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ON A RIGIDITY OF HARMONIC DIFFEOMORPHISM BETWEEN TWO RIEMANN SURFACES

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Abstract. One of the basic questions concerning harmonic map is on the existence of harmonic maps satisfying a certain condition. Rigidity of a certain harmonic map can be considered as an answer for this kinds of questions. In this article, we study a rigidity property of harmonic diffeomorphisms under the condition that the inverse map is also harmonic. We show that every such a harmonic diffeomorphism is totally geodesic or conformal in two dimensional case.

1. Introduction

The study of harmonic map has long history. Harmonic map is by definition a critical point of the energy functional. Since any harmonic map from a Riemann surface to a Riemannian manifold is automatically a minimal map, harmonic map were studied in connection with the theory of minimal surfaces from the beginning. Also it has been used to study the geometry of manifold such as Teichmüller theory and Kähler geometry. Böchner had singled out the theory of harmonic maps as generalized minimal surfaces and Morrey[6] had solved the famous Plateau problem in connection with harmonic map. Sacks-Uhlenbeck and R. Schoen had developed harmonic map theory related to minimal surface and Kähler geometry.

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In the midsixties, Eells and Sampson[2] had shown the existence of harmonic representative in each homotopy class when target manifold had negative curvature. They used heat flow to deform a map to an energy minimizing harmonic map in each homotopy class. This was a major breakthrough and have had deep influence on geometry. There are many works related to the heat flow of Eells-Sampson such as the characterization of those homotopy classes of maps between compact manifolds on which the energy functional takes arbitrary small values and a partial regularity theory for energy minimizing maps. The curvature assumption for the target manifold is the critical obstruction in higher dimensional case.

In two dimensional case, harmonic map has more nice properties. The energy functional is invariant under the conformal change of metric when the domain manifold is a surface. So harmonic map is closely related to conformal map in this case. For the existence of harmonic representative we don't need the curvature assumption. Sacks-Uhlenbeck[7] have shown that if the target manifold N is compact and $\pi_2(N) = 0$, then any homotopy class of maps from a compact surface to N contains an energy minimizing harmonic map amongst all maps in that class. Moreover there are some interesting results concerning harmonic diffeomorphism. Jost and Schoen[5] have shown that any diffeomorphism between two homeomorphic closed surfaces is homotopic to a harmonic diffeomorphism of least energy among all diffeomorphism homotopic to it.

There is another characterization of harmonic map given by Ishihara[3] which asserts that a map is harmonic iff it carries germs of convex functions to germs of subharmonic functions. As an immediate consequence of this assertion we know that arbitrary composition of two harmonic maps need not to be a harmonic map. A map is a harmonic morphism if it carries germs of harmonic functions to germs of harmonic functions. It had shown that a map is a harmonic morphism iff it is harmonic and horizontally conformal. Joining this to the Ishihara's characterization we know that composition of harmonic map and harmonic morphism is a harmonic map.

In this short article, we are interested in the harmonic diffeomorphism with some properties. Concerning to the composition property we consider harmonic map whose inverse is also harmonic. We first guess that it has some good properties such as composition invariant and a certain rigidity property and find out that it does not have composition invariant property but it has the following strong rigidity property; If $\phi : M, g \to N, h$ be a harmonic diffeomorphism between two Riemann surfaces whose inverse is also harmonic, ϕ is either a totally geodesic or a conformal diffeomorphism.

For the higher dimensional case, we cannot find out any meaningful results. Our method using in this article can be applied only to the two dimensional case and need some more conditions for the higher dimensional case.

2. Preliminaries

In this section, we recall some basic facts about the harmonic map and related topics, and introduce some examples of harmonic maps whose inverse is also harmonic.

For a map $\phi : M, g \to N, h$, the second fundamental form $\beta(\phi)$ is by definition the symmetric two tensor defined by $\beta(\phi) = \nabla d\phi$. A map is called harmonic when the trace of the second fundamental form $\tau(\phi) = trace_g \beta(\phi)$ of it is zero and totally geodesic when the second fundamental form is identically zero. Immediate from definition, totally geodesic map is harmonic but one can easily find harmonic maps which are not totally geodesic. In two dimensional case the conformal property is closely related to harmonic property. A map $\phi : M, g \to N, h$ is called conformal if $\phi^* h = \mu g$ for some nonnegative $\mu \in C(M)$.

The basic properties of harmonic maps and related totally geodesic maps are studied in many literatures, and we summarize some necessary results as follows.(See [1] for details.) The stress energy tensor is by definition the symmetric two tensor $S_\phi = e(\phi) - \phi^*h$, where $e(\phi) = \frac{1}{2}|d\phi|^2$ is the energy density of ϕ . By direct computation, $divS_\phi = -<\tau, d\phi>$ and so it vanishes if ϕ is harmonic. Conversely, if ϕ is a differentiable submersion almost everywhere and $divS_\phi = 0$ then ϕ is harmonic. When ϕ is nonconstant and dimM = m = 2, then $S_\phi = 0$ iff ϕ is conformal. If $m > 2$ and ϕ is harmonic and conformal, then ϕ is homothetic. When ϕ is totally geodesic, ϕ^*h is parallel and so $e(\phi)$ is constant.

Now let us introduce a condition for harmonic diffeomorphism which we will concern.

Definition 1. *A harmonic diffeomorphism is $B-$harmonic if its inverse is also harmonic.*

The isometry is the first example of $B-$harmonic diffeomorphism. In fact, since the inverse of totally geodesic map is also totally geodesic, every totally geodesic diffeomorphism is also $B-$harmonic. Moreover we have many interesting examples of B-harmonic diffeomorphism.

(1) Since every Lie group homomorphism between two Lie group with bi-invariant metric is harmonic, Lie group isomorphism is $B-$harmonic

(2) In Kähler case, every holomorphic map is harmonic and so biholomorphic map is $B-$harmonic in Kähler case.

(3) Especially in two dimensional case, the conformal map is harmonic. Since the inverse of conformal map is also conformal, the conformal diffeomorphism between two Riemann surfaces is $B-$harmonic.

One can expect that $B-$harmonic map has more strong rigidity or more good properties than the usual harmonic diffeomorphisms. We will introduce a partial answer about this question in the next section, which is complete in two dimensional case. It is remarkable that all of the above examples of $B-$harmonic maps form a group under composition when they are maps from a manifold onto itself. But the set of all harmonic diffeomorphisms does not form a group under composition in general. For two maps $\phi : M \to N$ and $\psi : N \to \bar{N}$, the tension field of their composition $\psi \circ \phi$ is

$$\tau(\psi \circ \phi) = \beta(\psi)(d\phi(e_i), d\phi(e_i)) + d\psi(\tau(\phi)),$$

where $\{e_i\}$ is an orthonormal frame field in M and β is the second fundamental form. Hence, the for the two harmonic map ϕ and ψ the composition $\psi \circ \phi$ is harmonic iff $\beta(\psi)(d\phi(e_i), d\phi(e_i)) = 0$. This assertion cannot be satisfied for the general harmonic maps and so the set of all harmonic diffeomorphisms does not form a group under composition.

3. Rigidity property

All manifolds are assumed to be a compact oriented Riemann surface without boundary throughout this section. We now start with a lemma which tells the relation between the second fundamental form of ϕ and that of its inverse ϕ^{-1}.

Lemma 1. *Let $\phi : (M, g) \to (N, h)$ be a diffeomorphism and X, Y be vector fields on M. Then*

$$d\phi^{-1}(\beta(\phi)(X, Y)) = -\beta(\phi^{-1})(d\phi(X), d\phi(Y)).$$

Proof. The second fundamental form of ϕ^{-1} is by definition

$$\begin{aligned}
\beta(\phi^{-1})(d\phi(X), d\phi(Y)) :&= (\nabla d\phi^{-1})(d\phi(X), d\phi(Y)) \\
&= (\nabla_{d\phi(X)} d\phi^{-1})(d\phi(Y)) \\
&= \nabla^{(\phi^{-1})^*(TM)}_{d\phi(X)} d\phi^{-1}(d\phi(Y)) - d\phi^{-1}(\nabla_{d\phi(X)} d\phi(Y)).
\end{aligned}$$

The first term of the above equation can be reduced as follows;

$$\nabla^{(\phi^{-1})^*(TM)}_{d\phi(X)} d\phi^{-1}(d\phi(Y)) = \nabla_{d\phi^{-1}(d\phi(X))} Y = \nabla_X Y = d\phi^{-1}(d\phi(\nabla_X Y)).$$

By substituting this to the first equation, we have

$$\begin{aligned}\beta(\phi^{-1})(d\phi(X), d\phi(Y)) &= \nabla^{(\phi^{-1})^*(TM)}_{d\phi(X)} d\phi^{-1}(d\phi(Y)) - d\phi^{-1}(\nabla_{d\phi(X)} d\phi(Y)) \\ &= d\phi^{-1}(d\phi(\nabla_X Y)) - d\phi^{-1}(\nabla_{d\phi(X)} d\phi(Y)) \\ &= -d\phi^{-1}(\nabla_{d\phi(X)} d\phi(Y) - d\phi(\nabla_X Y)) \\ &= -d\phi^{-1}(\beta(\phi)(X, Y)).\end{aligned}$$

□

The above lemma tells that the stress energy tensors of a map and its inverse are closely related. So there are some possibilities for $B-$harmonic map to have more strong rigidity.

Theorem 1. *Let $\phi : M, g \to N, h$ be a $B-$harmonic map between two Riemann surfaces. Then ϕ is either totally geodesic or a conformal diffeomorphism.*

Proof. First we begin with the pointwise argument. Fix $x \in M$ and choose an orthonorml frame field $\{e_1, e_2\}$ on a neighborhood of x such that $\{d\phi(e_1), d\phi(e_2)\}$ is an orthogonal frame field on a neighborhood of $\phi(x)$. One can find such a basis by diagonalizing $\phi^* h$ since ϕ is a diffeomorphism. Let $f_i = \frac{d\phi(e_i)}{\|d\phi(e_i)\|}$ and $d\phi(e_i) = \lambda_i f_i$, i.e., $\{f_1, f_2\}$ is an orthonormal frame field on a neighborhood of $\phi(x)$. Put $\mu_i = \frac{1}{\lambda_i}$. Since the second fundamental form is C^∞ bi-linear, by using the above two frames, the harmonic map equations of ϕ and ϕ^{-1} become

(1) $\tau(\phi) = \beta(\phi)(e_1, e_1) + \beta(\phi)(e_2, e_2) = 0$

(2)

$$\begin{aligned}\tau(\phi^{-1}) &= \beta(\phi^{-1})(f_1, f_1) + \beta(\phi^{-1})(f_2, f_2) \\ &= \mu_1^2 \beta(\phi^{-1})(d\phi(e_1), d\phi(e_1)) + \mu_2^2 \beta(\phi^{-1})(d\phi(e_2), d\phi(e_2)) \\ &= 0.\end{aligned}$$

Using Lemma 1, (2) can be written as

$$d\phi^{-1}(\mu_1^2\beta(\phi)(e_1,e_1)+\mu_2^2\beta(\phi)(e_2,e_2))=0,$$

which is equivalent to

(3) $\mu_1^2\beta(\phi)(e_1,e_1)+\mu_2^2\beta(\phi)(e_2,e_2)=0$

since ϕ is a diffeomorphism.

From the above two equations (1) and (3) it is easy to show that either $\mu_1^2=\mu_2^2$ or $\beta(\phi)(e_1,e_1)=\beta(\phi)(e_2,e_2)=0$ should hold. ϕ is conformal in the first case and totally geodesic in the other case.

It is known that any harmonic map from a compact Riemann surface of genus g which is not conformal is conformal at no more than $4g-4$ points(See p43 of [1]). Hence, by applying this to our case, we know that any $B-$harmonic map between two Riemann surfaces which is not conformal is totally geodesic with only $4g-4$ possible exceptions, which are finite especially. Hence it is totally geodesic on the whole space. □

Since totally geodesic diffeomorphism is $B-$harmonic in general and conformal diffeomorphism is $B-$harmonic in two dimensional case, the above theorem can be written as following.

Theorem 2. *Let (M,g), (N,h) be two compact oriented Riemann surfaces. A map $\phi: M,g\to N,h$ is $B-$harmonic iff it is either totally geodesic or conformal.*

Now we will finish our section with some remark on the composition property of the $B-$harmonic diffeomorphism. Let $\phi: M\to N$ and $\psi: N\to\bar{N}$ be two $B-$harmonic maps between compact oriented Riemann surfaces. By the above theorem, they are totally geodesic or conformal. When ψ is totally geodesic, the composition formula

$$\tau(\psi\circ\phi)=\beta(\psi)(d\phi(e_i),d\phi(e_i))+d\psi(\tau(\phi))$$

tells that the composition is also $B-$harmonic. When ψ is conformal, there are two cases (1) ϕ is conformal or (2) ϕ is totally geodesic. In the case of (1), the composition is $B-$harmonic since any composition of two conformal maps again becomes a conformal map and conformal map is automatically harmonic in two dimensional case. In the case of (2), the composition map is not always harmonic in general. If it were harmonic, then it should be either totally geodesic or conformal. When the composition $\psi\circ\phi$ is conformal, $\phi = \psi^{-1}(\psi\circ\phi)$ is a composition of two conformal maps and so is conformal. Joining this to the assumption that ϕ is totally geodesic, we get ϕ is both conformal and totally geodesic in this case. When the composition $\psi\circ\phi$ is totally geodesic, $\psi = (\psi\circ\phi)\phi^{-1}$ is a composition of two totally geodesic maps and so is totally geodesic. Joining this to the assumption that ψ is conformal we get ψ is both conformal and totally geodesic in this case. But the following simple proposition tells that it is not the general situation.

Proposition 1. *Every diffeomorphism which is both conformal and totally geodesic is homothetic.*

Proof. Let $\phi : M, g \to N, h$ be a conformal and totally geodesic map. Since ϕ is conformal, $\phi^*h = \mu g$ for some nonnegative $\mu \in C(M)$. Let $\{\theta_i\}$ and $\{e_i\}$ be an orthonormal basis such that $d\phi(e_i) = \mu\theta_i$. On the other hand, since ϕ is totally geodesic, we have

$$\begin{aligned} 0 &= \nabla d\phi(e_i, e_j) \\ &= \nabla_{d\phi(e_i)} d\phi(e_j) - d\phi(\nabla_{e_i} e_j) \\ &= \mu\theta_i(\mu)\theta_j + \mu^2 \nabla_{\theta_i}\theta_j \\ &= \mu\theta_i(\mu)\theta_j. \end{aligned}$$

Hence μ is a constant function, i.e., ϕ is homothetic. □

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