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A CHARACTERIZATION OF THE HYPERSPHERE

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Abstract. We study hypersurfaces in the Euclidean space with the following property: the tangential part of the position vector has constant length. As a result, we prove that among the connected and complete hypersurfaces in the Euclidean space, only the hypersphere centered at the origin satisfies the property.

We consider a smooth hypersurface M^n of $n+1$ dimensional Euclidean space E^{n+1}. Choose an origin O and denote by x, and U the position vector and the unit normal to M^n, respectively. Then the position vector x is decomposed as

$$x = x_T + < x, U > U, \tag{1}$$

where x_T denotes the tangential component.

If the distance function $r = |x|$ is constant, then M^n is part of a hypersphere with center O. If the support function $f =< x, U >$ is constant, then in addition to generalized cylinders $S^k \times E^{n-k}$, we have many other local examples([3]). In case M^n is complete and the support function f is identically 1, then either $M^n = E^n$ at a distance 1 from O, or $M^n = S^n$ with center O, or $M^n = S^k \times E^{n-k}$ for some $1 \le k \le n-1$ where S^k is a unit sphere centered at $O([3])$.

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What if the length function $g = |x_T|$ of the tangential part of x is constant? In addition to the hyperspheres $S^n(r)$ with center O, we have other examples. In E^3, the generic one is this: fix a unit circle C centered at O and denote by γ an involute of the unit circle C which starts from $P \in C$. Note that each normal line to γ is tangent to the unit circle C. Let M^2 be the rotation surface of γ with axis a line passing through O. Then every normal line to M^2 is tangent to the unit sphere centered at O, and hence we see that the function g is constant 1. Obviously, the rotation surface M^2 has many singularities. In this note we prove the following:

Theorem. *Let M^n be a connected and complete hypersurface in E^{n+1}, and O a point with respect to which the tangential part of the position vector has constant length. Then M^n is a hypersphere centered at O.*

Let ∇ be the standard connection of E^{n+1}, and D the induced connection on the hypersurface M^n. The equations of Gauss and Weingarten are, respectively,

$$\nabla_X Y = D_X Y + < SX, Y > U, \nabla_X U = -SX,$$

where X, Y are vectors tangent to M^n, and S is the shape operator of M^n. We use the decomposition (1) of the position vector field x with the support function $f = < x, U >$. Then we have

$$X = \nabla_X x = \nabla_X (x_T + fU) = D_X x_T + < SX, x_T > U + (Xf)U - fSX.$$

Taking the tangential part of this equation, we obtain

$$D_X x_T = X + fSX, \tag{2}$$

and taking the normal part we obtain

$$Sx_T = -\nabla f, \tag{3}$$

where ∇f denotes the gradient of f. The hypersurface M^n satisfies for a nonnegative constant d^2

$$(4) \qquad < x_T, x_T >= d^2.$$

By differentiating (4) with respect to an arbitrary tangent vector X to M^n, from (2) we get

$$(5) \qquad fS(x_T) + x_T = 0.$$

Hence (2) shows that

$$(6) \qquad D_{x_T} x_T = 0.$$

From (4) and (5) we obtain

$$(7) \qquad \nabla_{x_T} x_T =< S x_T, x_T > U = -\frac{d^2}{f} U.$$

By differentiating (7) once more, we have from (5)

$$(8) \qquad \nabla_{x_T} \nabla_{x_T} x_T = -x_T(\frac{d^2}{f})U - \frac{d^2}{f^2} x_T.$$

Suppose that the constant d^2 is positive. Let $x(t)$ denote an integral curve of x_T. The assumption together with (6) shows that $x(t)$ is a geodesic of M^n. Together with the equation

$$(1) \qquad x = x_T + < x, U > U,$$

(7) and (8) shows that $x(t)$ lies in a plane which passes through the origin. Note that $x(t)$ satisfies

$$(9) \qquad < x'(t), x'(t) >= d^2,$$

and

$$(10) \qquad < x(t), T(t) >= d,$$

where $T(t)$ denotes the unit tangent vector to $x(t)$.

Lemma. Let $x(t)$ be a constant speed plane curve which satisfies (9) and (10). Then the maximal domain of $x(t)$ is $(\frac{c}{d}, \infty)$ for some $c \in R$. Furthermore, if we denote by C the circle of radius d and centered at O, then $x(t)$ is an open segment of the involute of C which starts from $x(\frac{c}{d}) \in C$.

PROOF. We introduce the arc length parameter $s = dt$. Let $N(s)$ denote the unit normal vector to $x(s)$ such that $\{T(s), N(s)\}$ gives a right handed orthonormal basis of R^2 for each s. We decompose $x(s)$ as follows:

$$x(s) = dT(s) + f(s)N(s). \tag{11}$$

By differentiating (11), we obtain from the Frenet-Serret equation

$$1 + f(s)\kappa(s) = 0, f'(s) + d\kappa(s) = 0, \tag{12}$$

where $\kappa(s)$ is the curvature function of $x(s)$. From these we get for a constant c

$$f^2(s) = 2d(s-c), |x(s)|^2 = 2d(s-c) + d^2, \tag{13}$$

and

$$\kappa^2(s) = \frac{1}{2d(s-c)}, \tag{14}$$

which completes the proof of the first part.

For the proof of the second part, first note that (13) implies $P = x(c) \in C$. If we denote by $\theta(s)$ the angle between $T(s)$ and a line l passing through O and P, measured counter clockwise, then we have

$$T(s) = (\cos\theta(s), \sin\theta(s), N(s) = (-\sin\theta(s), \cos\theta(s)). \tag{15}$$

Since $\theta'(s) = \kappa(s) > 0$, $x(\theta)$ is a well-defined parametrization of $x(s)$. By differentiating with respect to θ, the initial condition and (12) show that $f(\theta) = -d\theta$. Thus, together with the fundamental theorem of plane curves, (11) and (14) complete the proof of the second part. □

Now we prove the theorem. Since the hypersurface is complete, the above lemma shows that the constant d^2 cannot be positive, which implies that the tangential part x_T vanishes. Hence the position vector x is normal to the hypersurface everywhere. Thus, by differentiating the function $r^2 = < x, x >$ with respect to an arbitrary tangent vector to M^n we see that the distance function $r = |x|$ is constant. This completes the proof of the theorem.

Remark. Recently, B. -Y. Chen defined and classified constant ratio hypersurfaces of $n+1$ dimensional Euclidean space E^{n+1}. A hypersurface is of constant ratio if the ratio $|x_T| : |x|$ is constant([1]). It was proved in [2] that M^n is a constant ratio hypersurface if and only if the gradient of the distance function $r = |x|$ has constant length.

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