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TAUT STRUCTURES IN AN IDEAL TRIANGULATION WITH TWO OR THREE TETRAHEDRA

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Abstract. In this paper, we classify all taut structures for ideal triangulations with two and three tetrahedra.

1. Introduction

Taut triangulations were introduced by Lackenby [2], based on Gabai's theory of taut foliations. Lackenby showed that any irreducible atoroidal orientable 3-manifold with tori boundary has an ideal triangulation which admits a taut structure. In this paper, we classify all taut structures in ideal triangulations with two or three tetrahedra.

An ideal triangulation is a simplicial structure obtained by gluing tetrahedra with all vertices deleted and some edges or faces identified (for details, see [3]). This gives a very efficient way of representing 3-manifolds with non-empty boundary. The ideal vertices deleted represents each boundary components of a given 3-manifold.

Definition 1. *Let $\Im$ be an ideal triangulation of M. A taut structure for $\Im$ is an assigment of angles 0 or π to the dihedral angles at edges between pairs of faces in each tetrahedron of $\Im$ to be satisfied the following two conditions;*
(i) For each tetrahedron τ of $\Im$, the angle sum of three edges at each vertex in τ is π.

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(ii)For each edge E of $\Im$, the sum of all dihedral angles around E is exactly 2π.

From the first condition for a taut structure, we can see that exactly one pair of opposite edges is of angle π and the other edges of angle 0 in each tetrahedron of a taut triangulation.

Let τ_1, τ_2, ..., and τ_n be given tetrahedra in an ideal triangulation. We denote a taut structure of the ideal triangulation by $\mathcal{T} = (A_1B_1, A_2B_2, ..., A_nB_n)$, where A_iB_i is the pair of opposite edges of angle π in τ_i, $i = 1, 2, ..., n$. We also give a notation for the degrees of a series of edges $A_1, A_2, ..., A_n$ by $deg(A_1, A_2, ..., A_n) = (a_1, a_2, ..., a_n)$, where the degree of A_i is a_i for $i = 1, 2, ..., n$. Here the degree of an edge E is given by the number of edges of tetrahedra glued along the edge E. Denote the triangle with three edges A, B and C by ABC. If some edges of a triangle is not specified with labels, we label the edges $*$ and the triangle $A**$ or $AB*$. Suppose that an ideal triangulation of a 3-manifold has no 2-simplex labeled with the same edge. Then at most four edges in a tetrahedron can be identified. Hence we have exactly six identification types of an edge A in a tetrahedron τ, $AA + AA$, $AA + A$, AA, $3A$, $2A$ and A, and denote by A_τ (see Figure 1).
These are all possible ways of identifying edges in a tetrahedron. If there is no edges labeled A in τ, denote $A_\tau = \emptyset$. Now we denote the identification types of an edge A in a series of tetrahedra τ_1, τ_2, ..., and τ_n as follows ;

$$A(\tau_1, \tau_2, ..., \tau_n) = (A_{\tau_1}, A_{\tau_2}, ..., A_{\tau_n}).$$

2. Taut structures for two tetrahedra

Let τ_1 and τ_2 denote given two tetrahedra. We consider all possible identifications of these two tetrahedra and find a taut structure for each identification expected to be an ideal triangulation. In an ideal triangu

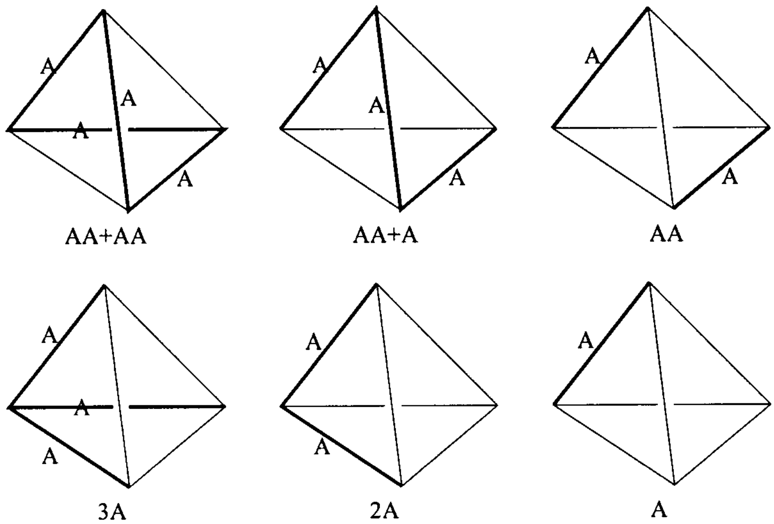

FIGURE 1. Six identification types of an edge A

-lation, the number of edges is the same as the number of tetrahedra. Here we only have two edges, labeled A and B. For the tautness, the 0-efficiency is necessary [1]. But in this article we deal with manifolds with more specific restrictions. Suppose that all edges of ideal triangulation have degree at least 4. Then there are three cases of $deg(A, B)$; $(4, 8)$, $(5, 7)$ or $(6, 6)$. If there is a pair of opposite edges AA in a tetrahedron, say in τ_1, then there must be a pair of opposite edges BB in τ_2. Otherwise, we have AAA face or $3A$ identification type of the edge A in τ_2. But in both cases, we cannot have even number of AAA or $AA*$ faces since the degree of A is less than equal to 6. Choose $\mathcal{T} = (AA, BB)$ as a taut structure. Suppose that there is no pairs of opposite edges AA. Then both tetrahedra τ_1 and τ_2 should have an edge labeled A and so have a pair of opposite edges AB. Choose $\mathcal{T} = (AB, AB)$. Therefore, we obtain the following theorem.

Theorem 1. *Let M be a 3-manifold with an ideal triangulation $\Im$ of two tetrahedra. Suppose that all edges of $\Im$ have degree at least 4. Then there always be a taut structure of the ideal triangulation.*

3. Taut structures for three tetrahdera

Let $\Im$ be an ideal triangulation of three tetrahedra, τ_1, τ_2 and τ_3. We denote three edges in $\Im$ by A, B and C and assume that these have degree at least 4. There are seven choices of $deg(A,B,C)$; (4,4,10), (4,5,9), (4,6,8), (4,7,7), (5,5,8), (5,6,7), (6,6,6). We can also classify the identification type $A(\tau_1,\tau_2,\tau_3)$ in each case regarding the degree of A. Since there must be even number of faces having the same labels of edges and cannot be identified all four face of a tetrahedron in itself, the followings are all the cases we should consider. For the case of $degA = 4$, $A(\tau_1,\tau_2,\tau_3) = (AA+A, A, \emptyset), (AA, A, A)$, or $(2A, 2A, \emptyset)$. For $degA = 5$, $A(\tau_1,\tau_2,\tau_3) = (AA+A, A, A), (AA, AA, A), (2A, 2A, A)$, or $(3A, 2A, \emptyset)$. Finally, for $degA = 6$, we have five cases of $A(\tau_1,\tau_2,\tau_3)$; $(AA+A, AA, A)$, (AA, AA, AA), $(AA, 2A, 2A)$, $(3A, 3A, \emptyset)$ or $(3A, 2A, A)$.

Case 1. $deg(A,B,C) = (4,4,10)$.

We divide into three subcases by the identification type $A(\tau_1,\tau_2,\tau_3)$.

(1) $A(\tau_1,\tau_2,\tau_3) = (AA+A, A, \emptyset)$ (see Figure 2).

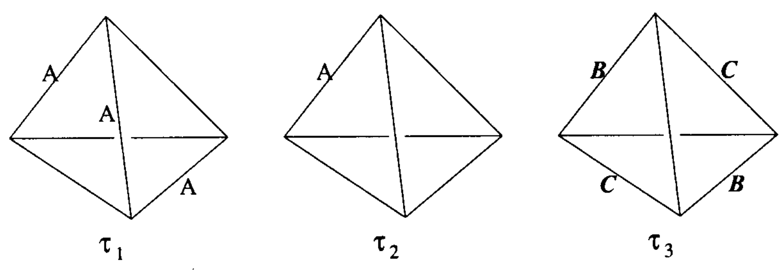

FIGURE 2. $A(\tau_1,\tau_2,\tau_3) = (AA+A, A, \emptyset)$

Since in an ideal triangulation we cannot have a face with all three edges of the same label, there must be both pairs of opposite edges BB and CC in the tetrahedron τ_3. Hence $B(\tau_1,\tau_2,\tau_3) = (*, *, BB+B)$ or

$(*, *, BB)$, where $*$ is either B or $\emptyset$. Then there always be the opposite pair CC in τ_2. Choose $\mathcal{T} = (AA, CC, BB)$.

(2) $A(\tau_1, \tau_2, \tau_3) = (AA, A, A)$ (see Figure 3).

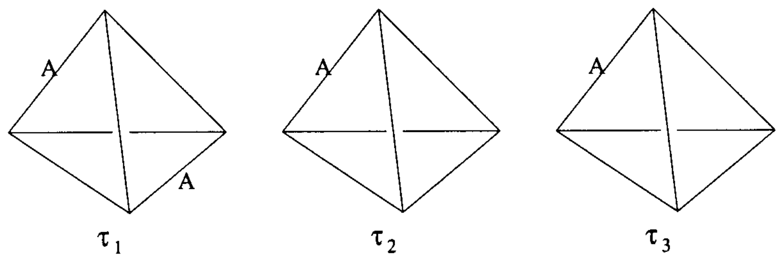

FIGURE 3. $A(\tau_1, \tau_2, \tau_3) = (AA, A, A)$

Since τ_2 and τ_3 have the same identification type of the edge A, by a symmetry we may consider only the cases based on τ_2. There are three cases for $B(\tau_1, \tau_2, \tau_3)$; (BB, B, B), (B, BB, B) and $(\emptyset, 2B, 2B)$. If $B(\tau_1, \tau_2, \tau_3) = (BB, B, B)$, then AB must be in both τ_2 and τ_3 since we cannot have a face with all edges labeled by C. Hence all four ABC faces in τ_1 must be identified each other. This is a contradiction. In the case of $B(\tau_1, \tau_2, \tau_3) = (B, BB, B)$, there is CC in τ_3. Then choose $\mathcal{T} = (AA, BB, CC)$. Finally, if $B(\tau_1, \tau_2, \tau_3) = (\emptyset, 2B, 3B)$, then BC is in both τ_2 and τ_3. Choose $\mathcal{T} = (AA, BC, BC)$.

(3)$A(\tau_1, \tau_2, \tau_3) = (2A, 2A, \emptyset)$ (see Figure 4).

In this case, we know that there are both BB and CC in τ_3. Then $B(\tau_1, \tau_2, \tau_3) = (B, B, BB)$ and AC is in both τ_1 and τ_2. Choose $\mathcal{T} = (AC, AC, BB)$.

Case 2. $deg(A, B, C) = (4, 5, 9)$.

We will use a similar argument as the previous case.

(1)$A(\tau_1, \tau_2, \tau_3) = (AA + A, A, \emptyset)$.

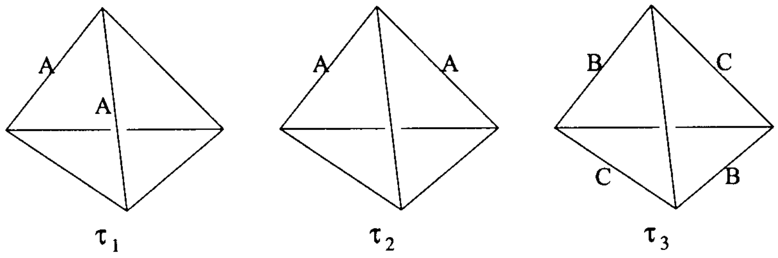

FIGURE 4. $A(\tau_1, \tau_2, \tau_3) = (2A, 2A, \emptyset)$

As in Case 1, we know that there are both of the opposite pairs BB and CC in τ_3. We consider two separated cases regarding whether CC is in τ_2 or not. If CC is in τ_2, then choose $\mathcal{T} = (AA, CC, BB)$. If there is no CC in τ_2, then $2B$ is in τ_2. But there is no way to have both BB and $2B$ identification types in an ideal triangulation since $degB = 5$.

(2)$A(\tau_1, \tau_2, \tau_3) = (AA, A, A)$.

In the same reason as in Case 1, we consider only the cases based on τ_2 rather than regarding all cases for both τ_2 and τ_3. If BB is in τ_2, $2B$ or $3B$ cannot be in τ_3. Hence CC is in τ_3. Choose $\mathcal{T} = (AA, BB, CC)$. By a symmetry, we can choose a taut structure in the case that BB is in τ_3 either. Now if BB is not in $\tau_2 \cup \tau_3$ but in τ_1, then $B(\tau_1, \tau_2, \tau_3) = (BB + B, B, B)$ and AB must be in both τ_2 and τ_3 (see Figure 5). But then all four faces of τ_1 should be identified each other in τ_1. This cannot happen. Suppose that there is no BB. If $B(\tau_1, \tau_2, \tau_3) = (2B, 2B, B)$ or $(2B, B, 2B)$, then there are AC in τ_2 and AB in τ_3 (see Figure 6). Choose $\mathcal{T} = (BC, AC, AB)$. Otherwise, $2B$ is in both τ_2 and τ_3 and so is BC. Choose $\mathcal{T} = (AA, BC, BC)$.

(3)$A(\tau_1, \tau_2, \tau_3) = (2A, 2A, \emptyset)$.

There must be the face BCC in both τ_1 and τ_2. Then AC is in both τ_1 and τ_2. Choose $\mathcal{T} = (AC, AC, BB)$.

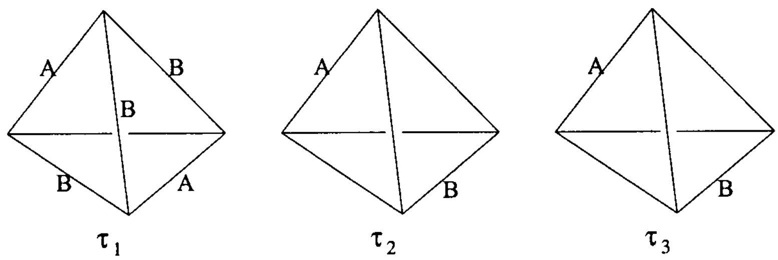

FIGURE 5. $A(\tau_1, \tau_2, \tau_3) = (AA, A, A)$ and $B(\tau_1, \tau_2, \tau_3) = (BB + B, B, B)$

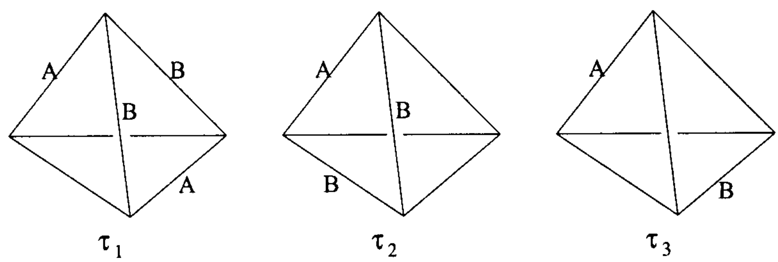

FIGURE 6. $A(\tau_1, \tau_2, \tau_3) = (AA, A, A)$ and $B(\tau_1, \tau_2, \tau_3) = (2B, 2B, B)$

Case 3. $deg(A, B, C) = (4, 6, 8)$.

(1)$A(\tau_1, \tau_2, \tau_3) = (AA + A, A, \emptyset)$.

If CC is in τ_2, choose (AA, CC, BB). If CC is not in τ_2, then $2B$ is in τ_2 and $B(\tau_1, \tau_2, \tau_3) = (2B, 2B, BB)$. But this does not happen because there is no face to be matched to AAB in τ_1.

(2)$A(\tau_1, \tau_2, \tau_3) = (AA, A, A)$.

If BB is in τ_2 and CC in τ_3, choose $\mathcal{T} = (AA, BB, CC)$. Suppose that BB is in τ_2 and CC is not in τ_3. Then $2B$ is in τ_3 and $B(\tau_1, \tau_2, \tau_3) = (2B, BB, 2B)$. Hence the faces ABB and ACC in τ_1 should be identified with the faces ABB and ACC in τ_3 (see Figure 7). Then the degree of the edge A placed between ABB and ACC must be 2, which is a contradiction. Now consider the case that BB is in neither τ_2 nor τ_3. If BB is in τ_1, then $B(\tau_1, \tau_2, \tau_3) = (BB, 2B, 2B)$. Choose $\mathcal{T} =$

(AA, BC, BC). Suppose that there is no opposite pair BB. If $2B$ is in τ_1, then $B(\tau_1, \tau_2, \tau_3) = (2B, 3B, B)$ and AB is in both τ_2 and τ_3. Choose $\mathcal{T} = (BC, AB, AB)$. Otherwise, $B(\tau_1, \tau_2, \tau_3) = (B, 3B, 2B)$ or $(\emptyset, 3B, 3B)$. In either case, BC is in both τ_2 and τ_3. Choose $\mathcal{T} = (AA, BC, BC)$.

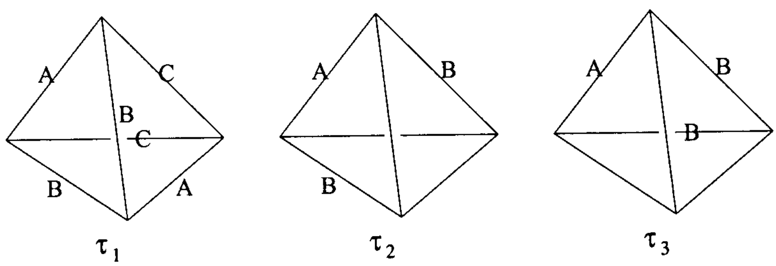

FIGURE 7. $A(\tau_1, \tau_2, \tau_3) = (AA, A, A)$ and $B(\tau_1, \tau_2, \tau_3) = (2B, BB, 2B)$

(3)$A(\tau_1, \tau_2, \tau_3) = (2A, 2A, \emptyset)$.

In this case, both BB and CC are in τ_3. If AC is in both τ_1 and τ_2, choose $\mathcal{T} = (AC, AC, BB)$. If AC is not in τ_1 or τ_2, say τ_1, then $2B$ is in τ_1 and $B(\tau_1, \tau_2, \tau_3) = (2B, 2B, BB)$. Choose $\mathcal{T} = (AB, AB, CC)$.

Case 4. $deg(A, B, C) = (4, 7, 7)$.

Since the degrees of B and C are the same, we only give arguments considering the identification types of the edge B rather than regarding both of cases for B and C.

(1)$A(\tau_1, \tau_2, \tau_3) = (AA + A, A, \emptyset)$.

Since the two faces in τ_1 with two edges labeled A are identified, there is either BB or CC in τ_1, say BB in τ_1 (see Figure 8). Then we need two $AB*$ in τ_2 to be identified with the faces $AB*$ in τ_1. Since the degree of B is 7, the remaining edge of τ_1 must be labeled C. If CC is in τ_2, choose $\mathcal{T} = (AA, CC, BB)$. If CC is not in τ_2, then τ_2 has the face BBC and the opposite pair AC. Choose $\mathcal{T} = (AC, AC, BB)$.

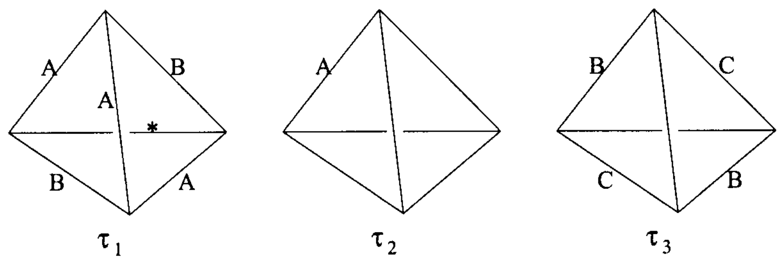

FIGURE 8. $A(\tau_1, \tau_2, \tau_3) = (AA + A, A, \emptyset)$ and BB in τ_1

(2)$A(\tau_1, \tau_2, \tau_3) = (AA, A, A)$.

First we consider the case that AB is in both τ_2 and τ_3. If CC is in τ_1, then choose $\mathcal{T} = (CC, AB, AB)$. Suppose that CC is not in τ_1. Then $B(\tau_1)$ is either $BB + B$ or $2B$. If $B(\tau_1) = BB + B$, we need at least two more faces labeled ABC in $\tau_2 \cup \tau_3$. If τ_2(resp. τ_3) has two faces of ABC, then BB is in τ_2(resp. τ_3). Choose $\mathcal{T} = (AA, BB, CC)$. If both τ_2 and τ_3 have the face ABC, then BC is in both τ_2 and τ_3. Choose $\mathcal{T} = (AA, BC, BC)$. Now suppose that $2B$ is in τ_1. Since there is only one ABB in τ_1, we need one more ABB in τ_2 or τ_3, say τ_2. Then $B(\tau_1, \tau_2, \tau_3) = (2B, 3B, 2B)$ and BC is in both τ_2 and τ_3. Choose $\mathcal{T} = (AA, BC, BC)$.

Secondly, we assume that AB is in τ_2 and AC in τ_3. If BC is in τ_1, choose $\mathcal{T} = (BC, AB, AC)$. If BC is not in τ_1, then both BB and CC are in τ_1. In the case that BC is in both τ_2 and τ_3, choose $\mathcal{T} = (AA, BC, BC)$. Otherwise, there are BB and CC in both τ_2 and τ_3. Choose $\mathcal{T} = (AA, BB, CC)$.

(3)$A(\tau_1, \tau_2, \tau_3) = (2A, 2A, \emptyset)$.

In this case, AAB is in both τ_1 and τ_2 and so is AC. Choose $\mathcal{T} = (AC, AC, BB)$.

Case 5. $deg(A, B, C) = (5, 5, 8)$.

Here we have four subcases divided by the identification type $A(\tau_1, \tau_2, \tau_3)$.

(1)$A(\tau_1, \tau_2, \tau_3) = (AA + A, A, A)$

Since τ_2 and τ_3 have the same identification type of A, we lead the argument based on τ_2. In this case, either BB or CC is in τ_1. If BB is in τ_1, then $B(\tau_1, \tau_2, \tau_3) = (BB + B, B, B)$ or (BB, BB, B). Choose $\mathcal{T} = (AA, BB, CC)$. If $CC + C$ is in τ_1, then $B(\tau_1, \tau_2, \tau_3) = (\emptyset, 3B, 2B)$. Choose $\mathcal{T} = (AA, BC, BC)$. If CC and AB are in τ_1, then $B(\tau_1, \tau_2, \tau_3) = (B, BB + B, B)$, (B, BB, BB) or $(B, 2B, 2B)$. For the case that $B(\tau_1, \tau_2, \tau_3) = (B, BB + B, B)$, there is only one face labeled ABC in the triangulation. This cannot happen. If $B(\tau_1, \tau_2, \tau_3) = (B, BB, BB)$, choose $\mathcal{T} = (AA, BB, CC)$. Finally, if $B(\tau_1, \tau_2, \tau_3) = (B, 2B, 2B)$, then choose $\mathcal{T} = (AA, BC, BC)$.

(2)$A(\tau_1, \tau_2, \tau_3) = (AA, AA, A)$.

Since τ_1 and τ_2 have the same identification type of A, we consider only the cases of τ_1 for a fixed type in τ_3. If BB is in τ_1 or τ_2, say τ_1, then $B(\tau_1, \tau_2, \tau_3) = (BB + B, B, B)$, (BB, BB, B) or (BB, B, BB). In either case, there is CC in τ_3. Choose $\mathcal{T} = (BB, AA, CC)$. Suppose that BB is neither τ_1 nor τ_2. If BB is in τ_3, then $B(\tau_1, \tau_2, \tau_3) = (B, B, BB + B)$ and CC is in both τ_1 and τ_2. Choose $\mathcal{T} = (AA, CC, BB)$. If there is no opposite pair BB, then BC is in τ_1 or τ_2, say τ_1. If there is BC in τ_3 either, then choose $\mathcal{T} = (BC, AA, BC)$. If BC is not in τ_3, $B(\tau_1, \tau_2, \tau_3) = (2B, 2B, B)$ with AB in τ_3 (see Figure 9). Then the two BBC faces in τ_3 should be identified and the degree of B in the face BCC must be 1 which is a contradiction.

(3)$A(\tau_1, \tau_2, \tau_3) = (2A, 2A, A)$.

If BB is in τ_1 or τ_2, say τ_1, then $B(\tau_1, \tau_2, \tau_3) = (BB + B, B, B)$, $(BB,$

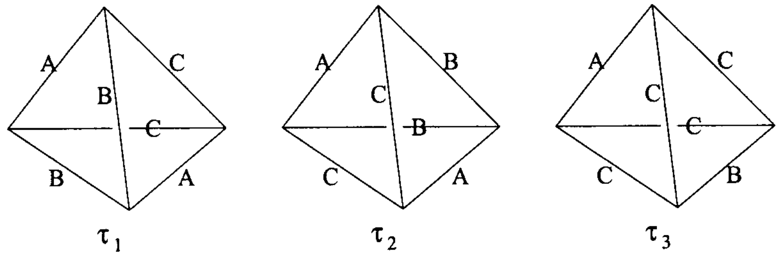

FIGURE 9. $A(\tau_1, \tau_2, \tau_3) = (AA, AA, A)$ and $B(\tau_1, \tau_2, \tau_3) = (2B, 2B, B)$

$BB, B)$ or (BB, B, BB). In the case that $B(\tau_1, \tau_2, \tau_3) = (BB{+}B, B, B)$, there is no face to be identified with ABB in τ_1. Suppose that $B(\tau_1, \tau_2, \tau_3) = (BB, BB, B)$. Then there are two ACC faces in τ_2 which must be identified. But since the degree of A is 5, this cannot occur. Consider the case that $B(\tau_1, \tau_2, \tau_3) = (BB, B, BB)$. In this case, both τ_2 and τ_3 have the opposite pair AC. Choose $\mathcal{T} = (BB, AC, AC)$. Now suppose that BB is neither τ_1 nor τ_2. There are three cases regarding $B(\tau_1, \tau_2, \tau_3)$. If $B(\tau_1, \tau_2, \tau_3) = (B, B, BB + B)$, choose $\mathcal{T} = (AC, AC, BB)$. If $B(\tau_1, \tau_2, \tau_3) = (2B, 2B, B)$, then AB is in both τ_1 and τ_2 and CC is in τ_3. Choose $\mathcal{T} = (AB, AB, CC)$. Finally, if $B(\tau_1, \tau_2, \tau_3) = (2B, B, 2B)$, then there are AB in τ_1, AC in τ_2 and BC in τ_3. Choose $\mathcal{T} = (AB, AC, BC)$.

(4)$A(\tau_1, \tau_2, \tau_3) = (3A, 2A, \emptyset)$.

In this case, there are AB and AC in τ_1, and BB and CC in τ_3. Choose $\mathcal{T} = (AB, AB, CC)$ if AB is in τ_2 and $\mathcal{T} = (AC, AC, BB)$ if AC is in τ_2.

Case 6. $deg(A, B, C) = (5, 6, 7)$.

(1)$A(\tau_1, \tau_2, \tau_3) = (AA + A, A, A)$.

If $BB + B$ is in τ_1, we need at least two more ABB in $\tau_2 \cup \tau_3$. But the degree of B is only 6. Suppose that BB and AC are in τ_1. Then $B(\tau_1, \tau_2, \tau_3)$ is either $(BB, BB + B, B)$ or (BB, BB, BB) or

$(BB, 2B, 2B)$. If $B(\tau_1, \tau_2, \tau_3) = (BB, BB+B, B)$, then AB is in τ_3 and the four faces, two ACC and two BCC, must be identified each other in τ_3. This cannot happen. If $B(\tau_1, \tau_2, \tau_3) = (BB, BB, BB)$, choose $\mathcal{T} = (AA, BB, CC)$, and if $\tau_3 = (BB, 2B, 2B)$, choose $\mathcal{T} = (AA, BC, BC)$. Now suppose that CC and AB are in τ_1. We have two choices of $B(\tau_1, \tau_2, \tau_3)$. If $B(\tau_1, \tau_2, \tau_3) = (B, BB+B, BB)$, choose $\mathcal{T} = (AA, BB, CC)$. If $B(\tau_1, \tau_2, \tau_3) = (B, 2B, 2B)$, choose $\mathcal{T} = (AA, BC, BC)$. Finally, we assume that $CC+C$ is in τ_1. Then $B(\tau_1, \tau_2, \tau_3) = (\emptyset, 3B, 3B)$. Choose $\mathcal{T} = (AA, BC, BC)$.

(2)$A(\tau_1, \tau_2, \tau_3) = (AA, AA, A)$.

If BB is in τ_1 or τ_2, say τ_1, then there are four choices of $B(\tau_1, \tau_2, \tau_3)$. First if $B(\tau_1, \tau_2, \tau_3) = (BB+B, BB, B)$, then two ACC and two BCC faces in τ_3 should be identified each other. This cannot happen. The other cases are $B(\tau_1, \tau_2, \tau_3) = (BB+B, B, BB)$, (BB, BB, BB) and $(BB, 2B, 2B)$. For each case, we can choose a taut structure as follows ; (BB, AA, CC), (AA, BB, CC) and (AA, BC, BC), respectively.

(3)$A(\tau_1, \tau_2, \tau_3) = (2A, 2A, A)$.

Here we have seven subcases regarding $B(\tau_1, \tau_2, \tau_3)$. First, if $B(\tau_1, \tau_2, \tau_3) = (BB+B, BB, B)$, there is no face to be identified with BBC in τ_1. So this cannot occur. If $B(\tau_1, \tau_2, \tau_3) = (BB, BB, BB)$, choose $\mathcal{T} = (BB, AC, AC)$. If $B(\tau_1, \tau_2, \tau_3) = (BB, 2B, 2B)$, then either AB or AC is in τ_3. Choose $\mathcal{T} = (AC, BC, AB)$ and $\mathcal{T} = (BB, AC, AC)$ in each case. For the case that $B(\tau_1, \tau_2, \tau_3) = (2B, 2B, BB)$ and $(3B, 2B, B)$, we can choose $\mathcal{T} = (AB, AB, CC)$. Finally suppose that $B(\tau_1, \tau_2, \tau_3) = (3B, B, 2B)$. Then AB and AC are in τ_2. Choose $\mathcal{T} = (BC, AC, AB)$ if AB is in τ_3 and $\mathcal{T} = (BC, AB, AC)$ if AC is in τ_3.

(4)$A(\tau_1, \tau_2, \tau_3) = (3A, 2A, \emptyset)$.

By the same argument as Case 5, we can choose a taut structure for each case.

Case 7. $deg(A, B, C) = (6, 6, 6)$.

Here all of the three edges have the same degree. Hence by symmetries, many cases are cancelled out of consideration. We have five subcases regarding $A(\tau_1, \tau_2, \tau_3)$.

(1)$A(\tau_1, \tau_2, \tau_3) = (AA + A, AA, A)$.

Since two $AA*$ faces in τ_1 must be identified, there is either BB or CC in τ_1, say BB in τ_1. Then $C(\tau_1) = C$ and either CC or $3C$ is in τ_3. Choose $\mathcal{T} = (BB, AA, CC)$ if CC is in τ_3, and $\mathcal{T} = (AA, BC, BC)$ if $3C$ is in τ_3.

(2)$A(\tau_1, \tau_2, \tau_3) = (AA, AA, AA)$.

There are three choices of $B(\tau_1, \tau_2, \tau_3)$. If $B(\tau_1, \tau_2, \tau_3) = (BB + B, BB, B)$ or (BB, BB, BB), then we can choose $\mathcal{T} = (AA, BB, CC)$. If $B(\tau_1, \tau_2, \tau_3) = (BB, 2B, 2B)$, choose $\mathcal{T} = (AA, BC, BC)$.

(3)$A(\tau_1, \tau_2, \tau_3) = (AA, 2A, 2A)$.

If BB is in τ_1, there are four choices of $B(\tau_1, \tau_2, \tau_3)$; $(BB+B, BB, B)$, $(BB, BB + B, B)$, (BB, BB, BB) and $(BB, 2B, 2B)$. For the former three cases, we can apply the previous arguments (1) and (2) by exchanging A and B. Let $B(\tau_1, \tau_2, \tau_3) = (BB, 2B, 2B)$, then AB is in both τ_2 and τ_3. Choose $\mathcal{T} = (CC, AB, AB)$. By a symmetry, we can also find a taut structure in the case that CC is in τ_1. Now assume that there is no BB or CC in τ_1. Then both $2B$ and $2C$ are in τ_1 and so that ABB and ACC are in τ_1. We need more ABB and ACC in $\tau_2 \cup \tau_3$. Then there are ABB in τ_2 and ACC in τ_3, or vice versa. Choose $\mathcal{T} = (BC, AB, AC)$ and (BC, AC, AB) in each case.

(4)$A(\tau_1, \tau_2, \tau_3) = (3A, 3A, \emptyset)$.

There are BB and CC in τ_3, and AB and AC in both τ_1 and τ_2. Choose $\mathcal{T} = (AB, AB, CC)$ or (AC, AC, BB).

(5)$A(\tau_1, \tau_2, \tau_3) = (3A, 2A, A)$.

We know that both AB and AC are in τ_1. In the case that BC is in τ_3, choose $\mathcal{T} = (AB, AC, BC)$ if AC is in τ_2 and $\mathcal{T} = (AC, AB, BC)$ if AB is in τ_2. Suppose that BC is not in τ_3. Then both BB and CC are in τ_3. Choose $\mathcal{T} = (AB, AB, CC)$ if AB is in τ_2 and $\mathcal{T} = (AC, AC, BB)$ if AC is in τ_2.

We close this section concluding that for an ideal triangulation of three tetrahedra, the following theorem holds.

Theorem 2. *Let M be a 3-manifold with an ideal triangulation of three tetrahedra. Suppose that $\Im$ has no 2-simplex with all three edges identified and has all edges of degree at least 4. Then there always be a taut structure of the ideal triangulation.*

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