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EIGHT-DIMENSIONAL EINSTEIN'S CONNECTION FOR THE SECOND CLASS II. THE EINSTEIN'S CONNECTION IN 8-g-UFT

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Abstract. Lower dimensional cases of Einstein's connection were already investigated by many authors for $n = 2, 3, 4, 5, 6, 7$. In the following series of two papers, we present a surveyable tensorial representation of 8-dimensional Einstein's connection in terms of the unified field tensor:

I. The recurrence relations in 8-g-UFT

II. The Einstein's connection in 8-g-UFT

In our previous paper [1], we investigated some algebraic structure in Einstein's 8-dimensional unified field theory (i.e., 8-g-UFT), with emphasis on the derivation of the recurrence relations of the third kind which hold in 8-g-UFT. This paper is a direct continuation of [1]. The purpose of the present paper is to prove a necessary and sufficient condition for a unique Einstein's connection to exist in 8-g-UFT and to display a surveyable tensorial representation of 8-dimensional Einstein's connection in terms of the unified field tensor, employing the powerful recurrence relations of the third kind obtained in the first paper [1].

All considerations in this paper are restricted to the second class only of the generalized 8-dimensional Riemannian manifold X_8, since the case of the first class are done in [2], [3] and the case of the third class, the simplest case, was already studied by many authors.

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This paper is a direct continuation of our previous paper [1], which will be denoted by I in the present paper. All considerations in this paper are based on the results and symbolism of I. Whenever necessary, they will be quoted in the present paper.

In the following theorem, we prove a necessary and sufficient condition for a unique Einstein's connection to exist in 8-g-UFT.

THEOREM 1 A necessary and sufficient condition for the existence and uniqueness of the solution of I-(2.4) or I-(2.19) in 8-g-UFT is given by

(For the second class with the third category).

$$gCD(E^2 - 4EF + G) \neq 0 \tag{1a}$$

where C, D, E, F, G are defined by

$$C = 1 - 2K_4 + (K_4)^2 - 4K_2K_6 - 8K_6 \tag{1b}$$

$$D = 1 - 3K_2 + 9K_4 - 27K_6 \tag{1c}$$

$$\begin{aligned} E = (1 - K_2 + K_4 - K_6)^2 - 4[K_4 - 6K_6 - (K_4)^2 + \\ +16(K_6)^2 + 10K_4K_6 + 2K_2K_6 - 2K_6(K_2)^2] \end{aligned} \tag{1d}$$

$$F = [2 - K_2 + 3K_6][3K_6 - K_2K_4] \tag{1e}$$

$$\begin{aligned} G = 16[(2 - K_2 + 4K_6)^2 + (K_6)^2][9(K_6)^2 - 6K_2K_4K_6 + \\ +(K_4)^3 + (K_2)^3K_6] - 16K_6(2 - K_2 + 4K_6) \times \\ \times[-2(K_2)^3K_6 + (K_2)^2(K_4)^2 + 6K_2K_4K_6 \\ -2(K_4)^3 - 9(K_6)^2] \end{aligned} \tag{1f}$$

(For the second class with the second category).

$$\begin{aligned} (1 + K_2 + K_4)(1 - K_2 + K_4)(1 - K_4)(1 - 3K_2 + 9K_4) \times \\ [(1 - K_2 - 3K_4)^2 - 4K_4((K_2)^2 - 4K_4)] \neq 0 \end{aligned} \tag{2}$$

(For the second class with the first category).

$$(1-(K_2)^2)(1-3K_2)\neq 0 \tag{3}$$

Proof. *The proof for the second class* may be obtained easily from (1a) and (1b) in [3] by simply substituting the corresponding conditions of each case.

Now, we are ready to establish a linear system containing the torsion tensor $S_{\omega\lambda\mu}$ of the Einstein's connection, employing the powerful recurrence relations of the third kind obtained in I-Theorem (4.5).

THEOREM 2a (For the second class with the third category). The system of equations I-(2.19) or I-(2.4) is reduced to the following linear system of 43 equations:

$$\begin{aligned}
B &= S + \overset{110}{S} + 2\overset{(10)1}{S} \\
\overset{(10)1}{B} &= \overset{(10)1}{S} + \overset{(21)1}{S} + \overset{(20)2}{S} + \overset{112}{S} \\
\overset{(12)1}{B} &= \overset{(12)1}{S} + \overset{(23)1}{S} + \overset{222}{S} + \overset{(13)2}{S} \\
\overset{(20)2}{B} &= \overset{(20)2}{S} + \overset{(31)2}{S} + \overset{(30)3}{S} + \overset{(21)3}{S} \\
\overset{(23)1}{B} &= \overset{(23)1}{S} + \overset{(34)1}{S} + \overset{332}{S} + \overset{(24)2}{S} \\
\overset{(13)2}{B} &= \overset{(13)2}{S} + \overset{(24)2}{S} + \overset{(23)3}{S} + \overset{(14)3}{S} \\
\overset{(30)3}{B} &= \overset{(30)3}{S} + \overset{(41)3}{S} + \overset{(40)4}{S} + \overset{(31)4}{S} \\
\overset{(12)3}{B} &= \overset{(12)3}{S} + \overset{(23)3}{S} + \overset{224}{S} + \overset{(13)4}{S} \\
2\overset{(34)1}{B} &= 2\overset{(34)1}{S} + 2K_4\overset{(32)1}{S} + 2K_6\overset{(21)1}{S} + 2K_6\overset{(30)1}{S} \\
&\quad K_4\overset{222}{S} + \overset{442}{S} + 2K_6\overset{(20)2}{S} - K_2\overset{332}{S} + K_6\overset{112}{S}
\end{aligned} \tag{4}$$

$$
\begin{aligned}
2\overset{(24)2}{B} &= 2\overset{(24)2}{S} - \overset{442}{S} - K_2\overset{332}{S} + 2K_6\overset{(20)2}{S} \\
&\quad \overset{222}{S} + K_6\overset{112}{S} - 2K_2\overset{(23)3}{S} + 2K_6\overset{(10)3}{S} \\
\overset{(23)3}{B} &= \overset{(23)3}{S} + \overset{(43)3}{S} + \overset{334}{S} + \overset{(24)4}{S} \\
2\overset{(14)3}{B} &= 2\overset{(14)3}{S} - 2\overset{(34)3}{S} - 2K_2\overset{(32)3}{S} + 2K_6\overset{(10)3}{S} \\
&\quad - \overset{334}{S} - 2K_2\overset{(13)4}{S} + 2\overset{(24)4}{S} - K_4\overset{114}{S} \\
2\overset{(40)4}{B} &= 2\overset{(40)4}{S} - 2K_2\overset{(31)4}{S} - \overset{334}{S} - K_4\overset{114}{S} \\
&\quad -2\overset{(32)5}{S} - 2K_2\overset{(30)5}{S} - 2K_2\overset{(21)5}{S} - 2K_4\overset{(10)5}{S} \\
\overset{(31)4}{B} &= \overset{(31)4}{S} + \overset{(42)4}{S} + \overset{(41)5}{S} + \overset{(32)5}{S} \\
\overset{(30)1}{B} &= \overset{(30)1}{S} + \overset{(41)1}{S} + \overset{(40)2}{S} + \overset{(31)2}{S} \\
\overset{(10)3}{B} &= \overset{(10)3}{S} + \overset{(21)3}{S} + \overset{(20)4}{S} + \overset{114}{S} \\
2\overset{(43)3}{B} &= 2\overset{(43)3}{S} + 2K_4\overset{(32)3}{S} + 2K_6\overset{(21)3}{S} + 2K_6\overset{(30)3}{S} \\
&\quad + \overset{444}{S} + K_4\overset{224}{S} + 2K_6\overset{(20)4}{S} + K_6\overset{114}{S} - K_2\overset{334}{S} \\
2\overset{(24)4}{B} &= 2\overset{(24)4}{S} - \overset{444}{S} - K_2\overset{334}{S} + 2K_6\overset{(20)4}{S} \\
&\quad + K_4\overset{224}{S} + K_6\overset{114}{S} - 2K_2\overset{(23)5}{S} + 2K_6\overset{(10)5}{S} \\
\overset{(32)5}{B} &= \overset{(32)5}{S} + \overset{(43)5}{S} - K_2\overset{(42)4}{S} - K_4\overset{(42)2}{S} \\
&\quad - K_6\overset{(42)0}{S} - K_2\overset{334}{S} - K_4\overset{332}{S} - K_6\overset{330}{S} \\
\overset{(30)5}{B} &= \overset{(30)5}{S} + \overset{(14)5}{S} - K_2\overset{(40)4}{S} - K_4\overset{(40)2}{S} \\
&\quad - K_6\overset{(40)0}{S} - K_2\overset{(31)4}{S} - K_4\overset{(31)2}{S} - K_6\overset{(13)0}{S} \\
\overset{(21)5}{B} &= \overset{(21)5}{S} + \overset{(32)5}{S} - K_2\overset{(31)4}{S} - K_4\overset{(31)2}{S} \\
&\quad - K_6\overset{(13)0}{S} - K_2\overset{224}{S} - K_4\overset{222}{S} - K_6\overset{220}{S} \\
\overset{(10)5}{B} &= \overset{(10)5}{S} + \overset{(21)5}{S} - K_2\overset{(20)4}{S} - K_4\overset{(20)2}{S}
\end{aligned}
$$

$$
\begin{aligned}
& && -K_6 \overset{(20)0}{S} - K_2 \overset{114}{S} - K_4 \overset{112}{S} - K_6 \overset{110}{S} \\
2 \overset{(14)5}{B} &=&& 2 \overset{(14)5}{S} - 2 \overset{(34)5}{S} - 2K_2 \overset{(23)5}{S} + 2K_6 \overset{(10)5}{S} \\
& && -2K_2 \overset{(24)4}{S} - 2K_4 \overset{(24)2}{S} - 2K_6 \overset{(24)0}{S} + 2(K_2)^2 \overset{(31)4}{S} \\
& && +(K_4)^2 \overset{112}{S} + 2K_2K_6 \overset{(31)0}{S} + K_6 \overset{330}{S} + K_4K_6 \overset{110}{S} \\
& && K_2 \overset{334}{S} + K_2K_4 \overset{114}{S} + 2K_2K_4 \overset{(31)2}{S} + K_4 \overset{332}{S} \\
2 \overset{(41)1}{B} &=&& 2 \overset{(41)1}{S} - 2 \overset{(43)1}{S} - 2K_2 \overset{(32)1}{S} + 2K_6 \overset{(10)1}{S} \\
& && -2K_2 \overset{(31)2}{S} - \overset{332}{S} - K_4 \overset{112}{S} + 2 \overset{(42)2}{S} \\
2 \overset{(40)2}{B} &=&& 2 \overset{(40)2}{S} - 2K_2 \overset{(31)2}{S} - \overset{332}{S} - K_4 \overset{112}{S} \\
& && -2 \overset{(32)3}{S} - 2K_2 \overset{(30)3}{S} - 2K_2 \overset{(21)3}{S} - 2K_4 \overset{(10)3}{S} \\
\overset{(20)4}{B} &=&& \overset{(20)4}{S} + \overset{(31)4}{S} + \overset{(30)5}{S} + \overset{(21)5}{S} \\
2 \overset{(43)5}{B} &=&& 2 \overset{(43)5}{S} + 2K_4 \overset{(32)5}{S} + 2K_6 \overset{(21)5}{S} + 2K_6 \overset{(30)5}{S} \\
& && +(K_2)^2 \overset{334}{S} - K_2K_4 \overset{224}{S} - 2K_2K_6 \overset{(20)4}{S}) - K_2K_6 \overset{114}{S} \\
& && +K_2K_4 \overset{332}{S} - (K_4)^2 \overset{222}{S} - 2K_4K_6 \overset{(20)2}{S} - K_4K_6 \overset{112}{S} \\
& && +K_2K_6 \overset{330}{S} - K_4K_6 \overset{220}{S} - 2(K_6)^2 \overset{(20)0}{S} - (K_6)^2 \overset{110}{S} \\
& && -K_2 \overset{444}{S} - K_4 \overset{442}{S} - K_6 \overset{440}{S} \\
2 \overset{(24)0}{B} &=&& 2 \overset{(24)0}{S} - \overset{440}{S} - K_2 \overset{330}{S} + 2K_6 \overset{(20)0}{S} \\
& && +K_4 \overset{220}{S} + K_6 \overset{110}{S} - 2K_2 \overset{(23)1}{S} + 2K_6 \overset{(10)1}{S}) \\
2 \overset{(40)0}{B} &=&& 2 \overset{(40)0}{S} - 2K_2 \overset{(31)0}{S} - \overset{330}{S} - K_4 \overset{110}{S} \\
& && -2 \overset{(32)1}{S} - 2K_2 \overset{(30)1}{S} - 2K_2 \overset{(21)1}{S} - 2K_4 \overset{(10)1}{S} \\
\overset{(13)0}{B} &=&& \overset{(13)0}{S} + \overset{(24)0}{S} + \overset{(23)1}{S} + \overset{(14)1}{S} \\
\overset{(20)0}{B} &=&& \overset{(20)0}{S} + \overset{(31)0}{S} + \overset{(30)1}{S} + \overset{(21)1}{S}
\end{aligned}
$$

$$\begin{aligned}
\overset{110}{B} &= \overset{110}{S} + \overset{220}{S} + 2\overset{(12)1}{S} \\
\overset{112}{B} &= \overset{112}{S} + \overset{222}{S} + 2\overset{(12)3}{S} \\
\overset{222}{B} &= \overset{222}{S} + \overset{332}{S} + 2\overset{(32)3}{S} \\
\overset{332}{B} &= \overset{332}{S} + \overset{442}{S} + 2\overset{(43)3}{S} \\
\overset{224}{B} &= \overset{224}{S} + \overset{334}{S} + 2\overset{(32)5}{S} \\
\overset{442}{B} &= (1+K_2)\overset{442}{S} + 2K_4\overset{(42)2}{S} + 2K_6\overset{(13)2}{S} + K_4\overset{332}{S} \\
&\quad + K_6\overset{222}{S} + 2K_6\overset{(40)2}{S} + 2K_4\overset{(23)3}{S} + 2K_6\overset{(21)3}{S} + 2K_6\overset{(30)3}{S} \\
\overset{334}{B} &= \overset{334}{S} + \overset{444}{S} + 2\overset{(43)5}{S} \\
\overset{114}{B} &= \overset{114}{S} + \overset{224}{S} + 2\overset{(12)5}{S} \\
\overset{444}{B} &= (1+K_2)\overset{444}{S} + 2K_4\overset{(42)4}{S} + 2K_6\overset{(13)4}{S} + K_4\overset{334}{S} \\
&\quad + K_6\overset{224}{S} + 2K_6\overset{(40)4}{S} + 2K_4\overset{(23)5}{S} + 2K_6\overset{(21)5}{S} + 2K_6\overset{(30)5}{S} \\
\overset{330}{B} &= \overset{330}{S} + \overset{440}{S} + 2\overset{(43)1}{S} \\
\overset{220}{B} &= \overset{220}{S} + \overset{330}{S} + 2\overset{(23)1}{\rightarrow} S \\
\overset{440}{B} &= (1+K_2)\overset{440}{S} + 2K_4\overset{(42)0}{S} + 2K_6\overset{(13)0}{S} + K_4\overset{330}{S} \\
&\quad + K_6\overset{220}{S} + 2K_6\overset{(40)0}{S} + 2K_4\overset{(23)1}{S} + 2K_6\overset{(21)1}{S} + 2K_6\overset{(30)1}{S}
\end{aligned}$$

THEOREM 2b (For the second class with the second category of 8-g-UFT). The system of equations I-(2.19) or I-(2.4) is reduced to the following linear system of 12 equations:

$$\begin{aligned}
(5) \qquad B &= S + \overset{110}{S} + 2\overset{(10)1}{S} \\
\overset{(10)1}{B} &= \overset{(10)1}{S} + \overset{(21)1}{S} + \overset{(20)2}{S} + \overset{112}{S} \\
2\overset{(12)1}{B} &= 2\overset{(12)1}{S} + 2K_4\overset{(10)1}{S} + \overset{222}{S} - K_2\overset{112}{S} + K_4\overset{002}{S}
\end{aligned}$$

$$
\begin{aligned}
2\overset{(02)2}{B} &= 2\overset{(02)2}{S} - \overset{222}{S} - K_2\overset{112}{S} + K_4\overset{002}{S} - 2K_2\overset{(10)3}{S}\\
\overset{(01)3}{B} &= \overset{(01)3}{S} + \overset{(12)3}{S} - K_2\overset{112}{S} - K_4\overset{110}{S} - K_2\overset{(02)2}{S}\\
&\quad - K_4\overset{(02)0}{S}\\
2\overset{(12)3}{B} &= 2\overset{(12)3}{S} + 2K_4\overset{(10)3}{S} - K_2\overset{222}{S} - K_4\overset{220}{S} + (K_2)^2\\
&\quad \overset{112}{S} + K_2K_4\overset{110}{S} - K_2K_4\overset{002}{S} - (K_4)^2 S\\
2\overset{(02)0}{B} &= 2\overset{(02)0}{S} - K_2\overset{110}{S} - 2K_2\overset{(01)1}{S} - \overset{220}{S} + K_4 S\\
\overset{110}{B} &= \overset{110}{S} + \overset{220}{S} + 2\overset{(12)1}{S}\\
\overset{112}{B} &= \overset{112}{S} + \overset{222}{S} + 2\overset{(21)3}{S}\\
\overset{222}{B} &= (1+K_2)\overset{222}{S} + K_4\overset{112}{S} + 2K_4\overset{(10)3}{S} + 2K_4\overset{(20)2}{S}\\
\overset{002}{B} &= \overset{002}{S} + \overset{112}{S} + 2\overset{(10)3}{\rightarrow} S\\
\overset{220}{B} &= (1+K_2)\overset{220}{S} + 2K_4\overset{(10)1}{S} + K_4\overset{110}{S} + 2K_4\overset{(20)0}{S}
\end{aligned}
$$

Proof. This assertion also follows from I-(3.9), using I-(4.2c) and I-(4.19).

THEOREM 2c (For the second class with the first category of 8-*g*-UFT). The system of equations I-(2.19) or I-(2.4) is reduced to the following linear system of 2 equations:

$$
\begin{aligned}
B &= (1+K_2)S + 2\overset{(10)1}{S}\\
\overset{(10)1}{B} &= (1-K_2)\overset{(10)1}{S}
\end{aligned}
\tag{6}
$$

Proof. This assertion follows from I-(3.9), using I-(4.2d) and I-(4.20).

Using the Gauss-Jordan Elimination Method, each system may be solved for $S = S_{\omega\lambda\mu}$. The solution for the first case was found by operating computer, but it's too lengthy. Therefore, in the following two theorems we just display the solutions for the last two cases only.

THEOREM 3a (For the second class with the second category of 8-g-UFT). If the condition (3) holds, the unique solution of the system I-(2.19) is given by

$$
\begin{aligned}
&(S - B)(1 + K_2 + K_4)[(1 - K_2 + 5K_4)^2 - 4K_4(2 - K_2)^2] \\
&= 4\underset{1}{B}(K_4 - 1) + \underset{2}{B}(1 - K_2 + 5K_4) + 2\underset{3}{B}[1 - 2K_2 + (K_2)^2 - 5K_4]
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
\underset{1}{B} = {} & (K_4)^2 B + 2\overset{(12)3}{B} + K_2 K_4 \overset{002}{B} + [2K_4 - (K_2)^2]\overset{112}{B} \\
& -2K_4 \overset{(12)1}{B} + K_4[2 + 2K_2 + (K_2)^2]\overset{110}{B} + K_2 \overset{222}{B} \\
& +2K_4 \overset{(20)2}{B} - K_4(1 + K_2)\overset{220}{B} - 2K_4(1 + K_2)\overset{(10)3}{B} \\
& -2K_4(1 + K_2)^2 \overset{(10)1}{B}
\end{aligned}
$$

$$
\begin{aligned}
\underset{2}{B} = {} & -(K_4)^2 B + 2[(K_2)^2 - 1 + K_4 + 2K_2 K_4]\overset{(10)1}{B} \\
& +(2 + K_2)\overset{112}{B} - \overset{222}{B} - K_4 \overset{002}{B} + 2\overset{(20)2}{B} + 2(K_2 + 2K_4)\overset{(10)3}{B} \\
& +2K_4 \overset{(20)0}{B} - [(K_2)^2 - 1 + K_4 + 2K_2 K_4]\overset{110}{B} \\
& +(K_2 - 1 + 2K_4)\overset{220}{B}
\end{aligned}
$$

$$
\begin{aligned}
\underset{3}{B} = {} & 2(K_4)^2 B + 2\overset{(12)3}{B} - K_4 \overset{002}{B} + K_2 \overset{112}{B} + 2(1 + K_2)\overset{(21)1}{B} \\
& - \overset{222}{B} + 2K_4 \overset{(10)3}{B} - (1 + K_4)(1 + K_2)\overset{110}{B} \\
& +(1 + K_4)\overset{220}{B} + 2K_4(1 + K_2)\overset{(10)1}{B} - 2K_4 \overset{(20)0}{B}
\end{aligned}
$$

THEOREM 3b (For the second class with the first category of 8-g-UFT). If the condition (4) holds, the unique solution of the system I-(2.19) is given by

$$(8)\qquad (B-S)[1-(K_2)^2] = K_2(1-K_2)B + 2\overset{(10)1}{B}$$

Now that we have obtained the tensor $S = S_{\omega\lambda\mu}$ in terms of $g_{\lambda\mu}$, it is possible to determine the tensor $U^{\nu}{}_{\lambda\mu}$ and eventually the 8-dimensional Einstein's connection $\Gamma^{\nu}_{\lambda\mu}$ in terms of $g_{\lambda\mu}$ by simply substituting for S into I-(2.18) and I-(2.17), respectively.

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