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A Note on Central Separable Cancellative Semialgebras

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Abstract. Here we define Central separable semialgebras and to prove some structure theorems for central separable cancellative, semialgebras over a commutative and cancellative semiring.

0. Introduction

In this paper our main aim is to define, Central separable seimalgebras and to develop partially, the structure theory for central separable cancellative, semialgebras over a commutative, cancellative semiring, so that one can give some computer applications in the theory of Brauer group of a commutative, cancellative semiring R.

While developing the structure theory for central separable cancellative, semialgebras, we use some results, which we have proved in [6].

1. Preliminaries

A set R together with two binary operations called addition $(+)$ and multiplication $(\cdot)$ is called a semiring, provided $(R,+)$ is an additive abelian monoid with identity element $O_R, (R,\cdot)$ is a semigroup and multiplication distributes over addition from left and from the right.

An element a of a semiring R is said to be cancellable if and only if $a+b = a+c \Rightarrow b = c$. Denote the set of all cancellable element of R by $K^+(R)$. If $K^+(R) = R$, then the semiring R is called a cancellative semiring.

An R-semialgebra A is a semiring which is also an R-semimodule satisfying the condition $a(xy) = (ax)y = x(ay)$ for any a in R and $x,\ y$ in A.

Let R be a semiring. An R-semialgebra A is said to be semisubtractive if it is a semisubtractive semimodule, that is, for any arbitrary $x \neq y$ in A, either $x+u = y$

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for some $u \in A$ or $x = y + v$ for some $v \in A$, and is said to be zerosumfree if it is a zerosumfree semimodule i.e., $m + m' = O_M$ implies that $m = m' = 0$.

Let R be any cancellative semiring. Define a relation "$(a, b) \sim (c, d)$" if and only if $a + d = b + c$. Then '$\sim$' is an equivalence relation on $R \times R$. Denote equivalence class of (a, b) by $[a, b]$ and

$$R^e = \{[a, b]/a, b \in R\}.$$

Define the binary operations addition and multiplication in R^e as

$$[a, b] + [c, d] = [a + c, b + d]$$

and

$$[a, b][c, d] = [ac + bd, ad + bc].$$

R^e forms a ring with respect to the above well defined operations, R^e is called the ring of differences with a zero element $[a, a]$ for any a in R and the additive inverse of $[a, b]$ is $[b, a]$ for any a, b in R. In this case if $f : R \Rightarrow R^e$ defined by $f(a) = [a, 0]$ for any a in R, then f is well defined and injective whenever R is cancellative. Similarly M^e is called module of differences of any cancellative semimodule M over any cancellative semiring R.

Let R be any cancellative semiring. If I is a left [right] ideal of R, then

$$I^e = \{[a, b]/a, b \in I\}$$

is a left [right] ideal of R^e. Conversely, if J is a left [right] ideal of R^e, then

$$J^c = \{a \in R/[a, 0] \in J\}$$

is a left [right] ideal of R.

Proposition 1.1 ([6]).

(a) *Let R be a cancellative semiring. Then for any k-ideal I of R, $I = (I^e)^c$.*

(b) *Let R be a cancellative semiring and I be a proper k-ideal of R then I^e is a proper ideal of R^e.*

(c) *If I, I' are any two k-ideals of a cancellative semiring R and $I \subset I'$, then $I^e \subset {I'}^e$.*

(d) *Let R be a cancellative semiring. Then for any two ideals J and J' of R^e, $J \subset J' \Rightarrow J^c \subset {J'}^c$.*

Proposition 1.2 ([6]). *Let R be a cancellative, semisubtractive semiring. Then for any ideal J of R^e, $J = (J^c)^e$.*

Proposition 1.3 ([2]). *Let R be a cancellative semiring, M be a cancellative right R-semimodule and N be a cancellative left R-semimodule. Then $M^e \otimes_{R^e} N^e \cong$*

$(M \otimes_R N)^e$.

Remark. For any modules M and N over a commutative ring R, one can easily verify that $(M \otimes_R N)^c = M^c \otimes N^c$.

Proposition 1.4 ([2]). *Let R be a semiring and P be a left R-semimodule. If P is R-projective then P^e is R^e-projective.*

Throughout A will denote a not necessarily commutative R-semialgebra and R is a commutative semiring with 1.

Proposition 1.5 ([2]). *Let R be a commutative semiring and let A and B be R-semialgebras. Let M be a finitely generated and projective A-semimodule and let N be a finitely generated and projective B-semimodule. Then $Hom_A(M, M) \otimes Hom_B(N, N) \cong Hom_{A\otimes B}(M \otimes N, M \otimes N)$ (where $\otimes = \otimes_R$).*

Generator and Progenerator are as defined in [1], [3].

Morita Theorem 1.6 ([1]). *Let R be any cancellative semiring, M be any cancellative left R-semimodule and left R progenerator. Set a cancellative semiring $S = Hom_R(M, M)$ and a cancellative semimodule $M^* = Hom_R(M, R)$. Then the functors*

$$(\quad) \otimes_R M : cs \;\; mod \;\; - R \Rightarrow S - cs \;\; mod,$$

$$M^* \otimes_S (\quad) : S - cs \;\; mod \;\; \Rightarrow cs \;\; mod \;\; - R$$

(*where cs mod $-R$ and $S - cs$ mod respectively denote the categories of cancellative right R-semimodules and cancellative left S-semimodules*) *are inverse equivalences.*

Corollary 1.7. *In the setting of the Proposition 1.6, we have $R \cong Hom_s(M, M)$* (*as semirings*) *under the mapping which associates to an element r of R the endomorphism of M induced by scalar multiplication by r.*

For any R-semialgebra A, we shall let A^0 denote the **opposite semialgebra** of A, whose underlying additive semigroup is A, multiplication is $a^0 b^0 = (ba)^0$ and the R-semimodule structure coincides with A (to avoid confusion, for any element $a \in A$, while considering an element in A^0 we shall denote it by a^0). The **enveloping semialgebra** is defined by $A \otimes A^0$.

For convenience we will write A^E for the enveloping semialgebra $A \otimes A^0$ of A.

Remark. $A \otimes A^0$ is a cancellative R-semimodule.

The semialgebra A has a structure as a left A^E-semimodule induced by $(a \otimes b^0)x = axb$. If A is a cancellative R-semialgebra then a map μ from semialgebra A^E onto A given by

$$\mu(\sum_i a_i \otimes b_i^0) = \sum_i a_i b_i,$$

μ is a left A^E-semimodule homomorphism, which in case A is commutative is a semiring homomorphism.

Definition 1.8. A cancellative R-semialgebra A is said to be R-separable if μ splits as an A^E-homomorphism or equivalently A is a retract of A^E (i.e., there exists an A^E-homomorphism $\rho : A \Rightarrow A^E$ such that $\mu\rho = Id_A$).

Proposition 1.9 ([2]). *Let A be a cancellative R-semialgebra. Then A is R-separable if and only if there exists an element e in A^E satisfying $\mu(e) = 1$ and $(1 \otimes a^0)e = (a \otimes 1^0)e$ for any a in A.*

The element e in A^E in the above proposition is called **separability idempotent** of A and is indeed idempotent.

Proposition 1.10 ([2]). *Let R be a cancellative semiring and let A be a cancellative, zerosumfree and semisubtractive R-semialgebra. Then A is A^E-projective if and only if μ splits as an A^E-homomorphism (i.e., A is a retract of A^E).*

Remark. From Proposition 1.10, we conclude that a cancellative, semisubtractive and zerosumfree R-semialgebra A is separable if A is A^E-projective.

Proposition 1.11 ([2]). *$Hom_{A^E}(A, A) \cong C(A)$ the center of A under the correspondence $f \Rightarrow f(1)$.*

2. Central separable seimalgebras

An R-semialgebra A is called central if A is faithful as an R-semimodule and $R.1$ coincides with the center of A. We call A is a central separable R-semialgebra if A is both central and separable.

For any cancellative R-semialgebra A, we have seen that A is naturally a left A^E-semimodule. This structure induces an R-semialgebra homomorphism ϕ from A^E to $Hom_R(A, A)$ by associating to any element α in A^E, the element $\phi(\alpha) \in Hom_R(A, A)$, which is scalar multiplication in A by α. If $\alpha = \sum_i a_i \otimes b_i^0$, then $\phi(\alpha)(a) = \alpha \cdot a = \sum_i a_i a b_i$.

Proposition 2.1 ([2]). *Let R be a cancellative semiring and let A and B be cancellative, R-semialgebras. If A and B are central separable over R, then $A \otimes B$ is a central separable R-semialgebra.*

Lemma 2.2. *Let R be a cancellative semiring and let A be a cancellative and semisubtractive R-semialgebra. Then $C(A^e) = (C(A))^e$.*

Proof. $C(A^e) = \{[a, b] \in A^e / [a, b][c, d] = [c, d][a, b] \ \ \forall \ \ [c, d] \in A^e\}$.

$$\begin{aligned} (C(A))^e &= \{[a, b] \in A^e / a, b \in C(A)\} \\ &= \{[a, b] \in A^e / ax = xa \text{ and by } \ by = yb \ \forall \ x, y \in A\}. \end{aligned}$$

Suppose that

$$\begin{aligned} [a, b] \in (C(A))^e &\Rightarrow [a, b] = [l, m] \text{ such that } l, m \in C(A) \\ &\Rightarrow lx = xl, \ \ my = ym \ \forall \ x, y \in A \end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad [l,m][x,y] &= [lx+my, ly+mx] \\
&= [xl+ym, yl+xm] \\
&= [x,y][l,m] \ \ \forall \ \ [x,y] \in A^e \\
\Rightarrow \quad [l,m] &\in C(A^e) \\
\Rightarrow \quad [a,b] &\in C(A^e).
\end{aligned}$$

Thus

$$(C(A))^e \subseteq C(A^e).$$

Conversely, suppose that $[a,b] \in C(A^e)$

$$\Rightarrow [a,b][x,y] = [x,y][a,b], \quad \forall [x,y] \in A^e = A \cup -A \cup \{0\} \qquad (*)$$

as A is semisubtractive.

Since A is semisubtractive, we will consider the following cases.

Case (1) : Suppose $[a,b] = [u,0]$ for some $u \in A$ and every $[x,y] = [v,0]$ or $[x,y] = [0,v], \quad \forall v \in A$.

Then from $(*)$

$$\begin{aligned}
& uv = vu, \quad \forall \ v \in A \\
& \Rightarrow u \in C(A) \\
& \Rightarrow [u,0] \in (C(A))^e \\
& \Rightarrow [a,b] \in (C(A))^e.
\end{aligned}$$

Case (2) : Suppose $[a,b] = [0,u]$ for some $u \in A$ and every $[x,y] = [0,v]$ or $[x,y] = [v,0], \quad \forall v \in A$.

Then from $(*)$

$$\begin{aligned}
& uv = vu, \quad \forall \ v \in A \\
& \Rightarrow u \in C(A) \\
& \Rightarrow [u,0] \in (C(A))^e \\
& \Rightarrow [a,b] \in (C(A))^e.
\end{aligned}$$

Thus for either cases

$$C(A^e) \subseteq (C(A))^e.$$

Hence $C(A^e) = (C(A))^e$. □

Lemma 2.3. *Let R be a cancellative semiring and let A be a cancellative, semisubtractive and zerosumfree R-semialgebra. If A is central separable over R then A^e is central separable over R^e.*

Proof. If A is A^E-projective then A^e is $(A^E)^e \cong (A^e)^E$-projective. This shows that

A^e is R^e-separable. Now we claim that, A^e is R^e-central if A is R-central. Let A be R-central. Then $C(A) = R$

$$\begin{aligned} &\Rightarrow (C(A))^e = R^e \\ &\Rightarrow (C(A^e) = R^e \\ &\Rightarrow A^e \text{ is } R^e - \text{central.} \end{aligned}$$

Also $Ann_{R^e}(A^e) = 0$ if $Ann_R(A) = 0$.

Hence A^e is central separable over R^e. □

Lemma 2.4. *Let R be a cancellative semiring and let A be a cancellative, semisubtractive and zerosumfree R-semialgebra. If A is an A^E-progenerator and R-central then A is central separable over R.*

Proof. If A is an A^E-progenerator, then A is A^E-projective. Since A is cancellative, semisubtractive and zerosumfree R-semialgebra, by Proposition 1.6, A is R-separable. Hence A is central separable over R. □

Lemma 2.5. *Let R be a cancellative semiring and let A be a cancellative R-semialgebra. If A is an A^E-progenerator and A is R-central then A is an R-progenerator and the map $\phi : A^E \Rightarrow Hom_R(A, A)$ is an isomorphism.*

Proof. If A is R-central then we have $Hom_{A^E}(A, A) \cong R$. If A is an A^E-progenerator, then

$$A^* \otimes_{Hom_{A^E}(A,A)} A \cong A^E$$

and

$$\begin{aligned} &A^* \otimes_{A^E} A \cong Hom_{A^E}(A, A) \\ &\Rightarrow Hom_{A^E}(A, A^E) \otimes_{Hom_{A^E}(A,A)} A \cong A^E \end{aligned}$$

and

$$\begin{aligned} &Hom_{A^E}(A, A^E) \otimes_{A^E} A \cong Hom_{A^E}(A, A) \\ &\Rightarrow A \text{ is an } Hom_{A^E}(A, A)\text{-progenerator} \\ &\Rightarrow A \text{ is an } R\text{-progenerator.} \end{aligned}$$

Moreover A^E would then by isomorphic to $Hom_R(A, A)$ by Corollary 1.7 to the Morita theorem, under left multiplication, which is precisely the map ϕ. □

Lemma 2.6. *Let R be a cancellative semiring and let A be a cancellative R-semialgebra. If A is an R-progenerator and the map $\phi : A^E \Rightarrow Hom_R(A, A)$ is an isomorphism, then A is an A^E-progenerator and A is R-central.*

Proof. A is an A^E-progenerator if A is an R-progenerator, and the map $\phi : A^E \Rightarrow Hom_R(A, A)$ is an isomorphism implies that $C(A) = Hom_{A^E}(A, A) \cong R$. Hence A is an A^E-progenerator and R-central. □

These lemmas now give the following important theorem.

Theorem 2.7. *Let R be a cancellative semiring and let A be a cancellative, semisubtractive and zerosumfree R-semialgebra. Then the following conditions are equivalent.*

(a) *A is central separable over R.*

(b) *A is an R-progenerator and the map ϕ from A^E to $Hom_R(A, A)$ is an isomorphism.*

Proof. A is an R-progenerator and $A^E \cong Hom_R(A, A)$ implies that A is central separable over R by above lemmas. Hence $b \Rightarrow a$.

Conversely, assume that A is central separable over R. Claim that A is an R-progenerator and $A^E \cong Hom_R(A, A)$. By Proposition 1.10, A is A^E-projective and it is obvious that 1 generates A over A^E. What remains to show that A is an A^E-generator. A is central separable over R, implies that A^e is central separable over R^e by Lemma 2.3.

Therefore A^e is an $(A^E)^e \cong (A^e)^E$-generator,

$$\Rightarrow (A^*)^e \otimes_{R^e} A^e \cong (A^E)^e,$$
$$\text{where } A^* = Hom_{A^E}(A, A^E) \text{ and } R = Hom_{A^E}(A, A)$$
$$\Rightarrow [(A^*)^e \otimes_{R^e} A^e]^c \cong [(A^E)^e]^c$$
$$\Rightarrow (A^*)^{e^c} \otimes_R A^{e^c} \cong (A^E)^{e^c}$$
$$\Rightarrow (A^*) \otimes_R A \cong (A^E),$$
$$\text{where } A^* = Hom_{A^E}(A, A^E) \text{ and } R = Hom_{A^E}(A, A)$$
$$\Rightarrow A \text{ is an } A^E\text{-generator.}$$

Thus A is an A^E-progenerator and R-central.

Hence by Lemma 2.5, A is an R-progenerator and $A^E \cong Hom_R(A, A)$, that is, $(a) \Rightarrow (b)$. □

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