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Time-delayed State Estimator for Linear Systems with Unknown Inputs

Jaehyun Jin and Min-Jea Tahk

Abstract: This paper deals with the state estimation of linear time-invariant discrete systems with unknown inputs. The forward sequences of the output are treated as additional outputs. In this case, the rank condition for designing the unknown input estimator is relaxed. The gain for minimal estimation error variance is presented, and a numerical example is given to verify the proposed unknown input estimator.

Keywords: Rank condition relaxation, time-delayed estimation, unknown input estimator.

1. INTRODUCTION

The unknown input observer (UIO), a unique observer that can estimate the states of dynamic systems in the presence of unknown inputs, has been studied for three decades. After Wang et al. [1] proposed a reduced-order UIO structure and Kudva et al. [2] derived conditions for the existence of a stable UIO, a number of studies have followed. Detailed surveys were presented at [3,4].

Recently, a filtering problem for systems with unknown inputs and known noises has been receiving a great deal of attention. Focus has been placed on estimating the states with minimal error variance in the presence of unknown inputs. Chen and Patton [5] as well as Hou and Patton [6] presented optimal filters similar to the Kalman filter. The gain was determined to allow the state estimation error to have minimum variance. Darouach et al. [7,8] presented an optimal estimator and predictor filters.

In this paper, a design method of an unknown input estimator is dealt with. In particular, the rank condition for design and filter gains are explored. The unknown inputs must appear separately in the output space for unknown-input decoupled estimation. The condition is mathematically represented as a rank condition in which the rank of the unknown input matrix multiplied by the output matrix has to be equal to its own rank. The authors [9] showed that the rank condition could be relaxed by using forward sequences of the output (y_{k+1}, y_{k+2}, ...), signifying time-delayed estimation. In this paper, the authors extend the result of [9] to the filtering problem, and derive an optimal gain for minimal estimation error variance for linear time-invariant discrete systems.

2. STATE ESTIMATION WITH UNKNOWN INPUTS

Consider the following linear time-invariant discrete system with unknown inputs

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + Mf_k + v_k, \\ y_k &= Cx_k + w_k, \end{aligned} \tag{1}$$

where $x_k \in \Re^n$, $y_k \in \Re^l$, $u_k \in \Re^m$, and $f_k \in \Re^p$ are the state, output, input, and unknown input variables. $v_k \in \Re^n$ and $w_k \in \Re^l$ are independent zero mean white noise with covariance matrices Q_k and R_k. The matrices written at (1) are assumed to be known and have appropriate dimensions. The estimator based on an unknown input observer is

$$\begin{aligned} \hat{x}_{k+1} = A\hat{x}_k + Bu_k + K_0(y_k - \hat{y}_k) \\ + K_1(y_{k+1} - \hat{y}_{k+1}), \end{aligned} \tag{2}$$

where $\hat{y}_k = C\hat{x}_k$ and $\hat{y}_{k+1} = CA\hat{x}_k + CBu_k$. The state estimation error equation is written as

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\ &= [(I - K_1C)A - K_0C]e_k + (M - K_1CM)f_k \\ &\quad + (I - K_1C)v_k - K_0w_k - K_1w_{k+1}. \end{aligned} \tag{3}$$

If $M - K_1CM = 0$, the estimation error is decoupled from unknown inputs. There is a solution matrix K_1 if the following rank condition is satisfied [2,10,11]:

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$$\text{rank}\left(CM\right) = \text{rank}\begin{bmatrix} CM \\ M \end{bmatrix} = p. \tag{4}$$

The remaining part is a familiar form to the Kalman filter problem. If the error system matrix ($[(I - K_1C)A - K_0C]$) is stable, the expectation of the estimation error converges to zero. The condition of the stability is that the original system of (1) must have no unstable transmission zero.

The authors [9] demonstrated that the rank condition could be relaxed, i.e., even though the rank condition is not satisfied, an unknown input observer can be designed by using the output's forward sequences (y_{k+2}, y_{k+3}, ...) if the system inversion condition is satisfied. It was also shown that the forward sequences did not change the transmission zeros of the original system.

Here, the concept is adopted for the estimation problem of (1). Let's assume that the system of (1) does not satisfy the rank condition but satisfies the system inversion condition [10] as

$$\text{rank}(H_d) = \text{rank}(H_{d-1}) + p, \tag{5}$$

where

$$H_k \equiv \begin{bmatrix} CM & & 0 \\ \vdots & \ddots & \\ CA^{k-1}M & \dots & CM \end{bmatrix}. \tag{6}$$

The proposed unknown input estimator for this system is

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + \begin{bmatrix} K_0 & \cdots & K_d \end{bmatrix}\begin{bmatrix} y_k - \hat{y}_k \\ \vdots \\ y_{k+d} - \hat{y}_{k+d} \end{bmatrix}, \tag{7}$$

where

$$\begin{aligned}\begin{bmatrix} y_k \\ y_{k+1} \\ \vdots \\ y_{k+d} \end{bmatrix} &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^d \end{bmatrix} x_k + \begin{bmatrix} 0 & \cdots & 0 \\ CB & & 0 \\ \vdots & \ddots & \\ CA^{d-1}B & \cdots & CB \end{bmatrix}\begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+d-1} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & \cdots & 0 \\ CM & & 0 \\ \vdots & \ddots & \\ CA^{d-1}M & \cdots & CM \end{bmatrix}\begin{bmatrix} f_k \\ f_{k+1} \\ \vdots \\ f_{k+d-1} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & \cdots & 0 \\ C & \ddots & \vdots \\ \vdots & \ddots & 0 \\ CA^{d-1} & \cdots & C \end{bmatrix}\begin{bmatrix} v_k \\ v_{k+1} \\ \vdots \\ v_{k+d-1} \end{bmatrix} + \begin{bmatrix} w_k \\ w_{k+1} \\ \vdots \\ w_{k+d} \end{bmatrix} \\ &\equiv \overline{C}x_k + \overline{D}\overline{u}_k + \overline{N}\overline{f}_k + \overline{T}\overline{v}_k + \overline{w}_k,\end{aligned} \tag{8}$$

$$\begin{aligned}\hat{\overline{y}}_k = \begin{bmatrix} \hat{y}_k \\ \hat{y}_{k+1} \\ \vdots \\ \hat{y}_{k+d} \end{bmatrix} &= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^d \end{bmatrix}\hat{x}_k + \begin{bmatrix} 0 & \cdots & 0 \\ CB & & 0 \\ \vdots & \ddots & \\ CA^{d-1}B & \cdots & CB \end{bmatrix}\begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+d-1} \end{bmatrix} \\ &\equiv \overline{C}\hat{x}_k + \overline{D}\overline{u}_k.\end{aligned} \tag{9}$$

The estimation error is given as

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\ &= \left[\left(I - K_1C - K_2CA \cdots - K_dCA^{d-1}\right)A - K_0C\right]e_k \\ &+ \left(\begin{bmatrix} M & 0 & \cdots \end{bmatrix} - \begin{bmatrix} K_1 & \cdots \end{bmatrix}\begin{bmatrix} CM & & 0 \\ \vdots & \ddots & \\ CA^{d-1}M & \dots & CM \end{bmatrix}\right)\overline{f}_k \\ &+ \left(\begin{bmatrix} I & 0 & \cdots \end{bmatrix} - \begin{bmatrix} K_1 & \cdots \end{bmatrix}\begin{bmatrix} C & & 0 \\ \vdots & \ddots & \\ CA^{d-1} & \dots & C \end{bmatrix}\right)\overline{v}_k \\ &- \begin{bmatrix} K_0 & \cdots & K_d \end{bmatrix}\overline{w}_k. \end{aligned} \tag{10}$$

The estimation error can be decoupled from unknown inputs $\overline{f}_k$ if

$$\text{rank}\left(H_d\right) = \text{rank}\begin{bmatrix} \overline{M} \\ H_d \end{bmatrix}, \tag{11}$$

where $\overline{M} = \begin{bmatrix} M & 0 & \cdots \end{bmatrix}$. Then,

$$\begin{aligned}\text{rank}\begin{bmatrix} \overline{M} \\ H_d \end{bmatrix} &= \text{rank}\begin{bmatrix} M & 0 & 0 & 0 \\ CM & 0 & \cdots & 0 \\ CAM & CM & & 0 \\ \vdots & \vdots & \ddots & \\ CA^{d-1}M & CA^{d-2}M & \cdots & CM \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & CM & & 0 \\ \vdots & \vdots & \ddots & \\ 0 & CA^{d-2}M & \cdots & CM \end{bmatrix} \\ &= \text{rank}\left[\text{M}\right] + \text{rank}\left[\text{H}_{\text{d-1}}\right].\end{aligned} \tag{12}$$

(11) is equal to the system inversion condition of (5). Finally, the estimation error is decoupled from unknown inputs. The remaining element is

$$\begin{aligned} e_{k+1} &= x_{k+1} - \hat{x}_{k+1} \\ &= \left[\left(I - K_1C - K_2CA \cdots - K_dCA^{d-1}\right)A - K_0C\right] e_k \end{aligned}$$

$$+\left(\begin{bmatrix} I & 0 & \cdots \end{bmatrix} - \begin{bmatrix} K_1 & \cdots \end{bmatrix}\begin{bmatrix} C & & 0 \\ \vdots & \ddots & \\ CA^{d-1} & \ldots & C \end{bmatrix}\right)\bar{v}_k$$
$$-\begin{bmatrix} K_0 & \cdots & K_d \end{bmatrix}\bar{w}_k$$
$$\equiv \left[\bar{A} - K_0 C\right] e_k - K_0 w_k - \bar{K}\bar{w}_{k/1} + \bar{S}\bar{v}_k, \quad (13)$$

where $\bar{w}_{k/1} = \begin{bmatrix} w_{k+1} & \cdots & w_{k+d} \end{bmatrix}^T$.

The next step is to determine the gain matrix K_0. The variance of the state estimation error is a common performance index for this selection. If the estimation error covariance matrix P_k is defined as

$$P_k = E\left[e_k e_k^T\right], \quad (14)$$

the update of the covariance matrix is given as

$$\begin{aligned} P_{k+1} &= E\left[e_{k+1}e_{k+1}^T\right] \\ &= \left[\bar{A} - K_0 C\right] P_k \left[\bar{A} - K_0 C\right]^T + K_0 R_k K_0^T \\ &\quad + \bar{K}\bar{R}_k\bar{K}^T + \bar{S}\bar{Q}_k\bar{S}^T \\ &= \bar{A}P_k\bar{A}^T + \bar{K}\bar{R}_k\bar{K}^T + \bar{S}\bar{Q}_k\bar{S}^T \\ &\quad + \left[K_0 V - \bar{A}P_k C^T\right] V^{-1}\left[K_0 V - \bar{A}P_k C^T\right]^T \\ &\quad - \bar{A}P_k C^T V^{-1} C P_k \bar{A}_k^T, \end{aligned} \quad (15)$$

where $V = \left(CP_kC^T + R_k\right)$. If we select K_0 as

$$K_0 = \bar{A}P_kC^T\left(CP_kC^T + R_k\right)^{-1} \quad (16)$$

the covariance matrix of e_{k+1} may be minimum as in

$$\begin{aligned} P_{k+1} &= E\left[e_{k+1}e_{k+1}^T\right] \\ &= \bar{A}\left(P_k - P_kC^TV^{-1}CP_k\right)\bar{A}^T \\ &\quad + \bar{K}\bar{R}_k\bar{K}^T + \bar{S}\bar{Q}_k\bar{S}^T. \end{aligned} \quad (17)$$

The steady state value of P_k and the resulting gain are given as

$$P = \bar{A}\left(P - PC^TV^{-1}CP\right)\bar{A}^T + \overline{KRK}^T + \overline{SQS}^T, \quad (18)$$
$$K_0 = \bar{A}PC^T\left(CPC^T + R\right)^{-1}. \quad (19)$$

This is the final result of the derivation. In the next section, a numerical example is presented to illustrate the design of an unknown input estimator and to verify the derived results.

3. NUMERICAL EXAMPLE

Let's consider the following example system. Known inputs are omitted without loss of generality.

$$A = \begin{bmatrix} 0.1 & & & \\ & 0.2 & & \\ & & 0.3 & \\ & & & 0.9 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \end{bmatrix},$$
$$Q = 0.01 I_4, \quad R = 0.01 I_2 \quad (20)$$

In this case, $\text{rank}(CM) = 1 \neq \text{rank}(M) = 2$ and $\text{rank}(H_2) = 3 = \text{rank}(H_1) + \text{rank}(M)$. So we propose the following estimator:

$$\hat{x}_{k+1} = A\hat{x}_k + \begin{bmatrix} K_0 & K_1 & K_2 \end{bmatrix}\begin{bmatrix} y_k - \hat{y}_k \\ y_{k+1} - \hat{y}_{k+1} \\ y_{k+2} - \hat{y}_{k+2} \end{bmatrix}. \quad (21)$$

The gains $\begin{bmatrix} K_1 & K_2 \end{bmatrix}$ are determined from the null space of $\begin{bmatrix} \bar{M}^T & H_2^T \end{bmatrix}^T$ as

$$K_1 = \begin{bmatrix} 1 & 0 \\ 2/7 & 0 \\ 1 & 0 \\ 2/7 & 0 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0 & 0 \\ 0 & -10/7 \\ 0 & 0 \\ 0 & -10/7 \end{bmatrix}. \quad (22)$$

(13) is written as

$$e_{k+1} = \left[\begin{bmatrix} 0 & 0 & 0 & 0 \\ -3/7 & 9/7 & 3/7 & -9/7 \\ -1 & 0 & 1 & 0 \\ -3/7 & 2/7 & 3/7 & -2/7 \end{bmatrix} A - K_0 C\right] e_k$$
$$+\frac{1}{7}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 9 & 3 & -9 & -10 & 10 & 10 & -10 \\ -7 & 0 & 7 & 0 & 0 & 0 & 0 & 0 \\ -3 & 2 & 3 & -2 & -10 & 10 & 10 & -10 \end{bmatrix}\bar{v}_k \quad (23)$$
$$-K_0 w_k - \begin{bmatrix} K_1 & K_2 \end{bmatrix}\begin{bmatrix} w_{k+1} \\ w_{k+2} \end{bmatrix}.$$

The steady state values of P_k and K_0 are given as

$$P = \begin{bmatrix} 0.0101 & 0.0029 & 0.0100 & 0.0029 \\ 0.0029 & 0.2236 & 0.0099 & 0.1320 \\ 0.0100 & 0.0099 & 0.0318 & 0.0117 \\ 0.0029 & 0.1320 & 0.0117 & 0.1123 \end{bmatrix}, \quad (24)$$

$$K_0 = \begin{bmatrix} 0 & 0 \\ -0.0857 & -0.0514 \\ 0.1000 & 0.0600 \\ 0.0143 & 0.0086 \end{bmatrix}.$$

The eigenvalues of $(\bar{A} - K_0 C)$ are 0, 0, 0, and 0.3. The assumed unknown inputs are

$$f_1(t) = \sin(0.1t) + 0.2\sin(0.3t),$$
$$f_2(t) = \sin(0.05t) + 0.1\sin(0.5t), \tag{25}$$

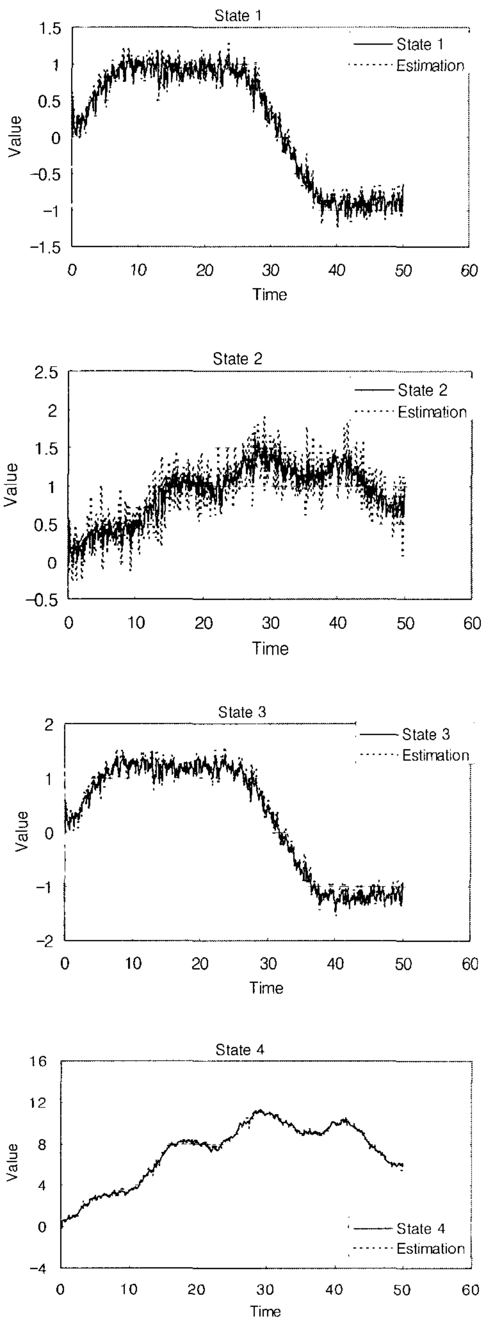

Fig. 1. States and their estimation.

and the states' initial conditions are 0.1, 0.1, 0.1, and 0.1. The time step size is set to 0.1. The following figures indicate the simulation results.

Discussion: The solid lines represent the states, and the dashed lines are their estimation. Actually, the states are delayed, i.e., we shifted the states left (backward) for comparison with estimation. This means that the current filter outputs are the estimation of the states at two steps prior (x_{k-2} vs. $\hat{x}_k$). The states are estimated satisfactorily. Even though the system does not satisfy the original rank condition for UIO design, the unknown inputs do not affect the state estimation. Also, we have obtained the minimum variance estimation. Unlike existing approaches, we can attain the minimum variance estimation decoupled from the unknown inputs if we relax the rank condition. The results verify the derivation of the paper.

4. CONCLUSION

In this paper, the authors have proposed a state estimator for linear discrete systems with unknown inputs. We have named it an unknown input estimator. Especially, we have relaxed the rank condition by augmenting the forward sequences of the outputs. We have illustrated that the system inversion condition is equal to the relaxed rank condition. Furthermore, the gain for the minimum variance of estimation error has been derived. The proposed method has been verified by a simulation study.

Unknown input observers are applicable to fault detection, disturbance decoupling, etc. We believe that the proposed design method would be highly beneficial in extending the applicability of unknown input estimators.

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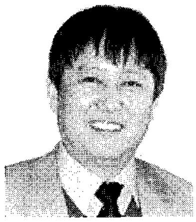

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