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The Tunnel Number One Knot with Bridge Number Three is a $(1, 1)$-knot

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Abstract. We call K a $(1, 1)$-knot in M if M is a union of two solid tori V_1 and V_2 glued along their boundary tori ∂V_1 and ∂V_2 and if K intersects each solid torus V_i in a trivial arc t_i for $i = 1$ and 2. Note that every $(1, 1)$-knot is a tunnel number one knot. In this article, we determine when a tunnel number one knot is a $(1, 1)$-knot. In other words, we show that any tunnel number one knot with bridge number 3 is a $(1, 1)$-knot.

1. Preliminaries

Let K be a knot in the 3-sphere, $N(K)$ the regular neighborhood of K and $E(K)$ the exterior of K. By tunnel number $t(K)$, we mean the minimum number of mutually disjoint arcs properly embedded in $E(K)$ such that the complementary space of a regular neighborhood of the arcs is a handlebody. We call the family of arcs satisfying this condition an unknotting tunnel system for K. In particular, we call it an unknotting tunnel if the system consists of a single arc.

Let M be a closed orientable 3-manifold, and K a knot in M. We say that K admits a (g, b)-decomposition if there is a genus g Heegaard splitting (V_1, V_2) of M such that K intersects V_i $(i = 1, 2)$ in a b-string trivial arc system. Occasionally, it is called a g-genus b-bridge knot or a (g, b)-knot for short.

Let K be a knot in the 3-sphere which admits a (g, b)-decomposition, then by taking the g cores of a handlebody of the Heegaard splitting together with $b - 1$ arcs connecting the b-string trivial arcs, we see that the knot K has at most tunnel number $g+b-1$. Hence if a knot K in the 3-sphere admits a $(1, 1)$-decomposition(or is a $(1, 1)$-knot), then we have $t(K) \leq 1$. Note that a $(1, 1)$-knot is a tunnel number one knot, but the converse is not true. In fact, Morimoto, Sakuma and Yokota showed that there is a tunnel number one knot which is not a $(1, 1)$-knot(see [15]). There are many papers on $(1, 1)$-knots. See [4], [5], [6], [7], [9], [10], [13], and [14].

For two knots K and K', the connected sum of them is denoted by $K\#K'$. Concerning a relation between the tunnel number and a connected sum of knots, we have the following basic inequality: $t(K_1\#K_2) \leq t(K_1) + t(K_2) + 1$ for any two knots K_1 and K_2. For a long time, it was asked if this estimate is the best

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possible, and if the tunnel number of knots can degenerate under a connected sum. Concerning the latter question, in [11], Morimoto showed that there are infinitely many tunnel number two knots K such that $t(K\#K') = 2$ for any 2-bridge knot K'. This shows that the tunnel numbers of knots can degenerate by 1. In [15], Morimoto, Sakuma and Yokota constructed more concrete examples of pairs of tunnel number one knots which go up under the connected sum. More generally, in [8], Moriah and Rubinstein showed that for any positive integers t_1 and t_2, there are infinitely many pairs of knots K_1 and K_2 such that $t(K_1) = t_1$, $t(K_2) = t_2$ and $t(K_1\#K_2) = t(K_1) + t(K_2) + 1$.

Since every tunnel number one knot is prime([16]), we have that $2 \leq t(K_1\#K_2) \leq 3$ for any tunnel number one knots K_1 and K_2. In [9], Morimoto showed that $t(K_1\#K_2) = 3$ if and only if neither K_1 nor K_2 admits a $(1,1)$-decomposition. See [15] for a more concrete example. There is a relationship between the tunnel number $t(K)$ and the Heegaard genus g of the 2-fold branched covering space $\Sigma_2(K)$ of the 3-sphere over a knot K so that $g \leq 2t(K) + 1$ (see [12]). In the special case of $t(K) = 1$, we have $g \leq 3$, and if K is a $(1,1)$-knot, then we have $g \leq 2$ from the result of [2]. Thus if there is a tunnel number one knot K such that $g = 3$, then this implies that the knot K does not admit a $(1,1)$-decomposition.

The purpose of this article is to consider when a knot K with $g \leq 2$ is a $(1,1)$-knot. We show that a tunnel number one knot K induced by a strong involution of a genus two Heehaard splitting of the covering space $\Sigma_2(K)$ is a $(1,1)$-knot; in other words, a tunnel number one knot with bridge number 3 is a $(1,1)$-knot.

2. On the tunnel number one knots with bridge number 3

Theorem 1([2]). *Let K be a knot in the 3-sphere, and $\Sigma_k(K)$ the k-fold branched covering space of the 3-sphere over K. Let g_k be the Heegaard genus of $\Sigma_k(K)$. Suppose that a knot K in the 3-sphere admits a (g,b)-decomposition. Then we have $g_k \leq 1 - k + k(g+b) - b$.*

Note that a genus one Heegaard splitting of lens space has a unique non-free involution. Thus, as we can see from Theorem 1 above, the 2-fold branched covering space over the 3-sphere branched along any $(1,1)$-knot except the 2-bridge knots admits a genus two Heegaard splitting. Conversely, we have the following conjecture: A tunnel number one knot K with 2-fold branched covering space of genus $g \leq 2$ admits a $(1,1)$-decomposition. For the above conjecture we show that a tunnel number one knot with bridge number 3 is a $(1,1)$-knot.

For definitions of bridge position and thin position, we refer to [3] for the convenience of explanation. The following Lemmas show that the tunnel of a tunnel number one knot may lie in a level sphere of a tunnel number one knot in a minimal bridge position.

Lemma 2 ([3]). *Let $K \subset S^3$ be a tunnel number one knot in a minimal bridge position and r a tunnel for the knot K. Then r may be slid and isotoped to lie* *entirely in a level sphere for the knot K.*

Lemma 3. *Let K be a tunnel number one with bridge number* 3*. Then the knot K in a* 3*-bridge position is in a minimal bridge position.*

Proof. Let K be a knot in a 3-bridge position. Then the complexity of the Morse position is 18, and this is a minimal complexity for the knot K since each complexity of trivial knot, 2-bridge knot and the connected sums of two 2-bridge knots in a case of less than 18, is 2, 8 and 14, respectively. Thus K is a knot in thin position. From Corollary 1.5 in [3], it is a knot in a minimal bridge position. □

Theorem 4. *Let K be a tunnel number one knot, and $\Sigma_2(K)$ the* 2*-fold cyclic branched covering space of the* 3*-sphere over K. Then a tunnel number one knot K induced by a strong involution of a genus two Heehaard splitting of $\Sigma_2(K)$ is a $(1,1)$-knot.*

Proof. By Birman-Hilden([1]), we have that $\Sigma_2(K)$ is the 2-fold cyclic branched covering space of the 3-sphere over a 3-bridge knot K and that the knot K is induced by a strong involution of a genus two Heegaard splitting of $\Sigma_2(K)$.

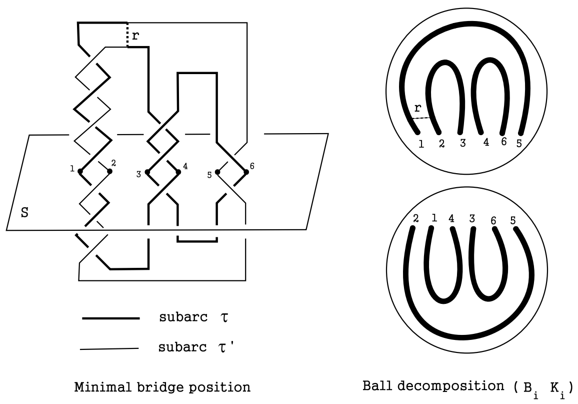

Fig.1. Arc r with different ends satisfying case (1)

Let K be a tunnel number one knot of the bridge number 3. From Lemma 3 we may assume that a pair of 3-balls $\{B_1, B_2\}$ gives the minimal bridge decomposition of K. That is, $S^3 = B_1 \cup B_2$, $B_1 \cap B_2 = \partial B_1 = \partial B_2$ and $(B_i, B_i \cap K)$ is a trivial tangle of 3 components $(i = 1, 2)$. Let $S = \partial B_1 = \partial B_2$ be the level sphere and for $i = 1, 2$, let K_i denote the collection of arcs $B_i \cap K$, parallel to a collection of arcs in S. By Lemma 2(or Theorem 6.1 in [3]), we have shown that any tunnel r for K can be slid and isotoped to lie in S as follows: for one of (B_i, K_i, r), $i = 1, 2$, either (1) r is an arc with its ends on different components of K_i and K_i is parallel to a collection of arcs in $S - r$, or (2) r is an arc with both ends on the same component of K_i. In this case, r can be slid and isotoped in B_i so that it lies in S as a loop with its ends at the same point of ∂K_i, or (3) r is an eyeglass and a disk that r

bounds in S containing exactly one end of each of the 2 components of K_i.
Let τ and τ' be two subarcs of (B_i, K_i) cut off by ∂r. Then at least one of $Cl(S^3 - N(r \cup \tau))$ and $Cl(S^3 - N(r \cup \tau'))$ in each case of (1), (2) and (3) is a solid torus. See Fig.1 for a typical example. By the Morimoto-Sakuma criterion for a $(1,1)$-tunnel(see Proposition 1.3 of [13]), we see that K is a 1-genus 1-bridge knot. Therefore, any 3-bridge knot with an unknotting tunnel is a knot admitting a $(1,1)$-decomposition. □

The following corollary is an immediate result of Theorem 4. We note that in the link case we cannot use the facts above.

Corollary 5. *Any tunnel number one knot with bridge number* 3 *admits a* $(1,1)$-*decomposition.*

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