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Intuitionistic Fuzzy Bi-ideals of Ordered Semigroups

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Abstract. The intuitionistic fuzzification of the notion of a bi-ideal in ordered semigroups is considered. In terms of intuitionistic fuzzy set, conditions for an ordered semigroup to be completely regular is provided. Characterizations of intuitionistic fuzzy bi-ideals in ordered semigroups are given. Using a collection of bi-ideals with additional conditions, an intuitionistic fuzzy bi-ideal is constructed. Natural equivalence relations on the set of all intuitionistic fuzzy bi-ideals of an ordered semigroup are investigated.

1. Introduction

The theory of fuzzy sets proposed by Zadeh [20] has achieved a great success in various fields. Out of several higher order fuzzy sets, intuitionistic fuzzy sets introduced by Atanassov [1], [2], [3] have been found to be highly useful to deal with vagueness. Gau and Buehrer [7] presented the concept of vague sets. But, Burillo and Bustince [4] showed that the notion of vague sets coincides with that of intuitionistic fuzzy sets. Szmidt and Kacprzyk [19] proposed a non-probabilistic-type entropy measure for intuitionistic fuzzy sets. De et al. [5] studied the Sanchez's approach for medical diagnosis and extended this concept with the notion of intuitionistic fuzzy set theory. Dengfeng and Chuntian [6] introduced the concept of the degree of similarity between intuitionistic fuzzy sets, presented several new similarity measures for measuring the degree of similarity between intuitionistic fuzzy sets, which may be finite or continuous, and gave corresponding proofs of these similariry measure and discussed applications of the similarity measure between intuitionistic fuzzy sets to pattern recofnition problems. Based on the terminology given by Zadeh, Kehayopulu and Tsingelis [14] first considered the fuzzy sets in ordered groupoids. They discussed fuzzy analogous for several notions that have been proved to be useful in the theory of ordered groupoids/semigroups. Moreover, each ordered groupoid can be embedded into an ordered groupoid having a greatest element in terms of fuzzy sets [15].

In this paper we consider the fuzzification of the notion of a bi-ideal in ordered

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semigroups. We show that every intuitionistic fuzzy bi-ideal of an ordered semigroup is an intuitionistic fuzzy subsemigroup. We prove that, in a regular, left and right simple ordered semigroup, every intuitionistic fuzzy bi-ideal is constant. We give conditions for an ordered semigroup to be completely regular in terms of intuitionistic fuzzy set. We provide characterizations of intuitionistic fuzzy bi-ideals in ordered semigroups. Using a collection of bi-ideals with additional conditions, we construct an intuitionistic fuzzy bi-ideal. We investigate some natural equivalence relations on the set of all intuitionistic fuzzy bi-ideals of an ordered semigroup.

2. Preliminaries

We include some elementary aspects of ordered semigroups that are necessary for this paper, and for more details we refer to [8], [9], [10] and [13].

By an *ordered semigroup* we mean an ordered set S at the same time a semigroup satisfying the following conditions:

$$(\forall a,\ b,\ x \in S)\,(a \leq b \Rightarrow xa \leq xb \text{ and } ax \leq bx)$$

A nonempty subset A of an ordered semigroup S is called a *left* (resp. *right*) *ideal* of S if it satisfies:

- $SA \subseteq A$ (resp. $AS \subseteq A$),
- $(\forall a \in A)$ $(\forall b \in S)$ $(b \leq a \Rightarrow b \in A)$.

Both a left and right ideal of S is called an *ideal* of S. A nonempty subset A of an ordered semigroup S is called a *bi-ideal* of S if it satisfies:

- $ASA \subseteq A$,
- $(\forall a \in A)$ $(\forall b \in S)$ $(b \leq a \Rightarrow b \in A)$.

An ordered semigroup S is said to be *left* (resp. *right*) *simple* if for every left (resp. right) ideal A of S, we have $A = S$. An ordered semigroup S is said to be *regular* if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$.

A mapping $\mu : S \to [0, 1]$, where S is an arbitrary non-empty set, is called a *fuzzy set* in S. The *complement* of μ, denoted by $\bar{\mu}$, is the fuzzy set in S given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in S$. Let $\mathbf{0}$ and $\mathbf{1}$ be fuzzy sets in S defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in S$. For any fuzzy set μ in S and any $t \in [0, 1]$ we define two sets

$$U(\mu; t) = \{x \in S \mid \mu(x) \geq t\} \quad \text{and} \quad L(\mu; t) = \{x \in S \mid \mu(x) \leq t\},$$

which are called an *upper* and *lower t-level cut* of μ and can be used to the characterization of μ.

A fuzzy set μ in an ordered semigroup S is called a *fuzzy subsemigroup* of S if $\mu(xy) \geq \min\{\mu(x),\ \mu(y)\}$ for all $x,\ y \in S$. A fuzzy set μ in an ordered semigroup S is called a *fuzzy bi-ideal* of S if it satisfies:

- $(\forall x, y \in S)\ (x \leq y \Rightarrow \mu(x) \geq \mu(y))$,
- $(\forall x, y, z \in S)\ (\mu(xyz) \geq \min\{\mu(x), \mu(z)\})$.

As an important generalization of the notion of fuzzy sets in S, Atanassov [1] introduced the concept of an *intuitionistic fuzzy set* (IFS for short) defined on a non-empty set S as objects having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in S\},$$

where the functions $\mu_A : S \to [0,1]$ and $\gamma_A : S \to [0,1]$ denote the *degree of membership* (namely $\mu_A(x)$) and the *degree of nonmembership* (namely $\gamma_A(x)$) of each element $x \in S$ to the set A respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in S$.

Such defined objects are studied by many authors (see for example two journals: 1. *Fuzzy Sets and Systems* and 2. *Notes on Intuitionistic Fuzzy Sets*) and have many interesting applications not only in mathematics (see Chapter 5 in the book [3]). In particular, Kim, Dudek and Jun in [16] introduced the notion of an intuitionistic fuzzy subquasigroup of a quasigroup. Also in [17], [18], Kim and Jun introduced the concept of intuitionistic fuzzy (interior) ideals of semigroups.

3. Intuitionistic fuzzy bi-ideals

For the sake of simplicity, we shall use the symbol $A = \langle \mu_A, \gamma_A\rangle$ for the intuitionistic fuzzy set $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in S\}$. Let $\mathbf{0}_\sim = \langle \mathbf{0}, \mathbf{1}\rangle$ and $\mathbf{1}_\sim = \langle \mathbf{1}, \mathbf{0}\rangle$ be intuitionistic fuzzy sets in S.

We first consider the intuitionistic fuzzification of the notion of bi-ideals in an ordered semigroup as follows.

Definition 3.1. An IFS $A = \langle \mu_A, \gamma_A\rangle$ in an ordered semigroup $(S, \cdot, \leq)$ is called an *intuitionistic fuzzy bi-ideal* of S if it satisfies:

(i) $(\forall x, y, z \in S)\ (\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\})$,

(ii) $(\forall x, y, z \in S)\ (\gamma_A(xyz) \leq \max\{\gamma_A(x), \gamma_A(z)\})$.

(iii) $(\forall x, y \in S)\ (x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y),\ \gamma_A(x) \leq \gamma_A(y))$,

Theorem 3.2. *Every intuitionistic fuzzy bi-ideal of a regular ordered semigroup $(S, \cdot, \leq)$ is an intuitionistic fuzzy subsemigroup of S.*

Proof. Let $A = \langle \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy bi-ideal of a regular ordered semigroup S and let a and b be any elements of S. Since S is regular, there exists $x \in S$ such that $b \leq bxb$. Then we have

$$\mu_A(ab) \geq \mu_A(a(bxb)) = \mu_A(a(bx)b) \geq \min\{\mu_A(a),\ \mu_A(b)\}$$

and

$$\gamma_A(ab) \leq \gamma_A(a(bxb)) = \gamma_A(a(bx)b) \leq \max\{\gamma_A(a),\ \gamma_A(b)\}.$$

This means that $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy subsemigroup of S. $\square$

Proposition 3.3. *Let $(S, \cdot, \leq)$ be a regular ordered semigroup and let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-ideal of S. Then we have*

$$(\forall a \in S)\,(a \leq a^2 \;\Rightarrow\; \mu_A(a) = \mu_A(a^2),\; \gamma_A(a) = \gamma_A(a^2)).$$

Proof. Let $a \in S$ be such that $a \leq a^2$. Then

$$\mu_A(a) \geq \mu_A(a^2) = \mu_A(aa) \geq \min\{\mu_A(a),\, \mu_A(a)\} = \mu_A(a)$$

and

$$\gamma_A(a) \leq \gamma_A(a^2) = \gamma_A(aa) \leq \max\{\gamma_A(a),\, \gamma_A(a)\} = \gamma_A(a).$$

Hence $\mu_A(a) = \mu_A(a^2)$ and $\gamma_A(a) = \gamma_A(a^2)$. $\square$

Proposition 3.4. *Let $(S, \cdot, \leq)$ be an ordered semigroup such that*

(i) $(\forall x \in S)\ (x \leq x^2)$,

(ii) $(\forall a, b \in S)\ (ab \in (baS] \cap (Sba])$.

Then every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S satisfies the following condition

$$(\forall a, b \in S)\,(\mu_A(ab) = \mu_A(ba),\; \gamma_A(ab) = \gamma_A(ba)).$$

Proof. Since $ab \in (baS] \cap (Sba]$, we have $ab \in (baS]$ and so $ab \leq bax$ for some $x \in S$. Using (ii), we get $(ba)x \in (xbaS] \cap (Sxba]$ and thus $bax \leq yxba$ for some $y \in S$. It follows from (i) that

$$ab \leq (ba)x \leq (ba)^2 x = ba(bax) \leq ba(yxba)$$

so that

$$\begin{aligned}\mu_A(ab) &\geq \mu_A(ba(yxba)) = \mu_A((ba)(yx)(ba)) \\ &\geq \min\{\mu_A(ba),\, \mu_A(ba)\} = \mu_A(ba)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(ab) &\leq \gamma_A(ba(yxba)) = \gamma_A((ba)(yx)(ba)) \\ &\leq \max\{\gamma_A(ba),\, \gamma_A(ba)\} = \gamma_A(ba).\end{aligned}$$

The reverse inequalities are by symmetry. This completes the proof. $\square$

An ordered semigroup $(S, \cdot, \leq)$ is said to be *left* (resp. *right*) *regular* [12] if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp. $a \leq a^2x$). An ordered semigroup $(S, \cdot, \leq)$ is said to be *completely regular* [12] if it is regular, left regular and right regular.

Lemma 3.5 ([13]). *An ordered semigroup $(S, \cdot, \leq)$ is completely regular if and only if for every $A \subseteq S$, we have $A \subseteq (A^2SA^2]$.*

Proposition 3.6. *Let $(S, \cdot, \leq)$ be a completely regular ordered semigroup. For every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S, we have*

$$(\forall x \in S)\,(\mu_A(x) = \mu_A(x^2),\ \gamma_A(x) = \gamma_A(x^2)).$$

Proof. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-ideal of S and let $x \in S$. Since $x \in (x^2Sx^2]$ by Lemma 3.5, there exists $a \in S$ such that $x \leq x^2ax^2$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S, it follows that

$$\begin{aligned}\mu_A(x) &\geq \mu_A(x^2ax^2) \geq \min\{\mu_A(x^2),\, \mu_A(x^2)\} \\ &= \mu_A(x^2) = \mu_A(xx) \geq \min\{\mu_A(x),\, \mu_A(x)\} = \mu_A(x)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(x) &\leq \gamma_A(x^2ax^2) \leq \max\{\gamma_A(x^2),\, \gamma_A(x^2)\} \\ &= \gamma_A(x^2) = \gamma_A(xx) \leq \max\{\gamma_A(x),\, \gamma_A(x)\} = \gamma_A(x)\end{aligned}$$

so that $\mu_A(x) = \mu_A(x^2)$ and $\gamma_A(x) = \gamma_A(x^2)$. □

Lemma 3.7 ([11]). *An ordered semigroup $(S, \cdot, \leq)$ is left (resp. right) simple if and only if $(Sa] = S$ (resp. $(aS] = S$) for every $a \in S$.*

Theorem 3.8. *Let $(S, \cdot, \leq)$ be an ordered semigroup. If S is regular, left and right simple, then every intuitionistic fuzzy bi-ideal of S is constant.*

Proof. Assume that S is regular, left and right simple. Let $A = \langle \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy bi-ideal of S. Consider the set

$$\Omega_S := \{e \in S \mid e \leq e^2\}.$$

Since S is regular, for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. For the element $ax \in S$, we have $ax \leq (axa)x = (ax)^2$, and so $ax \in \Omega_S$. This means that $\Omega_S \neq \emptyset$. Let $b \in \Omega_S$. Since $b \in S$, we have $\mu_A(b) \in [0, 1]$ and $\gamma_A(b) \in [0, 1]$. We first show that $\mu_A(b) = \mu_A(e)$ and $\gamma_A(b) = \gamma_A(e)$ for all $e \in \Omega_S$. Let $e \in \Omega_S$. Since S is left and right simple, it follows from Lemma 3.7 that $(Sb] = S$ and $(bS] = S$. Since $e \in S$, we have $e \in (Sb]$ and $e \in (bS]$. Thus $e \leq bx$ and $e \leq yb$ for some $x, y \in S$, which imply that $e^2 \leq (bx)e \leq (bx)(yb) = b(xy)b$. Since $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal, we obtain

$$\begin{aligned}(1) \qquad \mu_A(e^2) \geq \mu_A(b(xy)b) \geq \min\{\mu_A(b),\, \mu_A(b)\} &= \mu_A(b) \\ \gamma_A(e^2) \leq \gamma_A(b(xy)b) \leq \max\{\gamma_A(b),\, \gamma_A(b)\} &= \gamma_A(b).\end{aligned}$$

Since $e \in \Omega_S$, we get $e \leq e^2$, and so $\mu_A(e^2) \leq \mu_A(e)$ and $\gamma_A(e^2) \geq \gamma_A(e)$. It follows from (1) that $\mu_A(e) \geq \mu_A(b)$ and $\gamma_A(e) \leq \gamma_A(b)$. On the other hand, we have

$(Se] = S = (eS]$ by Lemma 3.7. Similar argument as in the previous case induce $\mu_A(e) \le \mu_A(b)$ and $\gamma_A(e) \ge \gamma_A(b)$. This shows that $A = \langle \mu_A, \gamma_A \rangle$ is constant on Ω_S. Now Let $a \in S$. Then $a \le axa$ for some $x \in S$ since S is regular. It follows that $ax \le (axa)x = (ax)^2$ and $xa \le x(axa) = (xa)^2$ so that $ax, xa \in \Omega_S$. Thus, by the previous arguments, we have $\mu_A(ax) = \mu_A(b) = \mu_A(xa)$ and $\gamma_A(ax) = \gamma_A(b) = \gamma_A(xa)$. Since $(ax)a(xa) = (axa)xa \ge axa \ge a$, we get

$$\begin{aligned}\mu_A(a) \ge \mu_A((ax)a(xa)) &\ge \min\{\mu_A(ax), \mu_A(xa)\} \\ &= \min\{\mu_A(b), \mu_A(b)\} = \mu_A(b)\end{aligned}$$

and

$$\begin{aligned}\gamma_A(a) \le \gamma_A((ax)a(xa)) &\le \max\{\gamma_A(ax), \gamma_A(xa)\} \\ &= \max\{\gamma_A(b), \gamma_A(b)\} = \gamma_A(b).\end{aligned}$$

Note that $b \in (Sa]$ and $b \in (aS]$ and hence $b \le xa$ and $b \le ay$ for some $x, y \in S$. Hence $b^2 \le a(yx)a$ which implies

$$\mu_A(b^2) \ge \mu_A(a(yx)a) \ge \min\{\mu_A(a), \mu_A(a)\} = \mu_A(a)$$

and

$$\gamma_A(b^2) \le \gamma_A(a(yx)a) \le \max\{\gamma_A(a), \gamma_A(a)\} = \gamma_A(a).$$

Since $b \in \Omega_S$, we have $b \le b^2$. It follows that $\mu_A(b) \ge \mu_A(b^2) \ge \mu_A(a)$ and $\gamma_A(b) \le \gamma_A(b^2) \le \gamma_A(a)$. Hence $A = \langle \mu_A, \gamma_A \rangle$ is constant on S. □

Lemma 3.9. *Let $(S, \cdot, \le)$ be an ordered semigroup. A nonempty subset K of S is a bi-ideal of S if and only if the characteristic intuitionistic fuzzy set $K_\sim := \langle \chi_K, \bar{\chi}_K \rangle$ is an intuitionistic fuzzy bi-ideal of S, where χ_K is the characteristic function of K.*

Proof. Straightforward. □

Theorem 3.10. *Let $(S, \cdot, \le)$ be an ordered semigroup. If every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S satisfies either $\mu_A(x) = \mu_A(x^2)$ or $\gamma_A(x) = \gamma_A(x^2)$ for all $x \in S$, then S is completely regular.*

Proof. Assume that every intuitionistic fuzzy bi-ideal $A = \langle \mu_A, \gamma_A \rangle$ of S satisfies either $\mu_A(x) = \mu_A(x^2)$ or $\gamma_A(x) = \gamma_A(x^2)$ for all $x \in S$. Let $a \in S$. Note that $K := (a^2 \cup a^4 \cup a^2Sa^2]$ is the bi-ideal of S generated by a^2. Thus, by Lemma 3.9, the characteristic intuitionistic fuzzy set $K_\sim := \langle \chi_K, \bar{\chi}_K \rangle$ is an intuitionistic fuzzy bi-ideal of S and so $K_\sim := \langle \chi_K, \bar{\chi}_K \rangle$ satisfies either $\chi_K(a) = \chi_K(a^2)$ or $\bar{\chi}_K(a) = \bar{\chi}_K(a^2)$. Suppose that $K_\sim := \langle \chi_K, \bar{\chi}_K \rangle$ satisfies $\chi_K(a) = \chi_K(a^2)$. Since $a^2 \in K$, we have $\chi_K(a^2) = 1$ and so $\chi_K(a) = 1$. Thus $a \in K = (a^2 \cup a^4 \cup a^2Sa^2]$, which implies that $a \le x$ for some $x \in a^2 \cup a^4 \cup a^2Sa^2$. If $x = a^2$, then

$$a \le x = a^2 = aa \le a^2a^2 = aaa^2 \le a^2aa^2,$$

and so $a \in (a^2Sa^2]$. If $x = a^4$, then

$$a \leq x = a^4 = aaa^2 \leq a^4aa^2 \in a^2Sa^2,$$

and thus $a \in (a^2Sa^2]$. If $x \in a^2Sa^2$, then obviously $a \in (a^2Sa^2]$. This shows that $K \subseteq (K^2SK^2]$. We obtain the same result in the case that $K_\sim := \langle \chi_K, \bar{\chi}_K \rangle$ satisfies $\bar{\chi}_K(a) = \bar{\chi}_K(a^2)$. Therefore S is completely regular by Lemma 3.5. □

Lemma 3.11. *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of an ordered semigroup $(S, \cdot, \leq)$, then so are $\Box A = \langle \mu_A, \bar{\mu}_A \rangle$ and $\Diamond A = \langle \bar{\gamma}_A, \gamma_A \rangle$.*

Proof. Let $x,\ y \in S$ be such that $x \leq y$. Then $\mu_A(x) \geq \mu_A(y)$, and thus

$$\bar{\mu}_A(x) = 1 - \mu_A(x) \leq 1 - \mu_A(y) = \bar{\mu}_A(y).$$

For any $x, y, z \in S$, we get $\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\}$, which implies

$$\begin{aligned}\bar{\mu}_A(xyz) &= 1 - \mu_A(xyz) \leq 1 - \min\{\mu_A(x), \mu_A(z)\} \\ &= \max\{1 - \mu_A(x), 1 - \mu_A(z)\} = \max\{\bar{\mu}_A(x), \bar{\mu}_A(z)\}.\end{aligned}$$

The proof of second part is similar to the first part. This completes the proof. □

According to the above lemma, it is not difficult to see that the following theorem is valid.

Theorem 3.12. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S if and only if $\Box A$ and $\Diamond A$ are intuitionistic fuzzy bi-ideals of S.*

Corollary 3.13. *Let $(S, \cdot, \leq)$ be an ordered semigroup. Then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S if and only if μ_A and $\bar{\gamma}_A$ are fuzzy bi-ideals of S.*

Theorem 3.14. *If $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of an ordered semigroup $(S, \cdot, \leq)$ then the upper t-level cut $U(\mu_A; t)$ of μ_A and the lower t-level cut $L(\gamma_A; t)$ of γ_A are bi-ideals of S for every $t \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A) \cap [0, 0.5]$.*

Proof. Let $t \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A) \cap [0, 0.5]$ and let $x, y, z \in S$ be such that $x, z \in U(\mu_A; t)$. Then $\mu_A(x) \geq t$ and $\mu_A(z) \geq t$. Hence

$$\mu_A(xyz) \geq \min\{\mu_A(x), \mu_A(z)\} \geq t$$

and so $xyz \in U(\mu_A; t)$. Now let $x \in U(\mu_A; t)$ and $y \in S$ be such that $y \leq x$. Then $\mu_A(x) \geq t$ and hence $\mu_A(y) \geq \mu_A(x) \geq t$. Thus $y \in U(\mu_A; t)$. Therefore $U(\mu_A; t)$ is a bi-ideal of S. Let $x, z \in L(\gamma_A; t)$ and $y \in S$. Then $\gamma_A(x) \leq t$ and $\gamma_A(z) \leq t$, which imply that

$$\gamma_A(xyz) \leq \max\{\gamma_A(x), \gamma_A(z)\} \leq t.$$

Hence $xyz \in L(\gamma_A; t)$. Let $x \in L(\gamma_A; t)$ and $y \in S$ be such that $y \leq x$. Then $\gamma_A(y) \leq \gamma_A(x) \leq t$, and so $y \in L(\gamma_A; t)$. Consequently $L(\gamma_A; t)$ is a bi-ideal of S. □

Theorem 3.15. *If $A = \langle \mu_A, \gamma_A \rangle$ is an IFS in an ordered semigroup $(S, \cdot, \leq)$ such that the nonempty sets $U(\mu_A; t)$ and $L(\gamma_A; t)$ are bi-ideals of S for all $t \in [0, 0.5]$, then $A = \langle \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy bi-ideal of S.*

Proof. For $t \in [0, 0.5]$, assume that $U(\mu_A; t) \neq \emptyset$ and $L(\gamma_A; t) \neq \emptyset$ are bi-ideals of S. Let $x, y, z \in S$. We put $t_3 = \min\{\mu_A(x), \mu_A(z)\}$ and $t_4 = \max\{\gamma_A(x), \gamma_A(z)\}$. Then $x, z \in U(\mu_A; t_3)$ and $x, z \in L(\gamma_A; t_4)$, which imply that $xyz \in U(\mu_A; t_3)$ and $xyz \in L(\gamma_A; t_4)$. It follows that

$$\mu_A(xyz) \geq t_3 = \min\{\mu_A(x), \mu_A(z)\}$$

and

$$\gamma_A(xyz) \leq t_4 = \max\{\gamma_A(x), \gamma_A(z)\}.$$

Let $x, y \in S$ be such that $x \leq y$. If $\mu_A(x) < \mu_A(y)$, then $\mu_A(x) < t_5 < \mu_A(y)$ for some t_5. Hence $y \in U(\mu_A; t_5)$ and $x \notin U(\mu_A; t_5)$, a contradiction. If $\gamma_A(x) > \gamma_A(y)$, then $\gamma_A(y) < t_6 < \gamma_A(x)$ for some t_6, and thus $y \in L(\gamma_A; t_6)$ but $x \notin L(\gamma_A; t_6)$. This is also a contradiction. Therefore $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$. This completes the proof. □

Corollary 3.16. *Let K be a bi-ideal of an ordered semigroup $(S, \cdot, \leq)$. If fuzzy sets μ and γ in S are defined by*

$$\mu(x) := \begin{cases} \alpha_0 & \text{if } x \in K, \\ \alpha_1 & \text{if } x \in S \setminus K, \end{cases} \qquad \gamma(x) := \begin{cases} \beta_0 & \text{if } x \in K, \\ \beta_1 & \text{if } x \in S \setminus K, \end{cases}$$

where $0 \leq \alpha_1 < \alpha_0$, $0 \leq \beta_0 < \beta_1$ and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$, then $A = \langle \mu, \gamma \rangle$ is an intuitionistic fuzzy bi-ideal of S and $U(\mu; \alpha_0) = K = L(\gamma; \beta_0)$.

Theorem 3.17. *Let Ω be a nonempty finite subset of $[0, 0.5]$. If $\{K_\alpha \mid \alpha \in \Omega\}$ is a collection of bi-ideals of S such that*

(i) $S = \bigcup_{\alpha \in \Omega} K_\alpha$,

(ii) $(\forall \alpha, \beta \in \Omega)$ $(\alpha > \beta \Leftrightarrow K_\alpha \subset K_\beta)$,

then an IFS $A = \langle \mu_A, \gamma_A \rangle$ in S defined by $\mu_A(x) = \bigvee\{\alpha \in \Omega \mid x \in K_\alpha\}$ and $\gamma_A(x) = \bigwedge\{\alpha \in \Omega \mid x \in K_\alpha\}$ is an intuitionistic fuzzy bi-ideal of S.

Proof. According to Theorem 3.15, it is sufficient to show that the nonempty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \beta)$ are bi-ideals of S, where $\alpha + \beta \leq 1$. We show that $U(\mu_A; \alpha) = K_\alpha$. Note that

$$\begin{aligned} x \in U(\mu_A; \alpha) &\iff \mu_A(x) \geq \alpha \\ &\iff \bigvee\{\delta \in \Omega \mid x \in K_\delta\} \geq \alpha \\ &\iff \exists \delta_0 \in \Omega,\ x \in K_{\delta_0},\ \delta_0 \geq \alpha \\ &\iff x \in K_\alpha \quad (\text{since } K_{\delta_0} \subseteq K_\alpha). \end{aligned}$$

Thus $U(\mu_A;\alpha) = K_\alpha$. Now, we prove that $L(\gamma_A;\beta) \neq \emptyset$ is a bi-ideal of S. We have

$$\begin{aligned} x \in L(\gamma_A;\beta) &\iff \gamma_A(x) \leq \beta \\ &\iff \bigwedge\{\delta \in \Omega \mid x \in K_\delta\} \leq \beta \\ &\iff \exists \delta_0 \in \Omega,\ x \in K_{\delta_0},\ \delta_0 \leq \beta \\ &\iff x \in \bigcup_{\delta \leq \beta} K_\delta \end{aligned}$$

and hence $L(\gamma_A;\beta) = \bigcup_{\delta\leq\beta} K_\delta$, which is a bi-ideal of S. This completes the proof.□

4. Relations

Let $\alpha \in [0,1]$ be fixed and let $IFBI(S)$ be the family of all intuitionistic fuzzy bi-ideals of an ordered semigroup $(S,\cdot,\leq)$. For any $A = (\mu_A,\gamma_A)$ and $B = (\mu_B,\gamma_B)$ from $IFBI(S)$ we define two binary relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ on $IFBI(S)$ as follows:

$$(A,B) \in \mathfrak{U}^\alpha \iff U(\mu_A;\alpha) = U(\mu_B;\alpha)$$

and

$$(A,B) \in \mathfrak{L}^\alpha \iff L(\gamma_A;\alpha) = L(\gamma_B;\alpha).$$

These two relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ are equivalence relations. Hence $IFBI(S)$ can be divided into the equivalence classes of $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$, denoted by $[A]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha}$ for any $A = (\mu_A,\gamma_A) \in IFBI(S)$, respectively. The corresponding quotient sets will be denoted by $IFBI(S)/\mathfrak{U}^\alpha$ and $IFBI(S)/\mathfrak{L}^\alpha$, respectively.

For the family $BI(S)$ of all bi-ideals of an ordered semigroup $(S,\cdot,\leq)$ we define two maps U_α and L_α from $IFBI(S)$ to $BI(S) \cup \{\emptyset\}$ by putting

$$U_\alpha(A) = U(\mu_A;\alpha) \quad \text{and} \quad L_\alpha(A) = L(\gamma_A;\alpha)$$

for each $A = (\mu_A,\gamma_A) \in IFBI(S)$.

It is not difficult to see that these maps are well-defined.

Lemma 4.1. *For any $\alpha \in (0,1)$ the maps U_α and L_α are surjective.*

Proof. Note that $\mathbf{0}_\sim = (\mathbf{0},\mathbf{1}) \in IFBI(S)$ and $U_\alpha(\mathbf{0}_\sim) = L_\alpha(\mathbf{0}_\sim) = \emptyset$ for any $\alpha \in (0,1)$. Moreover for any $K \in BI(S)$ we have $K_\sim = (\chi_K,\bar{\chi}_K) \in IFBI(S)$, $U_\alpha(K_\sim) = U(\chi_K;\alpha) = K$ and $L_\alpha(K_\sim) = L(\bar{\chi}_K;\alpha) = K$. Hence U_α and L_α are surjective. □

Theorem 4.2. *For any $\alpha \in (0,1)$ the sets $IFBI(S)/\mathfrak{U}^\alpha$ and $IFBI(S)/\mathfrak{L}^\alpha$ are equipotent to $BI(S) \cup \{\emptyset\}$.*

Proof. Let $\alpha \in (0,1)$. Putting $U^*_\alpha([A]_{\mathfrak{U}^\alpha}) = U_\alpha(A)$ and $L^*_\alpha([A]_{\mathfrak{L}^\alpha}) = L_\alpha(A)$ for any $A = (\mu_A,\gamma_A) \in IFBI(S)$, we obtain two maps

$$U^*_\alpha : IFBI(S)/\mathfrak{U}^\alpha \to BI(S) \cup \{\emptyset\} \ \text{ and } \ L^*_\alpha : IFBI(S)/\mathfrak{L}^\alpha \to BI(S) \cup \{\emptyset\}.$$

If $U(\mu_A;\alpha) = U(\mu_B;\alpha)$ and $L(\gamma_A;\alpha) = L(\gamma_B;\alpha)$ for some $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ from $IFBI(S)$, then $(A,B) \in \mathfrak{U}^\alpha$ and $(A,B) \in \mathfrak{L}^\alpha$, whence $[A]_{\mathfrak{U}^\alpha} = [B]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha} = [B]_{\mathfrak{L}^\alpha}$, which means that $U*_\alpha$ and L^*_α are injective.

To show that the maps U^*_α and L_α are surjective, let $K \in BI(S)$. Then for $K_\sim = (\chi_K, \bar{\chi}_K) \in IFBI(S)$ we have $U^*_\alpha([K_\sim]_{\mathfrak{U}^\alpha}) = U(\chi_K;\alpha) = K$ and $L^*_\alpha([K_\sim]_{\mathfrak{L}^\alpha}) = L(\bar{\chi}_K;\alpha) = K$. Also $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFBI(S)$. Moreover $U^*_\alpha([\mathbf{0}_\sim]_{\mathfrak{U}^\alpha}) = U(\mathbf{0};\alpha) = \emptyset$ and $L^*_\alpha([\mathbf{0}_\sim]_{\mathfrak{L}^\alpha}) = L(\mathbf{1};\alpha) = \emptyset$. Hence U^*_α and L^*_α are surjective. □

Now for any $\alpha \in [0,1]$ we define a new relation $\mathfrak{R}^\alpha$ on $IFBI(S)$ by putting:

$$(A,B) \in \mathfrak{R}^\alpha \Longleftrightarrow U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha),$$

where $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$. Obviously $\mathfrak{R}^\alpha$ is an equivalence relation.

Lemma 4.3. *The map* $I_\alpha : IFBI(S) \to BI(S) \cup \{\emptyset\}$ *defined by*

$$I_\alpha(A) = U(\mu_A;\alpha) \cap L(\gamma_A;\alpha),$$

where $A = (\mu_A, \gamma_A)$*, is surjective for any* $\alpha \in (0,1)$*.*

Proof. If $\alpha \in (0,1)$ is fixed, then for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFBI(S)$ we have

$$I_\alpha(\mathbf{0}_\sim) = U(\mathbf{0};\alpha) \cap L(\mathbf{1};\alpha) = \emptyset\,,$$

and for any $K \in BI(S)$ there exists $K_\sim = (\chi_K, \bar{\chi}_K) \in IFBI(S)$ such that $I_\alpha(K_\sim) = U(\chi_K;\alpha) \cap L(\bar{\chi}_K;\alpha) = K$. □

Theorem 4.4. *For any* $\alpha \in (0,1)$ *the quotient set* $IFBI(S)/\mathfrak{R}^\alpha$ *is equipotent to* $BI(S) \cup \{\emptyset\}$*.*

Proof. Let $I^*_\alpha : IFBI(S)/\mathfrak{R}^\alpha \to BI(S) \cup \{\emptyset\}$, where $\alpha \in (0,1)$, be defined by the formula:

$$I^*_\alpha([A]_{\mathfrak{R}^\alpha}) = I_\alpha(A) \quad \text{for each} \quad [A]_{\mathfrak{R}^\alpha} \in IFBI(S)/\mathfrak{R}^\alpha.$$

If $I^*_\alpha([A]_{\mathfrak{R}^\alpha}) = I^*_\alpha([B]_{\mathfrak{R}^\alpha})$ for some $[A]_{\mathfrak{R}^\alpha}$, $[B]_{\mathfrak{R}^\alpha} \in IFBI(S)/\mathfrak{R}^\alpha$, then

$$U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha),$$

which implies $(A,B) \in \mathfrak{R}^\alpha$ and, in the consequence, $[A]_{\mathfrak{R}^\alpha} = [B]_{\mathfrak{R}^\alpha}$. Thus I^*_α is injective. It is also onto because $I^*_\alpha(\mathbf{0}_\sim) = I_\alpha(\mathbf{0}_\sim) = \emptyset$ for $\mathbf{0}_\sim = (\mathbf{0}, \mathbf{1}) \in IFBI(S)$, and $I^*_\alpha(K_\sim) = I_\alpha(K) = K$ for $K \in BI(S)$ and $K_\sim = (\chi_K, \bar{\chi}_K) \in IFBI(S)$. □

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