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On a Class of Analytic Functions Related to the Starlike Functions

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Abstract. In this paper we discuss a class of analytic functions related to the starlike functions in the unit disk. We prove that this class belongs to the class of close-to-convex functions, we obtain the sharp coefficient upper bounds and distortion theorem of this class, we also get the convexity radius of this class.

1. Introduction

Sakaguchi [1] once introduced the concept of starlike functions with respect to the symmetric points, that is, the class of functions $f(z)$ analytic in $E = \{z : |z| < 1\}$ and satisfying $f(0) = 0$, $f'(0) = 1$, and

$$\mathrm{Re}\{\frac{zf'(z)}{f(z) - f(-z)}\} > 0 \quad (|z| < 1).$$

Following him, many mathematicians discussed this class and its subclasses, see paper [2]. In this paper, we introduce a class of analytic functions related to the starlike functions, and obtain some interesting results.

Let $f(z) = z+a_2z^2+\cdots$ be analytic in E, if there exists a function $g(z) \in S^*(\frac{1}{2})$, such that

$$\mathrm{Re}\{\frac{z^2f'(z)}{g(z)g(-z)}\} < 0 \quad (|z| < 1) \tag{1}$$

we say $f(z) \in K_s$, where $S^*(\frac{1}{2})$ denotes the class of starlike functions of order $\frac{1}{2}$.

We will prove that K_s is a subclass of close-to-convex functions, and give the sharp coefficient estimate of the functions, distortion theorems and the convexity radius of this class.

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2. Some results about starlike functions

It is well-known that, if $\phi(z) = z + b_2z^2 + \cdots$ is analytic in E and satisfies

$$\mathrm{Re}\{\frac{z\phi'(z)}{\phi(z)}\} > \alpha \quad (|z| < 1, 0 \leq \alpha < 1), \tag{2}$$

we say $\phi(z)$ is a starlike function of order α, denoted by $\phi(z) \in S^*(\alpha)$, especially denote $S^*(0) = S^*$. We have the following conclusions about starlike functions.

Theorem A. *Let $\phi(z) \in S^*(\alpha)$, $\psi(z) \in S^*(\beta)$. Then for $1 \leq \alpha + \beta < 2$, we have $(\phi(z) \cdot \psi(z))/z \in S^*(\alpha + \beta - 1)$.*

Proof. According to the definition, we have

$$\mathrm{Re}\{\frac{z\phi'(z)}{\phi(z)}\} > \alpha \text{ and } \mathrm{Re}\{\frac{z\psi'(z)}{\psi(z)}\} > \beta.$$

Now let $F(z) = (\phi(z) \cdot \psi(z))/z$, we have

$$\frac{zF'(z)}{F(z)} = \frac{z\phi'(z)}{\phi(z)} + \frac{z\psi'(z)}{\psi(z)} - 1$$

and

$$\mathrm{Re}\{\frac{zF'(z)}{F(z)}\} = \mathrm{Re}\{\frac{z\phi'(z)}{F\phi(z)}\} + \mathrm{Re}\{\frac{z\psi'(z)}{F\psi(z)}\} - 1 > \alpha + \beta - 1.$$

So, if $0 \leq \alpha + \beta - 1 < 1$, we know $F(z) = (\phi(z) \cdot \psi(z))/z \in S^*(\alpha + \beta - 1)$. □

Corollary. *Let $g(z) = z + b_2z^2 + \cdots \in S^*(\frac{1}{2})$, then $(-g(z) \cdot g(-z))/z \in S^*$.*

Proof. Let $\alpha = \beta = \frac{1}{2}$ in Theorem A. □

Theorem B. *Let $g(z) = z + a_2z^2 + \cdots \in S^*(\frac{1}{2})$, then we have*

$$\left|2b_{2n-1} - 2b_2b_{2n-2} + \cdots + (-1)^n 2b_{n-1}b_{n+1} + (-1)^{n+1}b_n^2\right| \leq 1 \quad (n \geq 2).$$

This estimates is sharp, the extreme function is $g(z) = z/(1 - z)$.

Proof. According to the Corollary of Theorem A, we have $(-g(z) \cdot g(-z))/z \in S^*$, and if let

$$G(z) = (-g(z) \cdot g(-z))/z, \tag{3}$$

we know $G(-z) = -G(z)$, so $G(z)$ is an odd starlike function. If let $G(z) = z + \sum_{n=2}^{\infty} B_{2n-1}z^{2n-1}$, it's well-known that

$$|B_{2n-1}| \leq 1 \quad (n = 2, 3, \cdots). \tag{4}$$

Substituting the series expressions of $g(z)$, $G(z)$ in (3) and compare the coefficients of two side of this equation, using (4) we can get the results of the theorem. Clearly,

$g(z) = z/(1-z)$ is the extreme function. $\square$

3. About class K_s

Defined as above, if $f(z) \in K_s$, we have inequality (1). Because $g(z) \in S^*(\frac{1}{2})$, using the Corollary of Theorem A, we can get

Theorem 1. *If $f(z) \in K_s$, then $f(z)$ is a close-to convex function, that is, K_s is a subclass of the class of close-to-convex functions K.*

Now we give the coefficient estimate, distortion theorems and the convexity radius of functions in K_s respectively.

3.1. On the coefficient estimate of functions in K_s

Theorem 2. *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in K_s$, then we have*

$$|a_n| \leq 1 \quad (n = 2, 3, \cdots).$$

The extreme function is $f(z) = z/(1-z)$.

Proof. According to the definition, there exists $g(z) \in S^*(\frac{1}{2})$ such that inequality (1) holds, and using Theorem B, we know

$$G(z) = \frac{1}{z}[-g(z) \cdot g(-z)] = z + \sum_{n=2}^{\infty} B_{2n-1} z^{2n-1}$$

is an odd starlike function and $|B_{2n-1}| \leq 1 \quad (n \geq 2)$, so there exists a function having positive real part

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

such that

$$\frac{zf'(z)}{G(z)} = p(z) \tag{5}$$

and

$$|c_n| \leq 2 \quad (n = 1, 2, 3, \cdots). \tag{6}$$

Substituting the series expressions of function $f(z)$, $G(z)$, $p(z)$ into equality (5) and compare the coefficients of two side of this equality we get

$$2na_{2n} = c_{2n-1} + c_{2n-3}B_3 + c_{2n-5}B_5 + \cdots + c_1 B_{2n-1}, \quad (n = 1, 2, \cdots)$$

and

$$(2n+1)a_{2n+1} = c_{2n} + c_{2n-2}B_3 + c_{2n-4}B_5 + \cdots + c_2 B_{2n-1} + B_{2n+1}, \quad (n = 1, 2, \cdots).$$

Using (4) and (6), we have

$$2n\,|a_{2n}| \le 2\cdot n \quad \text{so} \quad |a_{2n}| \le 1 \quad (n = 1, 2, \cdots),$$

and

$$(2n+1)\,|a_{2n+1}| \le 2\cdot n + 1 \quad \text{so} \quad |a_{2n+1}| \le 1 \quad (n = 1, 2, \cdots),$$

that is, $|a_n| \le 1$ $(n \ge 2)$. Clearly function $f(z) = z/(1-z) \in K_s$ and is the extreme function. $\square$

3.2. On the distortion theorems of the functions in K_s

Theorem 3. *Let $f(z) \in K_s$, then we have*

$$\frac{1-r}{(1+r)(1+r^2)} \le |f'(z)| \le \frac{1}{(1-r)^2} \tag{7}$$

and

$$\ln\frac{1+r}{\sqrt{1+r^2}} \le |f(z)| \le \frac{r}{1-r} \tag{8}$$

where $|z| = r$, $0 \le r < 1$, these estimates are sharp, the functions attain the equalities of left hand and right hand of (7) and (8) are

$$f(z) = \ln\frac{1+z}{\sqrt{1+z^2}} \quad \text{and} \quad f(z) = \frac{z}{1-z}$$

respectively, where logarithm function and power function both are main branch.

Proof. For $f(z) \in K_s$, there exists $g(z) \in S^*(1/2)$ such that we have (1), also $G(z) = (-g(z)\cdot g(-z))$ is an odd starlike function, and

$$\mathrm{Re}\{\frac{zf'(z)}{G(z)}\} > 0 \tag{9}$$

and

$$\frac{r}{1+r^2} \le |G(z)| \le \frac{r}{1-r^2} \quad (|z| = r, \quad 0 \le r < 1). \tag{10}$$

So there exists a function $p(z)$ having positive real part, such that

$$\frac{zf'(z)}{G(z)} = p(z) \tag{11}$$

and

$$\frac{1-r}{1+r} \le |p(z)| \le \frac{1+r}{1-r} \quad (|z| = r, \quad 0 \le r < 1). \tag{12}$$

Where (10) and (12) both are well-known results. From (10), (11), (12), we can get inequality (7). Using (7), we have

$$|f(z)| = \left|\int_0^z f'(z)dz\right| \le \int_0^r \left|f'(re^{i\theta})\right| dr \le \int_0^r \frac{1}{(1-r)^2}dr = \frac{r}{1-r}.$$

This is the right hand side inequality of (8), in order to obtain the left hand side one of (8), it is sufficient to prove that it holds for the nearest point $f(z_0)$ $(|z_0| = r)$ from zero, for some r $(0 < r < 1)$. otherwise we have $|f(z)| \ge |f(z_0)|$ $(|z| = r)$.

Because $f(z)$ is a close-to-convex function, it is univalent in the unit disk E, so the original image of the line segment $[0, f(z_0)]$ is a piece of arc R in $\{|z| \le r\}$, according to (7) we have

$$|f(z_0)| = \int_{f(R)} |dw| = \int_R |f'(z)|\,|dz| \ge \int_R \frac{1-r}{(1+r)(1+r^2)}dr = \ln\frac{1+r}{\sqrt{1+r^2}}.$$

Thus we have proved the left hand side one of (8). The extreme functions are clear. The proof of Theorem 3 is completed. □

3.3. On the convexity radius of the functions in K_s

We say a function $f(z)$ is convex in $\{z \;:\; |z| < r\}$, if $f(z)$ satisfies condition :

$$\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0.$$

When we give the convexity radius of the functions in K_s, we need the following lemmas.

Lemma 1. *If $G(z)$ is an odd starlike function, we have*

$$\mathrm{Re}\{\frac{zG'(z)}{G(z)}\} \ge \frac{1-r^2}{1+r^2} \quad (|z| = r).$$

Proof. As $G(z)$ is a starlike function, we know $\mathrm{Re}\{(zG'(z))/G(z)\} > 0$, also $G(z)$ is an odd function, so according to the subordination principle we have

$$\frac{zG'(z)}{G(z)} \prec \frac{1-z^2}{1+z^2},$$

so there exists a Schwarz function $\omega(z)$, $\omega(0) = 0$, $|\omega(z)| < 1$, such that

$$F(z) = \frac{zG'(z)}{G(z)} = \frac{1-[\omega(z)]^2}{1+[\omega(z)]^2},$$

that is,

$$[\omega(z)]^2 = \frac{1-F(z)}{1+F(z)},$$

so,

$$\left|\frac{1-F(z)}{1+F(z)}\right| = |\omega(z)|^2 \leq |z|^2\,,$$

this inequality can be written as

$$|F(z)|^2 - 2\mathrm{Re}\{F(z)\} + 1 \leq |z|^4 \{|F(z)|^2 + 2\mathrm{Re}\{F(z)\} + 1\}.$$

From this inequality we can get

$$\left|F(z) - \frac{1+|z|^4}{1-|z|^4}\right|^2 \leq \{\frac{1+|z|^4}{1-|z|^4}\}^2 - 1 = \frac{4\,|z|^4}{(1-|z|^4)^2},$$

that is,

$$\left|F(z) - \frac{1+|z|^4}{1-|z|^4}\right| \leq \frac{2\,|z|^2}{1-|z|^4}.$$

From this inequality we obtain

$$\mathrm{Re}\{F(z)\} \geq \frac{1-|z|^2}{1+|z|^2} = \frac{1-r^2}{1+r^2}.$$

This inequality is the one we need to prove. □

Lemma 2. *Let* $p(z)$ *satisfy* $p(0) = 1$, $\mathrm{Re}\{p(z)\} > 0$, *then we have*

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2r}{1-r^2},\ \ (|z| = r).$$

Proof. According to the condition we know that there exists a Schwarz function $\omega(z)$, $\omega(0) = 0$, $|\omega(z)| < 1$, such that

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)},$$

so,

$$\frac{zp'(z)}{p(z)} = \frac{2z\omega'(z)}{1-[\omega(z)]^2}.$$

Thus, using Schwarz-Pick Theorem, we obtain

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2\,|z|}{1-|\omega(z)|^2} \cdot |\omega'(z)| \leq \frac{2r}{1-|\omega(z)|^2} \cdot \frac{1-|\omega(z)|^2}{1-r^2} = \frac{2r}{1-r^2}.$$

The proof of Lemma 2 is completed. □

Theorem 4. *Let $f(z) \in K_s$, then $f(z)$ is convex in $|z| < r_0 = \frac{1}{2}\left[1+\sqrt{5}-\sqrt{2(1+\sqrt{5})}\right] = 0.346\cdots$, where r_0 can't be replaced by a bigger number.*

Proof. When $f(z) \in K_s$, there exists $g(z) \in S^*(1/2)$ such that (1) holds, also $G(z) = \frac{1}{z}[-g(z) \cdot g(-z)]$ is an odd starlike function, so from (1) we have

$$zf'(z) = G(z) \cdot p(z),$$

where $p(z)$ satisfies the condition of Lemma 2, and

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z)},$$

so use Lemmas we can get

$$\begin{aligned}
\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} &= \mathrm{Re}\{\frac{zG'(z)}{G(z)}\} + \mathrm{Re}\{\frac{zp'(z)}{p(z)}\} \\
&\geq \frac{1-r^2}{1+r^2} - \left|\frac{zp'(z)}{p(z)}\right| \\
&\geq \frac{1-r^2}{1+r^2} - \frac{2r}{1-r^2} = \frac{1-2r-2r^2-2r^3+r^4}{1-r^4}.
\end{aligned}$$

It is easy to know that if $1-2r-2r^2-2r^3+r^4 > 0$, we have $\mathrm{Re}\{1 + \frac{zf''(z)}{f'(z)}\} > 0$. Let

$$F(r) = 1 - 2r - 2r^2 - 2r^3 + r^4,$$

because $F(0) = 1$, $F(1) = -4$, and

$$F'(r) = -2 - 4r - 6r^2 + 4r^3 < 0, \quad (0 \leq r < 1),$$

we know that $F(r)$ is a monotonously decrease function of r, and equation $F(r) = 0$ has a root r_0 in interval $(0,\ 1)$, solve this equation we get $r_0 = \frac{1}{2}\left[1+\sqrt{5}-\sqrt{2(1+\sqrt{5})}\right] = 0.346\cdots$.

Thus when $r < r_0$, $\mathrm{Re}\{1 + \frac{zf''(z)}{f(z)}\} > 0$, that is, $f(z)$ is convex in $|z| < r_0$.

We know $f_0(z) = \ln \frac{1+z}{\sqrt{1+z^2}} \in K_s$, and $\mathrm{Re}\{1 + \frac{r_0 f''(r_0)}{f'(r_0)}\} = 0$, so clearly r_0 in the theorem can't be replaced by a bigger number. The proof of the theorem is completed. □

References

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