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Eventually Regular Regressive Generalized Transformation Semigroups

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Abstract. Necessary and sufficient conditions have been provided for some standard regressive transformation semigroups on a poset to be eventually regular. Our main purpose is to generalize this result by characterizing when their generalized semigroups are eventually regular.

1. Introduction

For a semigroup S, let $E(S)$ and $RegS$ denote respectively the set of all idempotents and the set of all regular elements of S. That is,

$$\begin{aligned} E(S) &= \{ x \in S \mid x^2 = x \} \quad \text{and} \\ RegS &= \{ x \in S \mid x = xyx \ \text{ for some } \ y \in S \}. \end{aligned}$$

Then $E(S) \subseteq RegS$. An element $a \in S$ is said to be *eventually regular* if $a^n \in RegS$ for some positive integer n, and we call S an *eventually regular semigroup* if every element of S is eventually regular. Note that every regular semigroup and every finite semigroup is eventually regular.

For a set X, let $P(X)$, $T(X)$ and $I(X)$ be respectively the partial transformation semigroup on X, the full transformation semigroup on X and the 1-1 partial transformation semigroup on X (the symmetric inverse semigroup on X). It is known that all $P(X)$, $T(X)$ and $I(X)$ are regular. We denote the domain and the image of $\alpha \in P(X)$ by $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$, respectively. For $\alpha \in P(X), \alpha$ is said to be *almost identical* if $S(\alpha)$ is finite where $S(\alpha) = \{ x \in \operatorname{dom}\alpha \mid x\alpha \neq x\}$. Let

$$\begin{aligned} AP(X) &= \{ \alpha \in P(X) \mid \alpha \text{ is almost identical } \}, \\ AT(X) &= \{ \alpha \in T(X) \mid \alpha \text{ is almost identical } \}, \\ AI(X) &= \{ \alpha \in I(X) \mid \alpha \text{ is almost identical } \}. \end{aligned}$$

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It is known that $AP(X)$, $AT(X)$ and $AI(X)$ are regular subsemigroups of $P(X)$, $T(X)$ and $I(X)$, respectively.

Next, let X be a poset. A point $x \in X$ is said to be *isolated* if for $y \in X$, $y \leq x$ or $x \leq y$ implies $x = y$, and we call X *isolated* if every point $x \in X$ is an isolated point of X. An element $\alpha \in P(X)$ is said to be *regressive* if $x\alpha \leq x$ for all $x \in$ dom α. Let

$$\begin{aligned} P_{RE}(X) &= \{ \alpha \in P(X) \mid \alpha \text{ is regressive } \}, \\ AP_{RE}(X) &= \{ \alpha \in AP(X) \mid \alpha \text{ is regressive } \}. \end{aligned}$$

and $T_{RE}(X)$, $AT_{RE}(X)$, $I_{RE}(X)$ and $AI_{RE}(X)$ are defined similarly. Then $AP_{RE}(X) \subseteq P_{RE}(X) \subseteq P(X)$ and $P_{RE}(X)$ and $AP_{RE}(X)$ are subsemigroups of $P(X)$ and $AP(X)$, respectively. We obtain similar results for $T_{RE}(X)$, $AT_{RE}(X)$, $I_{RE}(X)$ and $AI_{RE}(X)$. By a *regressive transformation semigroup* on X and a *regressive almost identical transformation semigroup* on X we mean a subsemigroup of $P_{RE}(X)$ and a subsemigroup of $AP_{RE}(X)$, respectively.

Some known results of regressive transformation semigroups and regressive almost identical transformation semigroups are as follows: A. Umar [3] has shown that if X is a finite chain, then the subsemigroup $S = \{ \alpha \in T_{RE}(X) \mid |\text{im}\,\alpha| < |X| \}$ of $T_{RE}(X)$ is generated by $E(S)$ and S is not regular if $|X| \geq 3$. A. Umar [4] proved that if X and Y are chains, then $T_{RE}(X) \cong T_{RE}(Y)$ if and only if X and Y are order-isomorphic. To generalize this result, T. Saitô, K. Aoki and K. Kajitori [2] have given a necessary and sufficient condition for any posets X and Y so that $T_{RE}(X) \cong T_{RE}(Y)$. Y. Kemprasit [1] characterized when $P_{RE}(X), T_{RE}(X), I_{RE}(X), AP_{RE}(X), AT_{RE}(X)$ and $AI_{RE}(X)$ are regular and eventually regular where X is any poset.

Our main purpose is to generalize the result in [1] mentioned above by considering the semigroup $(S(X), \theta)$ where X is a poset, $S(X)$ is a subsemigroup of $P_{RE}(X)$, $\theta \in S(X)$ and the operation is $*$ defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in S(X)$. We call the semigroup $(S(X), \theta)$ a *regressive generalized transformation semigroup* on X, and it is called a *regressive almost identical generalized transformation semigroup* on X if $S(X) \subseteq AP_{RE}(X)$. To distinguish α^n in the semigroup $S(X)$ and the product $\alpha * \alpha \cdots * \alpha$ (n times) in the semigroup $(S(X), *) = (S(X), \theta)$ where $\alpha \in S(X)$ and n is a positive integer, we shall use $(\alpha, \theta)^n$ to denote the later product. For examples, $(\alpha, \theta)^2$ and $(\alpha, \theta)^4$ denote $\alpha\theta\alpha$ and $\alpha\theta\alpha\theta\alpha\theta\alpha$, respectively.

In the remainder, let X be any poset and $\mathbb{N}$ the set of natural numbers (positive integers).

2. Regular regressive generalized transformation semigroups

It is known from [1] that $RegS(X) = E(S(X))$ for every regressive transformation semigroup $S(X)$ on X. The first proposition shows that this is also true for all regressive generalized transformation semigroups on X.

Proposition 2.1. *If $S(X)$ is a regressive transformation semigroup on X, then*

$Reg(S(X),\theta) = E(S(X),\theta)$ for every $\theta \in S(X)$.

Proof. Let $\alpha \in Reg(S(X),\theta)$. Then $\alpha = \alpha\theta\beta\theta\alpha$ for some $\beta \in S(X)$ and thus for $x \in \operatorname{dom}\alpha$, $x\alpha = x\alpha\theta\beta\theta\alpha = (x\alpha\theta\beta)\theta\alpha \leq x\alpha\theta\beta = (x\alpha)\theta\beta \leq x\alpha$ which implies that $x\alpha = x\alpha\theta\beta$ for every $x \in \operatorname{dom}\alpha$. Hence $x\alpha = (x\alpha\theta\beta)\theta\alpha = x\alpha\theta\alpha = x(\alpha,\theta)^2$ for all $x \in \operatorname{dom}\alpha$. But $\operatorname{dom}\alpha\theta\alpha \subseteq \operatorname{dom}\alpha$, so $\alpha = (\alpha,\theta)^2 \in E(S(X),\theta)$. □

The following two lemmas from [1] are useful to characterize regular regressive generalized transformation semigroups in the next two theorems.

Lemma 2.2 ([1]). *Let $S(X)$ be $P_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$ or $AI_{RE}(X)$. Then $S(X)$ is regular if and only if X is isolated.*

Lemma 2.3 ([1]). *Let $S(X)$ be $T_{RE}(X)$ or $AT_{RE}(X)$. Then $S(X)$ is regular if and only if $|C| \leq 2$ for every chain C of X.*

Theorem 2.4. *Let $S(X)$ be $P_{RE}(X)$, $I_{RE}(X)$, $AP_{RE}(X)$ or $AI_{RE}(X)$ and $\theta \in S(X)$. Then $(S(X),\theta)$ is regular if and only if $\theta = 1_X$ and X is isolated.*

Proof. Assume that $(S(X),\theta)$ is regular. But $Reg(S(X),\theta) = E(S(X),\theta)$ by Proposition 2.1, so $(S(X),\theta) = E(S(X),\theta)$. Since $1_X \in S(X), 1_X = 1_X\theta 1_X$, so $\theta = 1_X$. Then $(S(X),\theta) = S(X)$. It thus follows from Lemma 2.2 that X is isolated.

The converse follows directly from Lemma 2.2. □

Theorem 2.5. *Let $S(X)$ be $T_{RE}(X)$ or $AT_{RE}(X)$ and $\theta \in S(X)$. Then $(S(X),\theta)$ is regular if and only if $\theta = 1_X$ and $|C| \leq 2$ for every chain C of X.*

Proof. Using Proposition 2.1 and Lemma 2.3, the proof of the theorem can be given similarly to that of Theorem 2.4. □

3. Main results

To obtain the first main result, the following two lemmas are required.

Lemma 3.1. *Let $\theta \in P_{RE}(X)$. If there exists a positive integer n such that every chain of X of the form*

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq \cdots \geq x_m \geq x_m\theta$$

with $x_i \in$ dom θ has length at most n, then for every $\alpha \in P_{RE}(X)$, $(\alpha,\theta)^{n+2} \in E(P_{RE}(X),\theta)$.

Proof. Let $\alpha \in P_{RE}(X)$ and $x \in \operatorname{dom}(\alpha\theta)^{n+1}$. By assumption, the chain

$$x\alpha \geq x\alpha\theta \geq x\alpha\theta\alpha \geq \cdots \geq x(\alpha\theta)^n\alpha \geq x(\alpha\theta)^{n+1}$$

has length at most n, so its subchain

$$x\alpha\theta \geq x(\alpha\theta)^2 \geq \cdots \geq x(\alpha\theta)^{n+1}$$

has length at most n. This implies that $x(\alpha\theta)^i = x(\alpha\theta)^{i+1}$ for some $i \in \{1, 2, \cdots, n\}$. But $x \in \mathrm{dom}(\alpha\theta)^{n+1}$, so $x(\alpha\theta)^i \in \mathrm{dom}(\alpha\theta)^{n+1-i}$. We then deduce that $x(\alpha\theta)^{n+1} = x(\alpha\theta)^i(\alpha\theta)^{n+1-i} = x(\alpha\theta)^{i+1}(\alpha\theta)^{n+1-i} = x(\alpha\theta)^{n+2}$. Since x is arbitrary in $\mathrm{dom}(\alpha\theta)^{n+1}$ and $\mathrm{dom}(\alpha\theta)^{n+2} \subseteq \mathrm{dom}(\alpha\theta)^{n+1}$, it follows that $(\alpha\theta)^{n+1} = (\alpha\theta)^{n+2}$. Thus $(\alpha, \theta)^{n+2} = (\alpha\theta)^{n+1}\alpha = (\alpha\theta)^{n+2}\alpha = (\alpha, \theta)^{n+3}$. Consequently, $(\alpha, \theta)^{n+2} \in E(P_{RE}(X), \theta)$. $\square$

The following corollary is a direct consequence of Lemma 3.1.

Corollary 3.2. *Let $(S(X), \theta)$ be a regressive generalized transformation semigroup on X. If there exists a positive integer n such that every chain of X of the form*

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq \cdots \geq x_m \geq x_m\theta$$

with $x_i \in dom\ \theta$ has length at most n, then $(S(X), \theta)$ is eventually regular. In particular, if $im\,\theta$ is finite, then $(S(X), \theta)$ is eventually regular.

Lemma 3.3. *Let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S(X)$. If X contains a sequence of pairwise disjoint finite chains $C_1, C_2, C_3, \cdots$ such that each C_i is of the form*

$$x_1 \geq x_1\theta > x_2 \geq x_2\theta > \cdots > x_{k_i} \geq x_{k_i}\theta,$$

with $x_i \in dom\ \theta$ and $k_1 < k_2 < k_3 < \cdots$, then $(S(X), \theta)$ is not eventually regular.

Proof. For each $i \in \mathbb{N}$, let

$$\begin{aligned} C_i \;=\; & \{\, x_{i1}, x_{i1}\theta, x_{i2}, x_{i2}\theta, \cdots, x_{ik_i}, x_{ik_i}\theta \,\} \\ & \text{where } x_{i1} \geq x_{i1}\theta > x_{i2} \geq x_{i2}\theta > \cdots > x_{ik_i} \geq x_{ik_i}\theta. \end{aligned}$$

We assume that $k_1 \geq 2$, otherwise we consider the sequence $C_2, C_3, C_4, \cdots$ instead. To show that $(S(X), \theta)$ is not eventually regular, define α from the set $\bigcup_{i=1}^{\infty}\{x_{i1}\theta, x_{i2}\theta, \cdots, x_{i,k_i-1}\theta\}$ onto the set $\bigcup_{i=1}^{\infty}\{x_{i2}, x_{i3}, \cdots, x_{i,k_i}\}$ by

$$(x_{ij}\theta)\alpha = x_{i,j+1} \ \text{ for } \ i \in \mathbb{N} \ \text{ and } \ j \in \{1, 2, \cdots, k_i - 1\}.$$

The map α is well-defined because $C_1, C_2, C_3, \cdots$ are pairwise disjoint. Then $\alpha \in I_{RE}(X)$. Let $n \in \mathbb{N}$. Since the sequence $k_1, k_2, k_3, \cdots$ of positive integers is strictly increasing, $k_m > 2n$ for some $m \in \mathbb{N}$. We then deduce that

$$\begin{aligned} (x_{m1}\theta)(\alpha, \theta)^n \;&=\; (x_{m1}\theta)(\alpha\theta)^{n-1}\alpha \;=\; x_{m,n+1} \\ &>\; x_{m,2n+1} \;=\; (x_{m1}\theta)(\alpha, \theta)^{2n}. \end{aligned}$$

This proves that $(\alpha, \theta)^n \neq (\alpha, \theta)^{2n}$ for every $n \in \mathbb{N}$. Thus $(\alpha, \theta)^n \notin E(P_{RE}(X), \theta)$ for every $n \in \mathbb{N}$. By Proposition 2.1, α is not eventually regular in $(S(X), \theta)$ if $S(X)$ is $P_{RE}(X)$ or $I_{RE}(X)$.

Next, assume that $S(X) = T_{RE}(X)$. Then $\theta \in T_{RE}(X) \subseteq P_{RE}(X)$. Let $\beta\colon X \to$

X be defined by $x\beta = x\alpha$ if $x \in \operatorname{dom}\alpha$ and $x\beta = x$ if $x \in X \setminus \operatorname{dom}\alpha$. Then $\beta \in T_{RE}(X)$. If $n \in \mathbb{N}$, from the above proof, there exists an element $y \in \operatorname{dom}\alpha$ such that $y(\alpha,\theta)^n > y(\alpha,\theta)^{2n}$, that is, $y(\alpha\theta)^{n-1}\alpha > y(\alpha\theta)^{2n-1}\alpha$. Consequently, $y(\beta\theta)^{n-1}\beta = y(\alpha\theta)^{n-1}\alpha > y(\alpha\theta)^{2n-1}\alpha = y(\beta\theta)^{2n-1}\beta$ which implies that $(\beta,\theta)^n \neq (\beta,\theta)^{2n}$. We therefore have from Proposition 2.1 that β is not eventually regular in $(T_{RE}(X),\theta)$. $\square$

Lemma 3.3 gives a remarkable result as follows :

Corollary 3.4. *Let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$. If X does not have a minimal element, then $(S(X),\theta)$ is not eventually regular for every $\theta \in S(X)$ with $dom\theta = X$.*

Proof. Let $x_1 \in X$. Thus $x_1 \geq x_1\theta$. By assumption, $x_1\theta$ is not a minimal element, so $x_1\theta > x_2$ for some $x_2 \in X$. Then $x_1 \geq x_1\theta > x_2 \geq x_2\theta$. By this process, we obtain a sequence

$$x_1 \geq x_1\theta > x_2 \geq x_2\theta > x_3 \geq x_3\theta > \cdots .$$

Let (k_n) be a strictly increasing sequence of positive integers such that $k_1 > 1$ and let $l_i = k_1 + k_2 + \cdots + k_i$ for all $i \in \mathbb{N}$. Define the chain C_i for $i \in \mathbb{N}$ as follows:

$$\begin{aligned} C_1 &= \{x_1, x_1\theta, \cdots, x_{l_1}, x_{l_1}\theta\} \\ C_2 &= \{x_{l_1+1}, x_{l_1+1}\theta, \cdots, x_{l_2}, x_{l_2}\theta\} \\ C_3 &= \{x_{l_2+1}, x_{l_2+1}\theta, \cdots, x_{l_3}, x_{l_3}\theta\} \\ &\vdots \end{aligned}$$

Then each C_i is a finite chain of X, $C_i \cap C_j = \phi$ if $i \neq j$ and each C_i is of the form $y_1 \geq y_1\theta > y_2 \geq y_2\theta > \cdots > y_{k_i} \geq y_{k_i}\theta$. Therefore we have from Lemma 3.3 that $(S(X),\theta)$ is not eventually regular. $\square$

Now we are ready to give the first main result.

Theorem 3.5. *Let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$ and $\theta \in S(X)$. Then $(S(X),\theta)$ is eventually regular if and only if there exists a positive integer n such that every chain of X of the form*

$$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq \cdots \geq x_m \geq x_m\theta \tag{1}$$

with $x_i \in dom\ \theta$ has length at most n.

Proof. If there is an element $n \in \mathbb{N}$ such that every chain of X of the form (1) has length at most n, then by Corollary 3.2, $(S(X),\theta)$ is eventually regular.

It is clear that the chain (1) can be revised as follows : If there is an $i \in \mathbb{N}$ such that $x_i = x_i\theta = x_{i+1}$ in (1), then we can replace $x_i = x_i\theta = x_{i+1}$ by x_{i+1} and the revised chain is still of the form (1). Also, if there is an $i \in \mathbb{N}$ such that

$x_i\theta = x_{i+1} = x_{i+1}\theta$ in (1), then this can be replaced by $x_i\theta$ and the revised chain is still of the form (1). Hence (1) can be considered as

(2) $$x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq \cdots \geq x_m \geq x_m\theta$$
with $x_i \in \operatorname{dom}\theta$ and any three consecutive terms not identical.

To prove necessity by contrapositive, assume that

(3) for every $n \in \mathbb{N}$, there is a chain of X of the form (2) of length greater than n.

In the remainder of the proof of all x_i and x_{ij} which we use always belong to dom θ. First suppose that

(4) X does not contain any chain of the form $x_1 \geq x_1\theta > x_2 \geq x_2\theta$.

Let $C^{(1)}$ be a chain of X of the form (2). If $C^{(1)}$ contains a chain $x_1 \geq x_1\theta \geq x_2 \geq x_2\theta$, then by (2) and (4), $x_1 > x_1\theta = x_2 > x_2\theta$. If $C^{(1)}$ contains a chain $x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq \cdots \geq x_m \geq x_m\theta$ with $m \geq 3$, then by (2) and (4), $x_1 > x_1\theta = x_2 > x_2\theta = x_3 > x_3\theta$, and so $x_1 > x_1\theta > x_3 > x_3\theta$ which is contrary to (4). Hence $|C^{(1)}| \leq 3$. This shows that every chain of X of the form (2) has length at most 3 which contradicts (3). Then X contains a chain

$$C_1 = \{x_{11}, x_{11}\theta, x_{12}, x_{12}\theta\} \text{ where } x_{11} \geq x_{11}\theta > x_{12} \geq x_{12}\theta.$$

Next, consider the subposet $X \smallsetminus C_1$ of X and suppose

(5) $X \smallsetminus C_1$ does not contain any chain of the form
$$x_1 \geq x_1\theta > x_2 \geq x_2\theta > x_3 \geq x_3\theta.$$

Let $C^{(2)}$ be a chain of $X \smallsetminus C_1$ the form (2). If $C^{(2)}$ contains $x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq x_3 \geq x_3\theta \geq x_4 \geq x_4\theta$, then by (5), there are at least 2 of $x_1\theta \geq x_2$, $x_2\theta \geq x_3$ and $x_3\theta \geq x_4$ must be equalities. If $x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq \cdots \geq x_m \geq x_m\theta$ with $m \geq 5$ is a chain in $C^{(2)}$, then $x_1\theta > x_3$ and $x_3\theta > x_5$ since any three consecutive terms of the chain cannot be identical, and hence $x_1 \geq x_1\theta > x_3 \geq x_3\theta > x_5 \geq x_5\theta$ which contradicts (5). Thus $|C^{(2)}| \leq 6$. We therefore deduce that every chain of $X \smallsetminus C_1$ of the form (2) has length at most 6. Consequently, every chain of X of the form (2) has length at most $|C_1| + 6$ which is contrary to (3). Thus $X \smallsetminus C_1$ contains a chain

$$C_2 = \{x_{21}, x_{21}\theta, x_{22}, x_{22}\theta, x_{23}, x_{23}\theta\}$$
$$\text{where } x_{21} \geq x_{21}\theta > x_{22} \geq x_{22}\theta > x_{23} \geq x_{23}\theta.$$

Thus $C_1 \cap C_2 = \emptyset$. For one more step, consider the subposet $X \smallsetminus (C_1 \cup C_2)$. Suppose that

(6) $X \smallsetminus (C_1 \cup C_2)$ does not contain any chain of the form
$$x_1 \geq x_1\theta > x_2 \geq x_2\theta > x_3 \geq x_3\theta > x_4 \geq x_4\theta.$$

Let $C^{(3)}$ be a chain of $X \smallsetminus (C_1 \cup C_2)$ of the form (2). If $C^{(3)}$ contains a chain $x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq x_3 \geq x_3\theta \geq x_4 \geq x_4\theta \geq x_5 \geq x_5\theta \geq x_6 \geq x_6\theta$, then by (6), there are at least 3 of $x_1\theta \geq x_2$, $x_2\theta \geq x_3$, $x_3\theta \geq x_4$, $x_4\theta \geq x_5$ and $x_5\theta \geq x_6$ must be equalities. If $x_1 \geq x_1\theta \geq x_2 \geq x_2\theta \geq \cdots \geq x_m \geq x_m\theta$ with $m \geq 7$, then $x_1\theta > x_3$, $x_3\theta > x_5$ and $x_5\theta > x_7$ since any three consecutive terms of the chain cannot be identical. This implies that $X \smallsetminus (C_1 \cup C_2)$ contains a chain $x_1 \geq x_1\theta > x_3 \geq x_3\theta > x_5 \geq x_5\theta > x_7 \geq x_7\theta$ which contradicts (6). Hence $|C^{(3)}| \leq 9$. Hence every chain of $X \smallsetminus (C_1 \cup C_2)$ of the form (2) has length at most 9, and thus every chain of X of the form (2) has length at most $|C_1| + |C_2| + 9$. This is a contradiction because of (3), and therefore $X \smallsetminus (C_1 \cup C_2)$ contains a chain

$$\begin{aligned} &C_3 = \{x_{31}, x_{31}\theta, x_{32}, x_{32}\theta, x_{33}, x_{33}\theta, x_{34}, x_{34}\theta\} \\ &\text{where } x_{31} \geq x_{31}\theta > x_{32} \geq x_{32}\theta > x_{33} \geq x_{33}\theta > x_{34} \geq x_{34}\theta. \end{aligned}$$

Then C_1, C_2 and C_3 are pairwise disjoint. By this process, we obtain a sequence of pairwise disjoint finite chains $C_1,\ C_2,\ C_3, \cdots$ of X such that each C_i is the chain

$$x_{i1} \geq x_{i1}\theta > x_{i2} \geq x_{i2}\theta > \cdots > x_{i,i+1} \geq x_{i,i+1}\theta.$$

We then deduce from Lemma 3.3 that $(S(X), \theta)$ is not eventually regular.

Hence the theorem is completely proved. □

It is easily seen that the following two statements on X are equivalent.

(i) There is a positive integer n such that every chain of the form $x_1 \geq x_2 \geq \cdots \geq x_m$ has length at most n.

(ii) There is a positive integer n such that $|C| \leq n$ for every chain C of X.

Hence the following result given in [1] becomes our special case.

Corollary 3.6. *Let $S(X)$ be $P_{RE}(X)$, $T_{RE}(X)$ or $I_{RE}(X)$. Then $S(X)$ is eventually regular if and only if there is a positive integer n such that $|C| \leq n$ for every chain C of X.*

It was proved in [1] that every regressive almost identical transformation semigroup on X is always eventually regular. Our second main purpose is to show that this is also true for regressive almost identical generalized transformation semigroups. It then follows that the known result mentioned above is a consequence of our second main result.

Theorem 3.7. *If $S(X)$ is a regressive almost identical transformation semigroup on X, then $(S(X), \theta)$ is eventually regular for every $\theta \in S(X)$.*

Proof. Let $\alpha \in S(X)$. Then $\alpha\theta \in S(X)$, so $S(\alpha\theta)$ is finite, say $|S(\alpha\theta)| = n$. Let $x \in \operatorname{dom}(\alpha\theta)^{n+2}$. Then

$$x(\alpha\theta) \geq x(\alpha\theta)^2 \geq \cdots \geq x(\alpha\theta)^{n+2}.$$

If $x(\alpha\theta) > x(\alpha\theta)^2 > \cdots > x(\alpha\theta)^{n+2}$, then $\{x(\alpha\theta), x(\alpha\theta)^2, \cdots, x(\alpha\theta)^{n+1}\} \subseteq S(\alpha\theta)$ and $\left|\{x(\alpha\theta), x(\alpha\theta)^2, \cdots, x(\alpha\theta)^{n+1}\}\right| = n+1$, a contradiction. Thus $x(\alpha\theta)^i = x(\alpha\theta)^{i+1}$ for some $i \in \{1, 2, \cdots, n+1\}$. Since $x \in \operatorname{dom}(\alpha\theta)^{n+2}$, $x(\alpha\theta)^i \in$

dom $(\alpha\theta)^{n+2-i}$, so $x(\alpha\theta)^{n+2} = x(\alpha\theta)^{n+3}$. But x is arbitrary in dom $(\alpha\theta)^{n+2}$ and dom $(\alpha\theta)^{n+3} \subseteq$ dom $(\alpha\theta)^{n+2}$, so we have $(\alpha\theta)^{n+2} = (\alpha\theta)^{n+3}$. Hence $(\alpha,\theta)^{n+3} = (\alpha\theta)^{n+2}\alpha = (\alpha\theta)^{n+3}\alpha = (\alpha,\theta)^{n+4}$, and thus $(\alpha,\theta)^{n+3} \in E(S(X),\theta)$.

Therefore the proof is complete. □

Corollary 3.8. *If $S(X)$ is a regressive almost identical transformation semigroup on X , then $S(X)$ is eventually regular. In particular, $AP_{RE}(X)$, $AT_{RE}(X)$ and $AI_{RE}(X)$ are all eventually regular.*

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