - 18\rho^{3/2}\pi + 18\rho - 2 + 12\rho\ln(\rho) + 3\rho^{5/2}\pi\Big\}\Big] \Big/ \Big[2(1+\rho)^4\Big],$$

and

$$E\left(R^4\right) = -\Big[\rho\Big\{36\rho^2 + 62\rho^3 + 30\rho^{3/2}\pi - 3\rho^4 - 1 + 60\rho^2\ln(\rho) + 6\rho^{7/2}\pi - 6\rho\ln(\rho) - 30\rho^3\ln(\rho) - 30\rho - 60\rho^{5/2}\pi\Big\}\Big] \Big/ \Big[3(1+\rho)^5\Big].$$

3. ESTIMATION OF ρ IN THE RAYLEIGH CASE

Here, we consider estimation of the parameter ρ in (2.5). There are two avenues one could pursue. Firstly, suppose $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are mutually independent random samples from (??) with scale parameters β_x and β_y, respectively. Then, one possible estimator for ρ is $\hat{\rho} = (\hat{\beta}_x/\hat{\beta}_y)^2$, where $\hat{\beta}_x$ and $\hat{\beta}_y$ are the mles of β_x and β_y, respectively. Ali and Woo (2004a) have derived large sample properties of $\hat{\rho}$.

Sometimes, only data on $X/(X+Y)$ may be available. In these cases, the above method cannot be applied and thus one needs an alternative. Suppose $Z_1, Z_2, \ldots, Z_n$ is a random sample from (2.5). Then, it is easily seen that the maximum likelihood estimator (MLE) and the method of moments estimator (MME) of ρ are given by

$$\frac{1}{\hat{\rho}} = \frac{2}{n}\sum_{i=1}^{n} \frac{(1-z_i)^2}{z_i^2 + \hat{\rho}(1-z_i)^2}$$

and

$$\frac{\sqrt{\tilde{\rho}}\left\{2\tilde{\rho}^{3/2} - \pi\tilde{\rho} + 2\sqrt{\tilde{\rho}}\ln(\tilde{\rho}) + \pi + 2\sqrt{\tilde{\rho}}\right\}}{2(1+\tilde{\rho})^2} = \frac{1}{n}\sum_{i=1}^{n} Z_i,$$

respectively. We performed a simulation study of the behavior of these estimates by using the bisection method in Cheney and Kincaid (1994) and the representation

$$R = \left\{1 + \sqrt{\frac{1}{\rho}\left(\frac{1}{U} - 1\right)}\right\}^{-1},$$

TABLE 3.1 *Simulated averages and MSE's of $\hat{\rho}$ and $\tilde{\rho}$.*

ρ	n	Average Values		MSE	
		MME	MLE	MME	MLE
2	10	2.487902	2.232903	0.8137303	0.4777965
	15	2.446826	2.194685	0.7892309	0.4205423
	20	2.409786	2.119906	0.7621483	0.3989268
	25	2.340853	2.156713	0.7363174	0.3905537
	30	2.305748	2.101100	0.7348774	0.3639028
4	10	4.618302	4.415465	0.9348038	0.6171629
	15	4.590697	4.366880	0.9318193	0.5503754
	20	4.586234	4.286088	0.9231020	0.5439611
	25	4.5800958	4.314435	0.9159698	0.5066198
	30	4.569764	4.263897	0.9123034	0.4886514
8	10	8.491206	8.393456	0.2466729	0.2076182
	15	8.491089	8.372853	0.2466039	0.2035710
	20	8.487731	8.354048	0.2448812	0.1989511
	25	8.484030	8.337543	0.2442199	0.1925811
	30	8.478682	8.329282	0.2424818	0.1847957
10	10	10.499160	10.422820	0.2506164	0.2171517
	15	10.496260	10.390760	0.2497279	0.2106311
	20	10.488190	10.387750	0.2483347	0.2094065
	25	10.481400	10.364010	0.2373241	0.2027673
	30	10.476750	10.354840	0.2288843	0.1971724

(which follows from (2.6)), where U denotes a uniform random variable over $(0,1)$. Table 3.1 shows averages of the estimates $\hat{\rho}$ and $\tilde{\rho}$ (averages taken over 1000 values) and the corresponding mean squared errors for $n = 10(5)30$ and $\rho = 2, 4, 8$ and 10. We observe that the MLE has smaller mean squared errors compared to the MME. Also, MME appears to over-estimate ρ more than MLE does.

4. DISTRIBUTION OF R FOR LEVY CASE

The Levy distribution is a special case of the inverted gamma distribution (see O'Reilly and Rueda (1998)). It also arises as a limiting case of the inverse Gaussian distribution (see O'Reilly and Rueda (1998)). Formally, the pdf of the Levy distribution is given by

$$f(x;\sigma) = \sqrt{\frac{\sigma}{2\pi}} x^{-3/2} \exp\left(-\frac{\sigma}{2x}\right) \tag{4.1}$$

for $x > 0$ and $\sigma > 0$. This distribution does not have moments of all orders, but it has been useful in analysis of stock prices and has attracted applications in physics (Montroll and Shlesinger (1983)).

Let X and Y be independent Levy random variables with parameters σ_x and σ_y, respectively. To find the distribution of the ratio $R = X/(X+Y)$, let $S = X + Y$. Then the joint pdf of R and S can be obtained as

$$f_{R,S}(r,s) = \frac{\sqrt{\sigma_x\sigma_y}}{2\pi} s^{-2}(r(1-r))^{-3/2} \exp\left\{-\frac{1}{2s}\left(\frac{\sigma_x}{r} + \frac{\sigma_y}{1-r}\right)\right\}$$

for $0 < r < 1$ and $s > 0$. Standard integration shows that the marginal pdf and marginal cdf of R are

$$f_R(r) = \frac{\sqrt{\sigma_x\sigma_y}}{\pi}\frac{1}{r^2}\left(\frac{1-r}{r}\right)^{-1/2}\left(\sigma_y + \sigma_x\frac{1-r}{r}\right)^{-1} \tag{4.2}$$

and

$$F_R(r) = \frac{2}{\pi}\arcsin\left\{\left(1 + \frac{\sigma_x}{\sigma_y}\frac{1-r}{r}\right)^{-1/2}\right\}$$

for $0 < r < 1$. Furthermore, the kth moment of R can be calculated as follows: setting $y = 1/r - 1$, one can express

$$\begin{aligned} E\left(R^k\right) &= \frac{\sqrt{\sigma_x\sigma_y}}{\pi}\int_0^1 \frac{1}{r^2}\left(\frac{1-r}{r}\right)^{-1/2}\left(\sigma_y + \sigma_x\frac{1-r}{r}\right)^{-1} dr \\ &= \frac{1}{\pi}\sqrt{\frac{\sigma_y}{\sigma_x}}\int_0^\infty y^{-1/2}(1+y)^{-k}\left(y + \sigma_y/\sigma_x\right)^{-1} dy \\ &= \frac{1}{\pi}\sqrt{\frac{\sigma_y}{\sigma_x}} B\left(k+\frac{1}{2},\frac{1}{2}\right) F\left(1, k+\frac{1}{2}; k+1; 1-\frac{\sigma_y}{\sigma_x}\right), \end{aligned}$$

where the last step follows by application of the integration identity

$$\int_0^\infty x^{\lambda-1}(1+x)^{-\mu+\nu}(x+\beta)^{-\nu}dx = B(\mu-\lambda,\lambda)F(\nu,\mu-\lambda;\mu;1-\beta)$$

(see equation (3.197.9), Gradshteyn and Ryzhik, 1965). Using standard properties of the Gauss hypergeometric function, the first four moments of R can be expressed as

$$E(R) = \frac{\sqrt{\rho}\,(\sqrt{\rho}-1)}{\rho-1}, \tag{4.3}$$

$$E\left(R^2\right) = \frac{\sqrt{\rho}\left(2\rho^{3/2} - 3\rho + 1\right)}{2\left(\rho - 1\right)^2},$$

$$E\left(R^3\right) = \frac{\sqrt{\rho}\left(8\rho^{5/2} - 15\rho^2 + 10\rho - 3\right)}{8\left(\rho - 1\right)^3},$$

and

$$E\left(R^4\right) = \frac{\sqrt{\rho}\left(16\rho^{7/2} - 35\rho^3 + 35\rho^2 - 21\rho + 5\right)}{16\left(\rho - 1\right)^4}.$$

5. Estimation of ρ in Levy Case

As mentioned in Section 3, two methods for estimation of the parameter ρ in (4.2) are possible. They are:

1. If $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are mutually independent random samples from (4.1) with scale parameters σ_x and σ_y, respectively, then one possible estimator for ρ is $\hat{\rho} = \hat{\sigma}_x / \hat{\sigma}_y$, where $\hat{\sigma}_x$ and $\hat{\sigma}_y$ are the mles of σ_x and σ_y, respectively. Ali and Woo (2004b) have derived large sample properties of $\hat{\rho}$. This method cannot be applied when only data on $X/(X+Y)$ are available.

2. Since $T = \sqrt{\rho}/(1+\sqrt{\rho})$ is a monotone increasing function of ρ, inferences on ρ are equivalent to inferences on T (this means that probability statements about the estimator for ρ can be reexpressed as that of the estimator for T and vice versa – see McCool (1991)). If $Z_1, Z_2, \ldots, Z_n$ is a random sample from (4.2) then it is immediate from (4.3) that

$$\tilde{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$$

is an unbiased and MSE-consistent estimator of T with

$$\text{Var}\left(\tilde{Z}\right) = \frac{T(1-T)}{n}.$$

Moreover, it follows by the central limit and Slutsky's theorems that

$$\frac{\tilde{Z} - T}{\sqrt{\tilde{Z}(1-\tilde{Z})/n}}$$

has the limiting standard normal distribution. Hence, large sample inferences about T can be based on normality. For instance, a $100(1-\alpha)\%$ confidence interval for T is

$$\left(\tilde{z} - z_{\alpha/2}\sqrt{\frac{\tilde{z}(1-\tilde{z})}{n}}, \tilde{z} + z_{\alpha/2}\sqrt{\frac{\tilde{z}(1-\tilde{z})}{n}}\right),$$

where $z_{\alpha/2}$ denotes the $100(1-\alpha/2)\%$ percentile of the standard normal distribution.

ACKNOWLEDGEMENTS

The authors would like to thank the referees and the editor for carefully reading the paper and for their great help in improving the paper.

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