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A Multivariate Mixture of Linear Failure Rate Distribution in Reliability Models

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Abstract. This article provides a new class of multivariate linear failure rate distributions where every component is a mixture of linear failure rate distribution. The new class includes several multivariate and bivariate models including Marslall and Olkin type. The approach in this paper is based on the introducing a linear failure rate distributed latent random variable. The distribution of minimum in a competing risk model is discussed.

Key Words : *mixture of linear failure rate, competing risk models.*

1. INTRODUCTION

Multivariate lifetime data arise in many different types of industrial applications. For example, the measurements of the time to failure of machine components. The linear failure rate distribution is available family of life time distributions to model such life data interested in types of manufactured items and many other applications.

This paper presents in the reliability theory, since sometimes the failure rate occurs for more than one reason and the mixture distribution is a nice tool for modeling such situation.

The finite mixture distributions arise in a variety of applications from the length distribution of fish to the content of DNA in the nuclei of liver cells. The early development of this area was made by Karl Pearson (1894) since he published his well-known paper on estimating the five parameters in a mixture of two normal distributions. The most widely used finite mixture distribution are those involving

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normal components. Another general area where mixtures of distributions are important is in failure data and the observations are the times to failure of a sample of items. Often failure can occur for more than one reason and the failure distribution for each reason can be adequately approximated by simple density function such as the negative exponential (1952).

The reliability analysis and electronics widely use the univariate linear failure rate distribution, see for example Ahmad (2001), Pandey et al.(1993), Sarhan (1996), Zacks (1991) and El-Gohary (2004a, 2004b).

Several basic multivariate parametric families of distributions such as multivariate exponential, gamma and normal distributions, and shock models that give rise to them are considered by Barlow and Proschan (1981). Earlier, Marshall and Olkin (1967a)considered a shock model to derive a bivariate exponential distribution. Generalization of bivariate exponential distribution is proposed by Marshall and Olkin (1967b).

The model introduced in this paper is of some interest; in reliability theory, for example, sometimes failure may be occurred for more than one causes and a mixture distribution is a very nice tool for describing such situation.

The objective of this work is to introduce a new class of multivariate linear failure rate distributions, shortly (MLFRD). It is considered as a distribution of the life times of n- dependent components each has a univariate linear failure rate distribution. Also the mixture of bivariate linear rate distributions (BLFRD) is studied.

The paper is organized as follows. Section two presents the multivariate mixture of linear failure rate distributions with a latent random variable also linear failure rate. Section three introduces the mixture of bivariate linear failure rare distributions. Section four presents the joint moment generating function of mixture of bivariate linear failure rate distributions. Section five presents competing risk failure rate models. Finally, some properties of independence and bivariate dependence of the distribution are presented.

2. LINEAR FAILURE RATE MODEL

This section deals with the mixture of linear failure rate distributions and we derive a multivariate distribution where dependence among the components is characterized by a latent linear failure rate random variable independently distributed of the individual component. Also we develop a bivariate linear failure rate distribution with latent linear failure rate random variable as a special case.

2.1 The multivariate mixture model

We consider an n-component system where the lifetime of $i-th$ component, namely X_i has a mixture of linear failure rate distributions (LFRD), $i = 1, 2, \dots, n$. That is

$$X_i \sim \sum_{i=1}^{k} a_{ij}X_{ij},\ X_{ij} \sim LFRD(\alpha_{ij}, \beta_{ij}),\ j = 1, 2, \ldots, k \qquad (2.1)$$

where the nutation $LFRD(\alpha_{ij}, \beta_{ij})$ means a random variable, say X_{ij}, having a linear failure rate distribution with the parameters α_{ij}, β_{ij} and its density function is given as

$$f_{X_{ij}}(x) = (\alpha_{ij} + \beta_{ij}x)\bar{F}_{X_{ij}}(x),\ x \geq 0,\ \alpha_{ij} > 0,\ \beta_{ij} > 0, \qquad (2.2)$$

where $\bar{F}_{X_{ij}}(x)$ is the survival function of the random variable X_{ij} which is given by

$$\bar{F}_{X_{ij}}(x) = e^{-(\alpha_{ij}x + \frac{1}{2}\beta_{ij}x^2)},\ x \geq 0,\ \alpha_{ij} > 0,\ \beta_{ij} > 0, \qquad (2.3)$$

and $\vec{a_i} = (a_{i1}, \ldots, a_{ik})$ is the vector of mixing probabilities corresponding to $i-th$ component. That is $a_{ij} \geq 0$ for all i, j and $\sum_{j=1}^{k} a_{ij} = 1$.

Also we introduce a linear failure rate random variable, Z, with parameters α and β which is independent from X_{ij} for all i, j. The random variable Z will be used a latent variable to introduce dependence among $X_i's$. The density function of the latent variable Z is given by

$$f_Z(z) = (\alpha + \beta z)\bar{F}_Z(z),\ z \geq 0,\ \alpha > 0,\ \beta > 0. \qquad (2.4)$$

where the survival function $\bar{F}_Z(z)$ of Z is given by

$$\bar{F}_Z(z) = e^{-(\alpha z + \frac{1}{2}z^2)},\ z \geq 0,\ \alpha > 0,\ \beta > 0. \qquad (2.5)$$

Using the assumption of our model the latent variable Z is also independent of X_i for all $(i = 1, 2, \ldots, n)$.

Now, we define the vector of multivariate distribution $\vec{S} = (S_1, S_2, \ldots, S_n)$ where $S_i = \min(X_i, Z)$ for all $(i = 1, 2, \ldots, n)$ and obviously they are dependent as they commonly share the influence of the latent random variable Z. In what follows we introduce the joint of multivariate survival function of $S_1, S_2, \ldots, S_n$.

Corollary 2.1 The joint survival function of $S_1, S_2, \ldots, S_n$ is given by

$$\bar{F}(s_1, s_2, \ldots, s_n) = \bar{F}_Z(s_0) \prod_{i=1}^{n} \sum_{j=1}^{k} a_{ij}\bar{F}_{X_{ij}}(s_i), \qquad (2.6)$$

where

$$\bar{F}_{X_{ij}}(s_i) = e^{-(\alpha_{ij}s_i + \frac{1}{2}\beta_{ij}s_i^2)} \qquad (2.7)$$

and $s_0 = \max(s_1, s_2, \ldots, s_n) > 0$.

Proof. The survival function of $S_1, S_2, \ldots, S_n$ is defined by

$$\bar{F}(s_1, \ldots, s_n) = P(S_1 > s_1, \ldots, S_n > s_n)$$

Then using the definitions of S_i we get

$$\bar{F}(s_1,\ldots,s_n) = P(X_1 > s_1)P(X_2 > s_2)\ldots P(X_n > s_n)P(Z > s_0)$$

$$= e^{-(\alpha s_0+\beta s_0^2)} \prod_{i=1}^{n}\sum_{j=1}^{k} a_{ij} e^{-(\alpha_{ij}s_i+\frac{1}{2}\beta_{ij}s_i^2)} \tag{2.8}$$

One can write the above relation as given by (2.6), which completes the proof.

Note that the presence of s_0 makes it is very difficult to calculate the multivariate density function of $S_1, S_2, \ldots, S_n$ as we have to take mixed derivatives over all possible partitions of the sample space.

3. MIXTURE OF BIVARIATE LINEAR FAILURE RATE MODEL

In this section we will derived the mixture of bivariate linear failure rate distributions with latent variable also is a linear failure rate distribution. The bivariate linear failure rate distribution is discussed. Also, the mixture of bivariate exponential distributions may be derived as a special case from the mixture of bivariate linear rate.

3.1 Mixture of bivariate linear rate distributions

In this subsection we consider the case $n = k = 2$ for simplicity, that is under the plan of the bivariate two component mixture linear failure rate distributions. Then, from (2.6) it follows that the joint survival function of S_1 and S_2 will take the following form

$$\begin{aligned}\bar{F}(s_1,s_2) &= P(S_1 > s_1, S_2 > s_2) = P(X_1 > s_1)P(X_2 > s_2)P(Z > s_0)\\ &= \bar{F}_Z(s_0)\,\{p_{11}\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{21}}(s_2) + p_{12}\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{22}}(s_2)\\ &\quad +p_{21}\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{21}}(s_2) + p_{22}\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{22}}(s_2)\}\end{aligned} \tag{3.1}$$

where X_1 and X_2 have mixture of linear failure rate distributions that denoted by

$$\begin{aligned}X_1 &\sim [a_1 LFRD(\alpha_{11},\beta_{11}) + (1-a_1)LFRD(\alpha_{12},\beta_{12})],\\ X_1 &\sim [a_2 LFRD(\alpha_{21},\beta_{21}) + (1-a_2)LFRD(\alpha_{22},\beta_{22})],\end{aligned} \tag{3.2}$$

where $LFRD$ denotes the linear failure rate distribution, and

$$p_{ij} = a_1^{2-i}a_2^{2-j}(1-a_1)^{i-1}(1-a_2)^{j-1},\ \forall\, i,j \in \{1,2\}. \tag{3.3}$$

Form the relation (3.1) we can conclude that

1. For $i,j \in \{1,2\}$, $p_{ij} \geq 0$ and $p_{11} + p_{12} + p_{21} + p_{22} = 1$.
2. Every term of the right hand of $\bar{F}(s_1, s_2)$ which given by equation (3.1) has a survival function of a bivariate linear failure rate distributions.

This leads to the result that, the survival function given in (3.1) considers as the joint survival function of a mixture of four bivariate linear failure rate distributions.

Now the following Theorem gives the joint probability density function of S_1 and S_2.

Theorem 3.1 The joint pdf of S_1, S_2 say $f(s_1, s_2)$ is given by

$$f(s_1,s_2)=\begin{cases} f_1(s_1,s_2) & s_1>s_2 \\ f_2(s_1,s_2) & s_1<s_2 \\ f_0(s_0,s_0) & s_1=s_2=s_0 \end{cases} \tag{3.4}$$

where

$$\begin{aligned} f_1(s_1,s_2) = \bar{F}_Z(s_1)\{&p_{11}[\alpha+\alpha_{11}+(\beta+\beta_{11})s_1](\alpha_{21}+\beta_{21}s_2)\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{21}}(s_2) \\ &+p_{12}[\alpha+\alpha_{11}+(\beta+\beta_{11})s_1](\alpha_{22}+\beta_{22}s_2)\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{22}}(s_2) \\ &+p_{21}[\alpha+\alpha_{12}+(\beta+\beta_{12})s_1](\alpha_{21}+\beta_{21}s_2)\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{21}}(s_2) \\ &+p_{22}[\alpha+\alpha_{12}+(\beta+\beta_{12})s_1](\alpha_{22}+\beta_{22}s_2)\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{22}}(s_2)\}, \end{aligned}$$

$$\begin{aligned} f_2(s_1,s_2) = \bar{F}_Z(s_2)\{&p_{11}(\alpha+\alpha_{11}s_1)[\alpha+\alpha_{21}+(\beta+\beta_{21})s_2]\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{21}}(s_2) \\ &+p_{12}(\alpha_{11}+\beta_{11}s_1)[\alpha+\alpha_{22}+(\beta_{22}+\beta)s_2]\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{22}}(s_2) \\ &+p_{21}(\alpha_{12}+\beta_{12}s_1)[\alpha+\alpha_{21}+(\beta+\beta_{21})s_2]\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{21}}(s_2) \\ &+p_{22}(\alpha_{12}+\beta_{12}s_1)[\alpha+\alpha_{22}+(\beta+\beta_{22})s_2]\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{22}}(s_2)\} \end{aligned}$$

$$\begin{aligned} f_0(s_0,s_0) = (\alpha+\beta s_0)\bar{F}_Z(s_0)\{&p_{11}\bar{F}_{X_{11}}(s_0)\bar{F}_{X_{21}}(s_0)+p_{12}\bar{F}_{X_{11}}(s_0)\bar{F}_{X_{22}}(s_0) \\ &+p_{21}\bar{F}_{X_{12}}(s_0)\bar{F}_{X_{21}}(s_0)+p_{22}\bar{F}_{X_{12}}(s_0)\bar{F}_{X_{22}}(s_0)\} \end{aligned} \tag{3.5}$$

Proof. The forms of $f_1(s_1, s_2)$ and $f_2(s_1, s_2)$ can be obtained by differentiating the joint survival function $\bar{F}(s_1, s_2)$ with respect to s_1 and s_2. But the function $f_0(s_0, s_0)$ can not be derived in a similar method. In fact to derive the function $f_0(s_0, s_0)$ we will use the following identity

$$\int_0^\infty\int_0^{s_1} f_1(s_1,s_2)ds_2ds_1+\int_0^\infty\int_0^{s_2} f_2(s_1,s_2)ds_1ds_2+\int_0^\infty f_0(s_0,s_0)ds_0=1 \tag{3.6}$$

After obtaining the forms of $f_1(s_1, s_2)$ and $f_2(s_1, s_2)$ as given in (3.5), we will substitute into (3.6) and making some lengthy integral calculations we get

$$\begin{aligned} \int_0^\infty f_0(s_0,s_0)ds_0 = p_{11}&\int_0^\infty \{(\alpha+\beta s_0)\bar{F}_{X_{11}}(s_0)\bar{F}_{X_{21}}(s_0)\bar{F}_Z(s_0)\}\,ds_0 \\ +p_{12}&\int_0^\infty \{(\alpha+\beta s_0)\bar{F}_{X_{11}}(s_0)\bar{F}_{X_{22}}(s_0)\bar{F}_Z(s_0)\}\,ds_0 \\ +p_{21}&\int_0^\infty \{(\alpha+\beta s_0)\bar{F}_{X_{12}}(s_0)\bar{F}_{X_{21}}(s_0)\bar{F}_Z(s_0)\}\,ds_0 \\ +p_{22}&\int_0^\infty \{(\alpha+\beta s_0)\bar{F}_{X_{12}}(s_0)\bar{F}_{X_{22}}(s_0)\bar{F}_Z(s_0)\}\,ds_0 \end{aligned} \tag{3.7}$$

Hence we can easily obtained the function $f_0(s_0, s_0)$ as given by last equation of the system (3.5).

The following Corollary gives the marginal pdf's of S_1 and S_2.

Corollary 3.1 The marginal pdf's of S_1 and S_2 are given by

$$\begin{aligned} f_{S_1}(s_1) = \bar{F}_Z(s_1)\{a_1[\alpha + \alpha_{11} + (\beta + \beta_{11})s_1]\bar{F}_{X_{11}}(s_1) \\ +(1 - a_1)[\alpha + \alpha_{12} + (\beta + \beta_{12})s_1]\bar{F}_{X_{12}}(s_1)\}, \ s_1 > 0 \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} f_{S_2}(s_2) = \bar{F}_Z(s_1)\{a_2[\alpha + \alpha_{21} + (\beta + \beta_{21})s_2]\bar{F}_{X_{21}}(s_1) \\ +(1 - a_2)[\alpha + \alpha_{22} + (\beta + \beta_{22})s_2]\bar{F}_{X_{22}}(s_1)\}, \ s_2 > 0 \end{aligned} \tag{3.9}$$

The following Theorem provides an approach of obtaining the joint bivariate density when the component of the random variables can be equal with positive probability.

Theorem 3.2 If the bivariate survival function $\bar{F}(x, y)$ of X and Y takes the following form:

$$\bar{F}_{X,Y}(x, y) = \bar{F}_Z(z)\bar{F}_X(x)\bar{F}_Y(y), \text{ where } z = \max(x, y) \tag{3.10}$$

and

$$\bar{F}_X(x) = e^{-(\alpha_1 x + \frac{1}{2}\beta_1 x^2)}, \quad \bar{F}_Y(y) = e^{-(\alpha_2 y + \beta_2 y^2)}, \quad \bar{F}_Z(z) = e^{-(\alpha z + \beta z^2)}, \tag{3.11}$$

then the joint density function of X and Y is given by

$$f_{X,Y}(x, y) = \begin{cases} f_1(x, y) & x > y \\ f_2(x, y) & x < y \\ f_0(x, x) & x = y \end{cases} \tag{3.12}$$

where

$$\begin{aligned} f_1(x, y) &= (\alpha_2 + \beta_2 y)[\alpha + \alpha_1 + (\beta + \beta_1)x]\bar{F}_X(x)\bar{F}_Y(y)\bar{F}_Z(x) \\ f_2(x, y) &= (\alpha_1 + \beta_1 x)[\alpha + \alpha_2 + (\beta + \beta_2)y]\bar{F}_X(x)\bar{F}_Y(y)\bar{F}_Z(y) \\ f_0(x, x) &= (\alpha + \beta x)\bar{F}_X(x)\bar{F}_Y(x)\bar{F}_Z(x) \end{aligned} \tag{3.13}$$

Proof. The proof of this theorem is based on obtaining the forms of $f_1(x, y)$ and $f_2(x, y)$ by differentiating the joint survival $\bar{F}_{X,Y}(x, y)$ that given by (3.10) with respect to x and y twice times. While the function $f_0(x, x)$ will be obtained from the identity (3.6). Now, using the forms of $f_1(x, y)$ and $f_2(x, y)$ we have

$$\int_0^\infty \int_0^x f_1(x, y)dydx = 1 - \int_0^\infty [\alpha + \alpha_1 + (\beta + \beta_1)x]\bar{F}_X(x)\bar{F}_Y(x)\bar{F}_Z(x)dx, \tag{3.14}$$

and

$$\int_0^\infty \int_0^y f_2(x, y)dxdy = 1 - \int_0^\infty [\alpha + \alpha_2 + (\beta + \beta_2)y]\bar{F}_y(x)\bar{F}_Y(y)\bar{F}_Z(y)dy. \tag{3.15}$$

Substituting from (3.14) and (3.15) into (3.6) we get

$$\int_0^\infty \int_0^x f_1(x,y)dydx + \int_0^\infty \int_0^y f_2(x,y)dxdy = 1 - \int_0^\infty (\alpha + \beta x)\bar{F}_X(x)\bar{F}_Y(x)\bar{F}_x(x)dx \tag{3.16}$$

Comparing with the identity (3.6) with equation (3.16) we get

$$\int_0^\infty f_0(x,x)dx = \int_0^\infty (\alpha + \beta x)\bar{F}_X(x)\bar{F}_Y(x)\bar{F}_x(x)dx. \tag{3.17}$$

Thus, the function $f_0(x,x)$ is given by

$$f_0(x,x) = (\alpha + \beta x)\bar{F}_X(x)\bar{F}_Y(x)\bar{F}_x(x)dx, \ \ x > 0. \tag{3.18}$$

Corollary 3.2 The marginal pdf of X and Y are given by

$$f_X(x) = [\alpha + \alpha_1 + (\beta + \beta_1)x]\bar{F}_Z(x)\bar{F}_x(x), \ \ x > 0 \tag{3.19}$$

and

$$f_Y(y) = [\alpha + \alpha_2 + (\beta + \beta_2)y]\bar{F}_Z(y)\bar{F}_Y(y), \ y > 0. \tag{3.20}$$

Proof. The proof of this corollary can be done by integrating the joint pdf of (X,Y) with respect to x and y, respectively.

From (3.19) and (3.20) we find the marginal distributions of X and Y are also linear failure rate distributions.

Now, we consider the bivariate mixture of exponential distributions as a special case of the bivariate linear failure rate model.

3.2 Mixture of bivariate exponential distributions

In this subsection we will derived the bivariate mixture of exponential distributions as a special case of the mixture of bivariate linear failure rate model. The model of the mixture of bivariate exponential distribution can be obtained from (3.5)- (3.20) by simply putting $\beta_{ij} = 0$ and $\beta = 0$ for all i, j. Thus, the density function of S_1 and S_2 is given by

$$\begin{aligned} f_1(s_1,s_2) = \{&p_{11}\alpha_{21}(\alpha + \alpha_{11})\,\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{21}}(s_2) + p_{12}\alpha_{22}(\alpha + \alpha_{11})\,\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{22}}(s_2) + \\ &p_{21}\alpha_{21}(\alpha + \alpha_{12})\,\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{22}}(s_2) + p_{22}\alpha_{22}(\alpha + \alpha_{12})\,\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{21}}(s_2)\}\,\bar{F}_Z(s_1) \\ f_2(s_1,s_2) = \{&p_{11}\alpha_{11}(\alpha + \alpha_{21})\,\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{21}}(s_2) + p_{12}\alpha_{11}(\alpha + \alpha_{22})\,\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{22}}(s_2) + \\ &p_{21}\alpha_{12}(\alpha + \alpha_{21})\,\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{21}}(s_2) + p_{22}\alpha_{12}(\alpha + \alpha_{22})\,\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{22}}(s_2)\}\,\bar{F}_Z(s_2) \\ f_0(s_0,s_0) = \alpha\,\{&p_{11}\bar{F}_{X_{11}}(s_0)\bar{F}_{X_{21}}(s_0) + p_{12}\bar{F}_{X_{11}}(s_1)\bar{F}_{X_{22}}(s_2) + \\ &p_{21}\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{21}}(s_2) + p_{22}\bar{F}_{X_{12}}(s_1)\bar{F}_{X_{22}}(s_2)\}\,\bar{F}_Z(s_0) \end{aligned} \tag{3.21}$$

where the survival functions $\bar{F}_{ij}(s_i)$ and $\bar{F}_Z(z)$ are given by

$$\bar{F}_{X_{ij}}(s_i) = e^{-\alpha_{ij}s_i}, \ \ \bar{F}_Z(z) = e^{-\alpha z}, \ i,j = 1,2.$$

The marginal distributions of S_1 and S_2 for the exponential distribution take the for

$$f_{S_1}(s_1) = \{a_1(\alpha+\alpha_{11})\,\bar{F}_{X_{11}}(s_1)(1-a_1)(\alpha+\alpha_{12})\,\bar{F}_{X_{12}}(s_1)\}\,\bar{F}_Z(s_1),\ s_1 > 0$$

$$f_{S_2}(s_2) = \{a_2(\alpha+\alpha_{21})\,\bar{F}_{X_{21}}(s_2)(1-a_2)(\alpha+\alpha_{22})\,\bar{F}_{X_{22}}(s_1)\}\,\bar{F}_Z(s_2),\ s_2 > 0. \tag{3.22}$$

Next, we will obtained the moments generating function of the mixture of bivariate linear failure rate distributions.

4. THE MOMENT GENERATING FUNCTIONS AND EXPECTATIONS

In this section we will obtained the joint moment generating function of S_1 and S_2 and univariate moment generating functions of S_1 and S_2. Also we will obtained the expectations of S_i, S_i^2, $(i = 1,2)$ and S_1S_2.

Theorem 4.1 The joint moment generating function of S_1, S_2, say $M_{S_1,S_2}(t_1,t_2)$ is given by

$$\begin{aligned}
M_{S_1,S_2}(t_1,t_2) = 1 + \sqrt{\pi}\,t_1 & \left\{ \frac{a_1}{\sqrt{2(\beta+\beta_{11})}}\left[1-\Phi\left(\frac{\alpha+\alpha_{11}}{\sqrt{2(\beta+\beta_{11})}}\right)\right] e^{\frac{(\alpha+\alpha_{11}-t_1)^2}{2(\beta+\beta_{11})}} + \frac{1-a_1}{\sqrt{2(\beta+\beta_{12})}}\left[1-\right.\right. \\
& \left.\left.\Phi\left(\frac{\alpha+\alpha_{12}}{\sqrt{2(\beta+\beta_{12})}}\right)\right] e^{\frac{(\alpha+\alpha_{12}-t_1)^2}{2(\beta+\beta_{12})}} \right\} + \sqrt{\pi}\,t_2 \left\{ \frac{a_2}{\sqrt{2(\beta+\beta_{21})}}\left[1-\Phi\left(\frac{\alpha+\alpha_{21}}{\sqrt{2(\beta+\beta_{21})}}\right)\right] \times \right. \\
& \left. e^{\frac{(\alpha+\alpha_{21}-t_1)^2}{2(\beta+\beta_{21})}} + \frac{1-a_2}{\sqrt{2(\beta+\beta_{22})}}\left[1-\Phi\left(\frac{\alpha+\alpha_{22}}{\sqrt{2(\beta+\beta_{22})}}\right)\right] e^{\frac{(\alpha+\alpha_{22}-t_1)^2}{2(\beta+\beta_{22})}} \right\} + t_1t_2\sqrt{\pi}\,\times \\
& \left\{ \frac{a_1}{\sqrt{2(\beta+\beta_{11})}}\left[a_2I_1^{(11)}(t_1,t_2) + (1-a_2)I_1^{(12)}(t_1,t_2)\right] e^{\frac{(\alpha+\alpha_{11}-t_1)^2}{2(\beta+\beta_{11})}} + \frac{1-a_1}{\sqrt{2(\beta+\beta_{12})}} \times \right. \\
& \left[a_2I_1^{(21)}(t_1,t_2) + (1-a_2)I_1^{(22)}(t_1,t_2)\right] e^{\frac{(\alpha+\alpha_{12}-t_1)^2}{2(\beta+\beta_{12})}} + \frac{a_2}{\sqrt{2(\beta+\beta_{21})}}\left[a_1I_2^{(11)}(t_1,t_2)\right. \\
& \left. +(1-a_1)I_2^{(12)}(t_1,t_2)\right] e^{\frac{(\alpha+\alpha_{21}-t_1)^2}{2(\beta+\beta_{21})}} + \frac{1-a_2}{2(\beta+\beta_{22})}\left[a_1I_2^{(12)}(t_1,t_2) + (1-a_1)\times\right. \\
& \left.\left. I_2^{(22)}(t_1,t_2)\right] e^{\frac{(\alpha+\alpha_{22}-t_1)^2}{2(\beta+\beta_{22})}} \right\},
\end{aligned} \tag{4.1}$$

where

$$I_1^{(11)}(t_1,t_2)=\int_0^\infty\left\{1-\Phi\Big[\frac{\alpha_{11}+\alpha-t_1}{\sqrt{2(\beta+\beta_{11})}}+\sqrt{\frac{(\beta+\beta_{11})}{2}}\,u\Big]\right\}\,e^{-[(\alpha_{21}-t_2)u+\frac{1}{2}\beta_{21}u^2]}du\,, \tag{4.2}$$

$$I_2^{(11)}(t_1,t_2)=\int_0^\infty\left\{1-\Phi\Big[\frac{\alpha_{21}+\alpha-t_2}{\sqrt{2(\beta+\beta_{21})}}+\sqrt{\frac{(\beta+\beta_{21})}{2}}\,u\Big]\right\}\,e^{-[(\alpha_{11}-t_1)u+\frac{1}{2}\beta_{11}u^2]}du\,, \tag{4.3}$$

and

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du.$$

The integrals $I_1^{(12)},I_2^{(12)},I_1^{(21)},I_2^{(21)}$ and $I_1^{(22)},I_2^{(22)}$ can be obtained from (4.2)(4.3), by replacing $\alpha_{11},\alpha_{21},\beta_{11},\beta_{21}$ with $\alpha_{11},\alpha_{12},\beta_{11},\beta_{12}$; $\alpha_{11},\alpha_{22},\beta_{11},\beta_{22}$;and α_{12},α_{21}, β_{12},β_{21}.

Proof. The moment generating $M_{S_1,S_2}(s_1,s_2)$ is given as

$$M_{S_1,S_2}(s_1,s_2)=E[e^{(t_1S_1+t_2S_2)}]=\int_0^\infty\int_0^{s_1}e^{(t_1s_1+t_2s_2)}f_1(s_1,s_2)ds_2ds_1$$

$$+\int_0^\infty\int_0^{s_2}e^{(t_1s_1+t_2s_2)}f_1(s_1,s_2)ds_1ds_2+\int_0^\infty e^{[(t_1+t_2)s_0]}f_0(s_0,s_0)ds_0. \tag{4.4}$$

Then substituting (3.5) into (4.4) and after lengthy algebraic manipulation we can derive the moment $M_{S_1,S_2}(t_1,t_2)$ as by (4.1) that completes the proof.

Corollary 4.1. The moment generating functions of S_1 and S_2 are given by:

$$\begin{aligned}
M_{S_1}(t_1)&=a_1\left\{1+t_1\sqrt{\tfrac{\pi}{2(\beta+\beta_{11})}}e^{\left(\frac{\alpha_{11}+\alpha-t_1}{\sqrt{2(\beta+\beta_{11})}}\right)^2}\left[1-\Phi\left(\tfrac{\alpha_{11}+\alpha-t_1}{\sqrt{2(\beta+\beta_{11})}}\right)\right]\right\}\\
&\quad+(1-a_1)\left\{1+t_1\sqrt{\tfrac{\pi}{2(\beta+\beta_{12})}}e^{\left(\frac{\alpha_{12}+\alpha-t_1}{\sqrt{2(\beta+\beta_{12})}}\right)^2}\left[1-\Phi\left(\tfrac{\alpha_{12}+\alpha-t_1}{\sqrt{2(\beta+\beta_{12})}}\right)\right]\right\}\\
M_{S_2}(t_2)&=a_2\left\{1+t_2\sqrt{\tfrac{\pi}{2\beta_{21}}}e^{\left(\frac{\alpha_{21}+\theta_0-t_2}{\sqrt{2\beta_{21}}}\right)^2}\left[1-\Phi\left(\tfrac{\alpha_{21}+\theta_0-t_2}{\sqrt{2\beta_{21}}}\right)\right]\right\}\\
&\quad+(1-a_2)\left\{1+t_2\sqrt{\tfrac{\pi}{2\beta_{22}}}e^{\left(\frac{\alpha_{22}+\theta_0-t_2}{\sqrt{2\beta_{22}}}\right)^2}\left[1-\Phi\left(\tfrac{\alpha_{22}+\theta_0-t_2}{\sqrt{2\beta_{22}}}\right)\right]\right\}
\end{aligned} \tag{4.5}$$

Proof. This Corollary can be proved by using the joint moment generating function

(4.1) or by using the marginal pdf's (3.8) and (3.9) of S_1 and S_2 respectively.

Corollary 4.2 The expectations of S_i, S_i^2, $(i=1,2)$ and S_1S_2 are given by :

$$\begin{aligned} E[S_1] = &\ a_1\sqrt{\tfrac{\pi}{2(\beta+\beta_{11})}}\left\{1-\Phi\left(\tfrac{\alpha_{11}+\alpha}{\sqrt{2(\beta+\beta_{11})}}\right)\right\}\exp\left\{\tfrac{(\alpha_{11}+\alpha)^2}{2(\beta+\beta_{11})}\right\}\\ &+(1-a_1)\sqrt{\tfrac{\pi}{2(\beta+\beta_{12})}}\left\{1-\Phi\left(\tfrac{\alpha_{12}+\alpha}{\sqrt{2(\beta+\beta_{12})}}\right)\right\}\exp\left\{\tfrac{(\alpha_{12}+\alpha)^2}{2(\beta+\beta_{12})}\right\}, \end{aligned} \tag{4.6}$$

$$\begin{aligned} E[S_2] = &\ a_2\sqrt{\tfrac{\pi}{2(\beta+\beta_{21})}}\left\{1-\Phi\left(\tfrac{\alpha_{21}+\alpha}{\sqrt{2(\beta+\beta_{21})}}\right)\right\}\exp\left\{\tfrac{(\alpha_{21}+\alpha)^2}{2(\beta+\beta_{21})}\right\}\\ &+(1-a_2)\sqrt{\tfrac{\pi}{2(\beta+\beta_{22})}}\left\{1-\Phi\left(\tfrac{\alpha_{22}+\alpha}{\sqrt{2(\beta+\beta_{22})}}\right)\right\}\exp\left\{\tfrac{(\alpha_{22}+\alpha)^2}{2(\beta+\beta_{22})}\right\}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} E[S_1^2] = &\ a_1\left\{\tfrac{2}{\beta+\beta_{11}}+\tfrac{\sqrt{2\pi}\,(\alpha_{11}+\alpha)}{\beta+\beta_{11}\sqrt{\beta+\beta_{11}}}\left[1-\Phi\left(\tfrac{\alpha_{11}+\alpha}{\sqrt{2(\beta+\beta_{11})}}\right)\right]\exp\left\{\tfrac{(\alpha_{11}+\alpha)^2}{2(\beta+\beta_{11})}\right\}\right\}\\ &+(1-a_1)\left\{\tfrac{2}{\beta+\beta_{12}}+\tfrac{\sqrt{2\pi}\,(\alpha_{12}+\alpha)}{\beta+\beta_{12}\sqrt{\beta+\beta_{12}}}\left[1-\Phi\left(\tfrac{\alpha_{12}+\alpha}{\sqrt{2(\beta+\beta_{12})}}\right)\right]\exp\left\{\tfrac{(\alpha_{12}+\alpha)^2}{2(\beta+\beta_{12})}\right\}\right\} \end{aligned} \tag{4.8}$$

$$\begin{aligned} E[S_2^2] = &\ a_2\left\{\tfrac{2}{\beta+\beta_{21}}+\tfrac{\sqrt{2\pi}\,(\alpha_{21}+\alpha)}{\beta+\beta_{21}\sqrt{\beta+\beta_{21}}}\left[1-\Phi\left(\tfrac{\alpha_{21}+\alpha}{\sqrt{2(\beta+\beta_{21})}}\right)\right]\exp\left\{\tfrac{(\alpha_{21}+\alpha)^2}{2(\beta+\beta_{21})}\right\}\right\}\\ &+(1-a_2)\left\{\tfrac{2}{\beta+\beta_{22}}+\tfrac{\sqrt{2\pi}\,(\alpha_{22}+\alpha)}{\beta+\beta_{22}\sqrt{\beta+\beta_{22}}}\left[1-\Phi\left(\tfrac{\alpha_{22}+\alpha}{\sqrt{2(\beta+\beta_{22})}}\right)\right]\exp\left\{\tfrac{(\alpha_{22}+\alpha)^2}{2(\beta+\beta_{22})}\right\}\right\} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} E[S_1S_2] = &\sqrt{\tfrac{\pi}{2}}\Big\{\tfrac{1}{\beta+\beta_{11}}\Big[p_{11}I_1^{(11)}(0,0)+p_{12}I_1^{(12)}(0,0)\Big]e^{\frac{(\alpha_{11}+\alpha)^2}{2(\beta+\beta_{11})}}+\tfrac{1}{\beta+\beta_{12}}\Big[p_{21}I_1^{(21)}(0,0)\\ &+p_{22}I_1^{(22)}(0,0)\Big]e^{\frac{(\alpha_{12}+\alpha)^2}{2(\beta+\beta_{12})}}+\tfrac{1}{\beta+\beta_{21}}\Big[p_{11}I_2^{(11)}(0,0)+p_{21}I_2^{(21)}(0,0)\Big]e^{\frac{(\alpha_{21}+\alpha)^2}{2(\beta+\beta_{21})}}\\ &+\tfrac{1}{\beta+\beta_{22}}\Big[p_{12}I_2^{(12)}(0,0)+p_{22}I_2^{(22)}(0,0)\Big]e^{\frac{(\alpha_{22}+\alpha)^2}{2(\beta+\beta_{22})}}\Big\}. \end{aligned} \tag{4.10}$$

Proof. This Corollary can be proved by using the joint moment function (4.1).

Finally, the mixture of bivariate linear failure rate distributions with exponential latent random variable and its properties can be obtained as a special case from the derived results by setting $\beta=0$.

The moment generating function of bivariate exponential can be obtained from (4.1) by setting $\beta_{ij}=0$ for all i,j and $\beta=0$. That is

$$\begin{aligned} M_{S_1,S_2}(s_1,s_2) = &\ \frac{p_{11}}{\alpha+\alpha_{11}+\alpha_{21}-t_1-t_2}\left[\frac{\alpha_{11}(\alpha+\alpha_{21})}{\alpha+\alpha_{21}-t_2}+\frac{\alpha_{21}(\alpha+\alpha_{11})}{\alpha+\alpha_{11}-t_1}+\alpha\right]\\ &+\frac{p_{12}}{\alpha+\alpha_{11}+\alpha_{22}-t_1-t_2}\left[\frac{\alpha_{11}(\alpha+\alpha_{22})}{\alpha+\alpha_{22}-t_2}+\frac{\alpha_{22}(\alpha+\alpha_{11})}{\alpha+\alpha_{22}-t_1}+\alpha\right] \end{aligned}$$

$$+\frac{p_{21}}{\alpha+\alpha_{12}+\alpha_{21}-t_1-t_2}\Big[\frac{\alpha_{12}(\alpha+\alpha_{21})}{\alpha+\alpha_{12}-t_2}+\frac{\alpha_{21}(\alpha+\alpha_{12})}{\alpha+\alpha_{12}-t_1}+\alpha\Big]$$
$$+\frac{p_{22}}{\alpha+\alpha_{12}+\alpha_{22}-t_1-t_2}\Big[\frac{\alpha_{12}(\alpha+\alpha_{22})}{\alpha+\alpha_{22}-t_2}+\frac{\alpha_{22}(\alpha+\alpha_{12})}{\alpha+\alpha_{12}-t_1}+\alpha\Big] \tag{4.11}$$

Also, we can obtained the moment generating functions of S_1 and S_2 from (4.5) by setting $\beta_{ij}=0$ for all i,j and $\beta=0$.

5. COMPETING RISK LINEAR FAILURE RATE MODEL

In this section we propose a linear failure rate competing risk models. These models arise in situation in which fail of the components is due to several different causes. In such situations every system failure is caused by only of the competing risks. In the present work we consider each competing risk has a mixture of linear failure rate and also the latent variable has a linear failure rate distribution.

Now we develop the distribution of the minimum. Assume that the random variable X_i be a mixture of $X_{i1}, X_{i2}, \ldots, X_{ik}$ each of them has a linear failure rare distribution with parameters $(\alpha_{ij}, \beta_{ij})$ and the mixing probability are $a_{i1}, a_{i2}, \ldots, a_{ik}$, that is $\sum_{j=}^{k} a_{ij} = 1 \forall i = 1, 2, \ldots, n$. Now, consider one latent random variable Z with linear failure rate distribution with parameters α, β which is independent of $X_1, X_2, \ldots, X_n$. Also we define the lifetime of the system T as

$$T = \min(X_1, X_2, \ldots, X_n, Z) \tag{5.1}$$

Thus

$$\begin{aligned} P(T>t) &= P(\min(X_1, X_2, \ldots, X_n, Z) > t) \\ &= P(X_1>t)P(X_2>t)\ldots P(X_n>t)P(Z)>t) \\ &= e^{-(\alpha t+\frac{1}{2}\beta t^2)} \prod_{i=1}^{n} \sum_{j=1}^{k} a_{ij} e^{-(\alpha_{ij}t+\frac{1}{2}\beta_{ij}t^2)} \\ &= \prod_{i=1}^{n} \sum_{j=1}^{k} a_{ij} e^{-[(\frac{\alpha}{n}+\alpha_{ij})t+\frac{1}{2}(\frac{\beta}{n}+\beta_{ij}t^2)]} = \prod_{i=1}^{n} P(T_i>t) \end{aligned} \tag{5.2}$$

where T_i is mixture of linear failure rate distributions with parameters $(\frac{\alpha}{n}+\alpha_{ij}, \frac{\beta}{n}+\beta_{ij})$ and mixing probability $a_{i1}, a_{2k}, \ldots, a_{ik}$.

Thus, we find that

$$\bar{F}_T(t) = \prod_{i=1}^{n} F_{T_i}(t). \tag{5.3}$$

Therefore, the probability density function of T is given by

$$f_t(t) = \frac{d}{dt}P(T>t) = -P(T>t)\sum_{i=1}^{n}\frac{1}{P(T_i>t)}\frac{d}{dt}P(T_i>t) = \sum_{i=1}^{n}\frac{P(T>t)}{P(T_i>t)}f_{T_i}(t) \tag{5.4}$$

and hazard rate function of is given by

$$h_T(t) = \frac{f_T(t)}{\bar{F}_T(t)} = \sum_{i=1}^{n} h_{T_i}(t). \tag{5.5}$$

The average of the mean time to failure of the system is defined

$$E(T) = \int_0^\infty t f_T(t)dt = -\int_0^\infty t\frac{d}{dt}\bar{F}_T(t)dt = \int_0^\infty \bar{F}_T(t)dt \tag{5.6}$$

$$= \int_0^\infty \prod_{i=1}^{n} e^{-H_{T_i}(t)}dt = \int_0^\infty e^{-\sum_{i=1}^{n} H_{T_i}(t)}dt, \tag{5.7}$$

where $H_{T_i}(t)$ is the integrated hazard rate function of T_i, $(i = 1, \ldots, n)$.

5.1 Mixture failure linear rate model

In this subsection we will develop the distribution of the minimum and the corresponding hazard function for the mixture of linear failure rate distributions.

Now we define $S = \min(S_1, \ldots, S_n)$, then the survival function of S is given by

$$\bar{F}(s) = P(S > s) = P\{\min(S_1, \ldots, S_n) > s\} = P\{S_1 > s, S_2 > s, \ldots, S_r > s_n\}$$

$$= P(Z > s)\prod_{i=1}^{n} P(X_i > s) = \prod_{i=1}^{n}\sum_{j=1}^{k} a_{ij}e^{-(\alpha_{ij}s_i + \frac{1}{2}\beta_{ij}s_i^2 + \frac{\alpha}{n}s + \frac{\beta}{2n}s^2)} = \prod_{i=1}^{n} P(T_i > s)$$

$$= \prod_{i=1}^{n} \bar{F}_{T_i}(s)$$

where $f_{T_i}(t) = \sum_{j=1}^{k} a_{ij}[\alpha_{ij} + \frac{\alpha}{n} + (\frac{\beta}{n} + \beta_{ij})t]e^{-(\alpha_{ij}t + \frac{1}{2}\beta_{ij}t^2 + \frac{\alpha}{n}s + \frac{\beta}{2n}s^2)}$, $t > 0$. Hence, the density function of S takes the form

$$f_T(s) = \bar{F}(t)\sum_{i=1}^{n}\frac{f_{T_i}(s)}{\bar{F}_{T_i}(s)}. \tag{5.8}$$

Thus the hazard function corresponding to S reduces to

$$h_S(s) = \sum_{i=1}^{n} h_{T_i}(s)$$

where

$$h_{T_i}(s) = \frac{\sum_{j=1}^{k} a_{ij}[\alpha_{ij} + \frac{\alpha}{n} + (\frac{\beta}{n} + \beta_{ij})s]e^{-(\alpha_{ij}s + \frac{1}{2}\beta_{ij}s^2 + \frac{\alpha}{n}s + \frac{\beta}{2n}s^2)}}{\sum_{j=1}^{k} a_{ij}e^{-(\alpha_{ij}s + \frac{1}{2}\beta_{ij}s_i^2 + \frac{\alpha}{n}s + \frac{\beta}{2n}s^2)}} \tag{5.9}$$

is the hazard of T_i, $i = 1, \ldots, n$.

In what follows we will obtained some of special cases, the first special case occurs when $\beta_{ij} = 0$ for all i, j and $k = 2$ and $n = 2$. This case represents a mixture of bivariate exponential distributions with linear failure rate latent random variable. Then the density function of T_1 and T_2 are given by

$$f_{T_1}(t) = a_1(\alpha_{11} + \tfrac{\alpha}{2} + \tfrac{\beta}{2}t)e^{-[(\alpha_{11}+\alpha)t+\frac{\beta}{2}t^2]} + (1 - a_1)(\alpha_{12} + \tfrac{\alpha}{2} + \tfrac{\beta}{2}t)e^{-[(\alpha_{12}+\alpha)t+\frac{\beta}{2}t^2]}$$
$$f_{T_2}(t) = a_2(\alpha_{21} + \tfrac{\alpha}{2} + \tfrac{\beta}{2}t)e^{-[(\alpha_{11}+\alpha)t+\frac{\beta}{2}t^2]} + (1 - a_2)(\alpha_{22} + \tfrac{\alpha}{2} + \tfrac{\beta}{2}t)e^{-[(\alpha_{22}+\alpha)t+\frac{\beta}{2}t^2]} \quad (5.10)$$

and the survival function of T is given by

$$\begin{aligned}\bar{F}_T(t) = p_{11}e^{-[(\alpha_{11}+\alpha_{12}+\alpha)t+\frac{\beta}{2}t^2]} + p_{12}e^{-[(\alpha_{11}+\alpha_{22}+\alpha)t+\frac{\beta}{2}t^2]} + \\ p_{21}e^{-[(\alpha_{12}+\alpha_{21}+\alpha)t+\frac{\beta}{2}t^2]} + p_{22}e^{-[(\alpha_{12}+\alpha_{22}+\alpha)t+\frac{\beta}{2}t^2]}.\end{aligned} \quad (5.11)$$

The second special case when $\beta_{ij} = 0$ for all i, j and $\beta = 0$. This case represents a mixture of bivariate exponential distributions with exponential latent random variable. In this case when $k = 2$ and $n = 2$ the density functions of T_1, T_2 and the survival function of T can be obtained from (5.10) and (5.11) by putting $\beta = 0$.

The Marshal-Olkin model can be obtained as a special case from the last model by putting $a_1 = a_2 = 1$, that is $p_{11} = 1$ and $p_{ij} = 0$ for all $i, j \neq 1$ and $\alpha_{11} = \alpha_1, \alpha_{21} = \alpha_2$ the joint pdf of S_1 and S_2 is given by

$$f(s_1, s_2) = \begin{cases} \alpha_2(\alpha + \alpha_1)e^{-(\alpha_1 s_1+\alpha_2 s_2+\alpha s_1)}, & s_1 > s_2 \\ \alpha_1(\alpha + \alpha_2)e^{-(\alpha_1 s_1+\alpha_2 s_2+\alpha s_2)}, & s_1 < s_2 \\ \alpha e^{-(\alpha_1+\alpha_2+\alpha)s_0}, & s_1 = s_2 = s_0. \end{cases} \quad (5.12)$$

It follows that the joint moment generating function of S_1 and S_2 is given by

$$M_{S_1,S_2}(s_1, s_2) = \frac{1}{\alpha + \alpha_1 + \alpha_2 - t_1 - t_2}\left[\alpha + \frac{\alpha_1(\alpha + \alpha_2)}{\alpha + \alpha_1 - t_1} + \frac{\alpha_2(\alpha + \alpha_1)}{\alpha + \alpha_2 - t_2}\right]. \quad (5.13)$$

It is very easy to calculate the covariance and correlation of S_1 and S_2.

Now, under the conditions of Lee (1979), we have $k = 1$ and $n = 2$. Thus the joint survival and density functions are given by

$$\bar{F}_{S_1,S_2}(s_1, s_2) = \bar{F}_Z(s_0)\prod_{i=1}^{2}\bar{F}_{S_i}(s_i), \quad (5.14)$$

where

$$\bar{F}_{S_i}(s_i) = e^{-(\alpha_i s_i+\frac{1}{2}\beta_i s_i^2)}, \text{ and } \bar{F}_Z(s_0) = e^{-(\alpha s_0+\frac{1}{2}s_0^2)}, \ i = 1, 2 \quad (5.15)$$

and

$$f_{S_1,S_2}(s_1, s_2) = \begin{cases} (\alpha_2 + \beta_2 s_2)[\alpha + \alpha_1 + (\beta + \beta_1)s_1]\bar{F}_{S_1}(s_1)\bar{F}_{S_2}(s_2)\bar{F}_Z(s_1), & s_1 > s_2 > 0 \\ (\alpha_1 + \beta_1 s_1)[\alpha + \alpha_2 + (\beta + \beta_2)s_2]\bar{F}_{S_1}(s_1)\bar{F}_{S_2}(s_2)\bar{F}_Z(s_2), & s_2 > s_1 > 0 \\ (\alpha + \beta s_0)\bar{F}_{S_1}(s_0)\bar{F}_{S_2}(s_0)\bar{F}_Z(s_0), & s_1 = s_2 = s_0 > 0 \end{cases} \quad (5.16)$$

Thus, the marginal pdf of the minimum is

$$\begin{aligned}f_S(s) &= p_{11}[\alpha+\alpha_{11}+\alpha_{21}+(\beta+\beta_{11}+\beta_{21})s]e^{-[(\alpha+\alpha_{11}+\alpha_{21})s+\frac{1}{2}(\beta+\beta_{11}+\beta_{21})s^2]}\\ &+p_{12}[\alpha+\alpha_{11}+\alpha_{22}+(\beta+\beta_{11}+\beta_{22})s]e^{-[(\alpha+\alpha_{11}+\alpha_{22})s+\frac{1}{2}(\beta+\beta_{11}+\beta_{22})s^2]}\\ &+p_{21}[\alpha+\alpha_{12}+\alpha_{21}+(\beta+\beta_{12}+\beta_{21})s]e^{-[(\alpha+\alpha_{12}+\alpha_{21})s+\frac{1}{2}(\beta+\beta_{12}+\beta_{21})s^2]}\\ &+p_{22}[\alpha+\alpha_{12}+\alpha_{22}+(\beta+\beta_{12}+\beta_{22})s]e^{-[(\alpha+\alpha_{12}+\alpha_{22})s+\frac{1}{2}(\beta+\beta_{12}+\beta_{22})s^2]},\\ &s>0\end{aligned} \tag{5.17}$$

Now, the moment generating function of the minimum is

$$\begin{aligned}M_S(t) = \int_0^\infty e^{st} f_S(s)ds = 1+\sqrt{\pi}t\Bigg\{ & p_{11}\left\{\frac{e^{\frac{\alpha+\alpha_{11}+\alpha_{21}-t}{2(\beta+\beta_{11}+\beta_{21})}}}{\sqrt{2(\beta+\beta_{11}+\beta_{21})}}\left[1-\Phi\left(\frac{\alpha+\alpha_{11}+\alpha_{21}-t}{2(\beta+\beta_{11}+\beta_{21})}\right)\right]\right\}\\ &+p_{12}\left\{\frac{e^{\frac{\alpha+\alpha_{11}+\alpha_{22}-t}{2(\beta+\beta_{11}+\beta_{22})}}}{\sqrt{2(\beta+\beta_{11}+\beta_{22})}}\left[1-\Phi\left(\frac{\alpha+\alpha_{11}+\alpha_{22}}{2(\beta+\beta_{11}+\beta_{22})}\right)\right]\right\}\\ &+p_{21}\left\{\frac{e^{\frac{\alpha+\alpha_{12}+\alpha_{21}-t}{2(\beta+\beta_{12}+\beta_{21})}}}{\sqrt{2(\beta+\beta_{12}+\beta_{21})}}\left[1-\Phi\left(\frac{\alpha+\alpha_{12}+\alpha_{21}}{2(\beta+\beta_{12}+\beta_{21})}\right)\right]\right\}\\ &+p_{22}\left\{\frac{e^{\frac{\alpha+\alpha_{12}+\alpha_{22}-t}{2(\beta+\beta_{12}+\beta_{22})}}}{\sqrt{2(\beta+\beta_{12}+\beta_{22})}}\left[1-\Phi\left(\frac{\alpha+\alpha_{12}+\alpha_{22}}{2(\beta+\beta_{12}+\beta_{22})}\right)\right]\right\}\Bigg\}\end{aligned} \tag{5.18}$$

Using the form (5.18) one can derive the expectation of the minimum distribution and the expected residual life of the system.

6. CONCLUSION

The class of models developed in this paper has many different applications in industrial and medical field. In this paper we present a new class of multivariate linear failure rate distributions. The obtained class includes multivariate and bivariate models including Marslall and Olkin type. The approach in this paper is based on the introducing a linear failure rate distributed latent random variable. The distribution of minimum in a competing risk reliability model is discussed.

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