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Locally Polynomial Rings over PVMD's

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Abstract. Let an integral domain R be locally polynomial over an integral domain D and let R be a content module over D. We show that if D is a PVMD, then $Cl_t(R) \cong Cl_t(D)$. This generalizes the polynomial case. We also show that R is a PVMD if and only if D is a PVMD if and only if R_{N_v} is a PVMD.

1. Introduction and preliminaries

P. Eakin and J. Silver in [1] introduced the concept of a locally polynomial ring, which is a generalization of a polynomial ring as follows. They examined the relationship between a ring D and an algebra which is locally polynomial over D.

Definition 1.1. If a ring R is an algebra over a ring D, then R is said to be *locally polynomial over* D provided that, for every prime ideal P of D, $R_P = R \otimes_D D_P$ is a polynomial ring over D_P.

For example, let $\mathbb{Z}$ denotes the ring of integers, X an indeterminate over $\mathbb{Z}$ and $\{p_i\}_{i=1}^{\infty}$ the set of prime numbers. Set $R = \mathbb{Z}[\{X/p_i\}_{i=1}^{\infty}]$. Although the ring R is neither Noetherian nor Krull, it is locally polynomial ring over $\mathbb{Z}$. For, if (p) denotes a prime ideal of $\mathbb{Z}$ then $R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[X/p]$.

Note that the class of polynomial rings is properly contained in the class of locally polynomial rings.

The following results are useful in our study.

• If R is locally polynomial over a ring D, then R is faithfully flat over D [1, (1.2)].

• If R is locally polynomial over an integral domain D, then R is an integral

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domain [1, (1.7)].

If R is locally polynomial over a ring D, then P. Eakin and J. Silver defined the content of any element in R_S, where S is the set of all regular elements of D.

Definition 1.2. Suppose that R is locally polynomial over a ring D and let S be the set of all regular elements of D. For $f \in R_S$, define the *content of* f to be the smallest fractional ideal $c(f)$ of D such that $f \in c(f)R$.

More generally, let M be an R-module and let $x \in M$. The *content* $c(x)$ of x is defined as the intersection of those ideals A of R such that $x \in AM$. If $x \in c(x)M$ for every $x \in M$, then M is called a *content R-module*.

Several domains, including PVMD's, are related to the notion of $*$-operation. For a detailed study of $*$-operations, we refer the reader to [5, Sections 32 and 34]. For a quicker definition and for other purposes we note the following.

Let $F(R)$ be the set of all fractional ideals of an integral domain R. The operation $I \mapsto (I^{-1})^{-1} = I_v$, on $F(R)$, is called the v-operation. An ideal $I \in F(R)$ is a *v-ideal* (or *divisorial*) if $I = I_v$. Then I_t is defined to be $\bigcup\{J_v : J$ is a nonzero finitely generated subideal of $I\}$. An ideal I is called a *t-ideal* if $I = I_t$. For any star-operation $*$, a fraction ideal I of R is said to be *$*$-invertible* if $(IJ)_* = R$ for some fractional ideal J of R. The set $TI(R)$ is of t-invertible t-ideals of R is a group under the t-product $I * J = (IJ)_t$, and the set $P(R)$ of nonzero principal fractional ideals of R under multiplication is a subgroup of $TI(R)$. The quotient group $Cl_t(R) = TI(R)/P(R)$ is called the *t-class group* of R; unlike the divisor class group, the t-class group is defined for arbitrary integral domains. When R is a Krull domain, the t- and the v-operations coincide and $Cl_t(R) = Cl(R)$, the usual divisor class group of R.

Throughout this paper, we shall assume that R is locally polynomial over an integral domain D and that R is a content module over D.

In this paper, we show that if D is a PVMD, then $Cl_t(R) \cong Cl_t(D)$. This generalizes the polynomial case. We also show that R is a PVMD if and only if D is a PVMD if and only if R_{N_v} is a PVMD.

2. Main results

The following lemmas are well-known in the case of polynomial rings.

Lemma 2.1. *Suppose that R is locally polynomial over an integral domain D and that R is a content module over D. Consider the following statements.*

(1) *D is integrally closed.*

(2) *If $0 \neq f, g \in R_{D^*}$, then $[c(f)c(g)]_v = [c(fg)]_v$.*

(3) *If $0 \neq f \in R$, then $fR_{D^*} \cap R = f[c(f)]^{-1}R$.*

Then $(1) \Leftarrow (2) \Leftrightarrow (3)$.

Proof. (2) $\Rightarrow$ (1): Let M be any maximal ideal of D. Denote by $c_{D_M}(h)$ the content of any $h \in R_{D^*}$ over D_M. Then $c_{D_M}(h) = c(h)D_M$ by [1, (2.24)]. Thus for

$0 \neq f, g \in R_{D^*}$, $[c_{D_M}(fg)]_v = [c(fg)D_M]_v = [(c(fg))_v D_M]_v = [(c(f)c(g))_v D_M]_v = [(c(f)c(g))D_M]_v = [c_{D_M}(f)c_{D_M}(g)]_v$. Since $R_{D^*} \cong (R_M)_{D_M^*} = D_M[\{\mathbf{X}_M\}]_{D_M^*} = K[\{\mathbf{X}_M\}]$, where $K = q.f.(D) = q.f.(D_M)$, D_M is integrally closed. Thus $D = \bigcap D_M$ is integrally closed.

(2) $\Leftrightarrow$ (3). The proofs of these are similar to those of the polynomial case. □

An integral domain D is called a *Prüfer v-multiplication domain* (for short, PVMD) if every nonzero finitely generated ideal I of D is t-invertible.

Lemma 2.2. *Suppose that R is locally polynomial over an integral domain D and that R is a content module over D. If D is a PVMD, then $[c(f)c(g)]_v = [c(fg)]_v$ for any $0 \neq f, g \in R_{D^*}$.*

Proof. We first note that $c(h)$ is f.g. for any $0 \neq h \in R_{D^*}$. Let M be a maximal t-ideal of D and let $0 \neq f, g \in R_{D^*}$. Then D_M is a valuation domain, and hence a Prüfer domain. Thus $[c(f)c(g)]D_M = c_{D_M}(f)c_{D_M}(g) = c_{D_M}(fg) = [c(fg)]D_M$. Therefore $[c(f)c(g)]_v = [c(f)c(g)]_t = \bigcap [c(f)c(g)]D_M = \bigcap [c(fg)]D_M = [c(fg)]_t = [c(fg)]_v$, where the second and the forth equalities follow from [6, Theorem 3.5].□

Lemma 2.3. *If J is a divisorial ideal of R and $I = J \cap K \neq (0)$, then $I = \bigcap \{dc(g)^{-1} \mid J \subseteq \frac{d}{g}R,\ d \in D,\ g \in R\}$.*

Proof. The proof of this is similar to that of the polynomial case. □

Lemma 2.4. *Suppose that D is a PVMD and let J be an integral divisorial ideal of R.*

(1) *If $J \cap D = I \neq (0)$, then $J = IR$ and I is a divisorial ideal of D.*

(2) *If $J \cap D = (0)$, then there exists $f \in R$ and a divisorial ideal H of D such that $J = fHR$.*

Moreover, if J is v-invertible, then I and H are v-invertible, and if J is a t-invertible t-ideal of R, then I and H are t-invertible t-ideals.

Proof. (1) If $J \cap D = I \neq (0)$, then by Lemma 2.3, $I = \bigcap \{dc(g)^{-1} \mid J \subseteq \frac{d}{g}R,\ d \in D,\ g \in R\}$. If $f \in J$, then $fg \in dR$ for any $d \in D$ and $g \in R$ such that $J \subseteq \frac{d}{g}R$. Thus $c(fg) \subseteq dD$. Since D is a PVMD, by Lemma 2.2 $c(f)c(g) \subseteq [c(fg)]_v \subseteq dD$ and so $c(f) \subseteq I$. Thus $f \in IR$. Since the reverse inclusion always holds, $J = IR$.

(2) If $J \cap D = (0)$, then $JR_{D^*} = fR_{D^*}$ for some $f \in R$. By Lemma 2.1, $J \subseteq JR_{D^*} \cap R = fR_{D^*} \cap R = fc(f)^{-1}R$. Note that $dc(f)^{-1} \subseteq D$ for some $0 \neq d \in D$. Thus $J' := \frac{d}{f}J$ is an integral divisorial ideal of R and $J'R_{D^*} = R_{D^*}$. Since $J' \cap D := I' \neq (0)$, by the case (1) $J' = I'R$ and thus $J = fd^{-1}I'R$ and $H := d^{-1}I'$ is a divisorial ideal of D. □

Theorem 2.5. *Suppose that R is locally polynomial over an integral domain D and that R is a content module over D. If D is a PVMD, then $Cl_t(R) \cong Cl_t(D)$.*

Proof. Define $\varphi : TI(D) \to TI(R)$ by $\varphi(I) = IR$. Then φ is a well-defined homomorphism. This induces a homomorphism $\psi : Cl_t(D) \to Cl_t(R)$. To prove the injectivity of φ, it is enough to consider integral ideals. Let I and I' be integral t-invertible t-ideals of D such that $\varphi(I) = \varphi(I')$. Since R is faithfully flat over D, $I = IR \cap D = I'R \cap D = I'$. Thus φ is monic. Moreover ψ is also injective. Indeed, suppose that I' a t-invertible t-ideal of D such that $\varphi(I') = I'R$ is principal. Then $dI' := I \subseteq D$ for some $0 \neq d \in D$. Thus $\varphi(I) = \varphi(dI') = d\varphi(I')$ is principal, say $IR = fR$ for some $f \in R$. Denote by K the quotient field of D. If $0 \neq g \in I$, then $gR \subseteq fR$ implies that $gR_{D^*} \subseteq fR_{D^*}$. Since R_{D^*} is a polynomial ring over K, there is a well-defined degree. Thus $deg(f) \leq deg(g) = 0$ and so $f \in R \cap K = D$. Since R is faithfully flat over D, $IR = fR$ implies that $I = fD$. Thus ψ is monic. To prove the surjectivity of ψ, it is enough to consider integral ideals. Denote by $[I]$ the class of I in $Cl_t(D)$. If J is an integral t-invertible t-ideal of R such that $J \cap D \neq (0)$, then by Lemma 2.4(1), $J \cap D$ is a t-invertible t-ideal of D such that $\psi([J \cap D]) = [J]$. Suppose that $J \cap D = (0)$. Then by Lemma 2.4(2), there exists an $f \in R$ and a t-invertible t-ideal H of D such that $J = fHR$. That is, $\psi([H]) = [J]$. □

Corollary 2.6. *Suppose that R is locally polynomial over an integral domain D and that R is a content module over D. If D is a Krull domain, then $Cl(R) \cong Cl(D)$.*

Proof. This follows from [1, (2.29)] and Theorem 2.5. □

Lemma 2.7. *Suppose that D is a PVMD and let T be a multiplicatively closed subset of R contained in $N_v = \{f \in R \mid c(f)_v = D\}$. Let I be a nonzero fractional ideal of D. Then*

(1) $(IR_T)^{-1} = I^{-1}R_T$,

(2) $(IR_T)_v = I_vR_T$, *and*

(3) $(IR_T)_t = I_tR_T$.

Proof. The proof of this is similar to that of the polynomial case. □

Corollary 2.8. *Suppose that D is a PVMD. Let I be a nonzero ideal of D. Then*

(1) $(IR)_v = I_vR$, $(IR)_t = I_tR$;

(2) $(IR_N)_v = I_vR_N$, $(IR_N)_t = I_tR_N$;

(3) $(IR_{N_v})_v = I_vR_{N_v}$, $(IR_{N_v})_t = I_tR_{N_v}$.

Lemma 2.9. *Let $*$ be a $*$-operation on D. Let I be a nonzero ideal of D. If I is a $*$-ideal, then $IR_{N_*} \cap K = IR_{N_*} \cap D = I$.*

Proof. Let I be a $*$-ideal of D. It suffices to show that $IR_{N_*} \cap K = I$. Let $k \in IR_{N_*} \cap K$. Then $kg = f$ for some $g \in N_*$ and $f \in IR$. Thus $kD = (kc(g))_* = (c(kg))_* = (c(f))_* \subseteq I_* = I$. Hence $k \in I$. Therefore $IR_{N_*} \cap K = I$ since the other inclusion is clear. □

Let $\mathcal{D}_f(R)$ be the set of finite type v-ideals of an integral domain R.

Lemma 2.10. *Suppose that D is a PVMD. Let I be a nonzero ideal of D. Then $IR \in \mathcal{D}_f(R)$ if and only if $I \in \mathcal{D}_f(D)$.*

Proof. Suppose that $IR \in \mathcal{D}_f(R)$, say $IR = (f_1, \ldots, f_n)_v$. Since IR is a v-ideal, $IR = (IR)_v = I_vR$. Since R is faithfully flat over D, $I = IR \cap D = I_vR \cap D = I_v$. Now $IR \subseteq [(c(f_1) + \cdots + c(f_n))R]_v = (c(f_1) + \cdots + c(f_n))_vR \subseteq I_vR = IR$. Hence $IR = (c(f_1) + \cdots + c(f_n))_vR$. Thus $I = (c(f_1) + \cdots + c(f_n))_v$. Hence $I \in \mathcal{D}_f(D)$. Conversely, suppose that $I \in \mathcal{D}_f(D)$. Let $I = (I')_v$, where I' is a f.g. ideal contained in I. Then $IR = (I')_vR = (I'R)_v$. Thus $IR \in \mathcal{D}_f(R)$. □

Theorem 2.11. *The following statements are equivalent.*

(1) *D is a PVMD.*

(2) *R is a PVMD.*

(3) *R_{N_v} is a PVMD.*

Proof. (1) ⇒ (2). Suppose that D is a PVMD. Let $J \in \mathcal{D}_f(R)$. By Lemma 2.4, $J = fIR$ for some $f \in R_{D^*}$ and some v-ideal I of D. Then $IR = f^{-1}J \in \mathcal{D}_f(R)$. Hence $I \in \mathcal{D}_f(D)$ by Lemma 2.10. Thus I is a t-invertible t-ideal of D. By Lemma 2.7, $J^{-1} = f^{-1}I^{-1}R$ and hence $(JJ^{-1})_t = (fIRf^{-1}I^{-1}R)_t = (II^{-1}R)_t = R$. Thus every $J \in \mathcal{D}_f(R)$ is t-invertible. Hence R is a PVMD.

(2) ⇒ (3). This follows from the fact that any localization of a PVMD is also a PVMD.

(3) ⇒ (1). Let I be a nonzero f.g. ideal of D. Then IR_{N_v} is also a f.g. ideal of R_{N_v}. Thus $R_{N_v} = (IR_{N_v}(IR_{N_v})^{-1})_t = (IR_{N_v}I^{-1}R_{N_v})_t = (II^{-1}R_{N_v})_t = (II^{-1})_tR_{N_v}$. By Lemma 2.9, $D = (II^{-1})_t$ and so I is t-invertible. Therefore D is a PVMD. □

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