KYUNGPOOK Math. J. 45(2005), 579-594

Oscillation Criteria of Second-order Half-linear Delay Difference Equations

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Abstract. In this paper, we will establish some new oscillation criteria for the second-order half-linear delay difference equation

$$\Delta(p_n (\Delta x_n)^\gamma) + q_n x_{n-\sigma}^\gamma = 0, \qquad n \geq n_0,$$

where $\gamma > 0$ is a quotient of odd positive integers. Our results in this paper are sharp and improve some of the well known oscillation results in the literature. Some examples are considered to illustrate our main results.

1. Introduction

In this paper, we are concerned with oscillation of the second order half-linear delay difference equation

$$\Delta(p_n (\Delta x_n)^\gamma) + q_n x_{n-\sigma}^\gamma = 0, \qquad n \geq n_0, \tag{1.1}$$

where Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$ for any sequence $\{x_n\}$ of real numbers. Throughout this paper we assume that: σ is a fixed nonnegative integer, $\gamma > 0$ is quotient of odd positive integers, $\{p_n\}_{n=n_0}^\infty$ and $\{q_n\}_{n=n_0}^\infty$ are sequences of real numbers such that $p_n > 0$, $q_n \geq 0$ and $\{q_n\}_{n=n_0}^\infty$ has a positive subsequence, and

$$\sum_{n=n_0}^\infty \left(\frac{1}{p_n}\right)^{\frac{1}{\gamma}} = \infty, \tag{1.2}$$

or

$$\sum_{n=n_0}^\infty \left(\frac{1}{p_n}\right)^{\frac{1}{\gamma}} < \infty. \tag{1.3}$$

By a solution of (1.1) we mean a nontrivial sequence $\{x_n\}$ which is defined for $n \geq -\sigma$ and satisfies Eq.(1.1) for $n = 0, 1, 2 \cdots$. Clearly if

$$x_n = A_n \quad \text{for } n = -\sigma, \cdots, -1, n_0 - 1, \tag{1.4}$$

Received May 11, 2004, and, in revised form, June 29, 2004.
2000 Mathematics Subject Classification: 39A10.
Key words and phrases: oscillation, half-linear, second order delay difference equations.

are given, then Eq.(1.1) has a unique solution satisfying the initial conditions (1.4). A solution $\{x_n\}$ of (1.1) is said to be oscillatory if for every $n_1 > n_0$ there exists an $n \geq n_1$ such that $x_n x_{n+1} \leq 0$. Otherwise it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The problem of determining oscillation and nonoscillation of solutions of second order difference equations has been a very active area of research in the last ten years, and for surveys of recent results we refer to the monographs of Agrawal [2] and Agarwal and Wong [3].

Half-linear equations derive their name from the fact that if $\{x_n\}$ is a solution, then so is $\{cx_n\}$ for any constant c. Half-linear equations of the form

$$\Delta(p_n\,(\Delta x_n)^\gamma) + q_{n+1}x_{n+1}^\gamma = 0, \qquad n = 0, 1, 2\cdots, \tag{1.5}$$

and their generalizations have received a good bit of attention in the literature in the last few years, and we cite as recent contributions the papers of Cheng [4], Chcchi et al. [6], Dosly and Rehak [7], Liu and Cheng [15], Rehak [20], [21], Thandapani et al. [27]-[29] and Wong and Agarwal [31]. Many of these approaches in the above mentioned papers employ Riccati equations of various types to obtain criteria which guarantee that any nontrivial solution is oscillatory or nonoscillatory.

When $p_n = 1$, Eq.(1.1) reduces to the difference equation

$$\Delta\,(\Delta x_n)^\gamma + q_n x_{n-\sigma}^\gamma = 0, \qquad n \geq n_0, \tag{1.5}$$

which has been considered by Thandapani et al. [30] and proved that: If

$$\sum_{l=n_0}^{\infty} q_l = \infty, \tag{1.6}$$

then every solution of Eq.(1.5).

For oscillation and nonoscillation of different classes of second order difference equations we refer the reader to [5], [8], [10], [11], [14], [16]-[18], [24]-[27], [33]-[37]. In the oscillation of second order differential equations, the equation

$$x^{''}(t) + q(t)f(x(t)) = 0,\ t \geq t_0, \tag{1.7}$$

has been tackled by many authors, see the survey papers [13], [32] which give over 300 references. It is known that, due to Kamenev [12] the average function $A_\lambda(t)$ defined by

$$A_\lambda(t) = \frac{1}{t^\lambda}\int_{t_0}^{t}(t-s)^\lambda q(s)ds, \quad \lambda \geq 1, \tag{1.8}$$

plays a crucial role in the oscillation of Eq.(1.7).

Philos [19] further extends Kamenev's result by proving the following: Suppose there exist continuous functions H, $h : D \equiv \{(t,s) : t \geq s \geq t_0\} \to R$ such that

(i) $H(t,t) = 0, \;\; t \geq t_0$,

(ii) $H(t,s) > 0, \;\; t > s \geq t_0$,

and H has a continuous partial derivative on D with respect to the second variable and satisfies

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \geq 0. \tag{1.9}$$

Further, suppose that

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} [H(t,s)q(s) - \frac{1}{4}h^2(t,s)]\, ds = \infty, \tag{1.10}$$

then every solution of equation (1.7) oscillates.

By means of Riccati transformation techniques, we establish some new oscillation criteria and Kamanev-type oscillation criteria for Eq.(1.1) which can be considered as the discrete analogues of (1.8) and (1.10). In Section 2, we consider the case when (1.2) holds and establish some sufficient conditions for oscillation of all solutions of (1.1). In Section 3, we consider the case when (1.3) holds and establish some sufficient conditions which guarantee that every solution $\{x_n\}$ of (1.1) oscillates or converges to zero. Our results when (1.2) holds extend as well as improve the results by Cheng [4], Dosly and Rehak [7], Liu and Cheng [15], Rehak [20], [21] and Thandapani et al. [29], [30], and when (1.3) holds our results are essentially new. Some examples are considered to illustrate the main results.

2. Oscillation criteria when (1.2) holds

In this section, we consider the case when (1.2) holds and establish some sufficient conditions for oscillation of all solutions of (1.1).

First we consider the case when $\gamma > 0$ and $\Delta p_n \geq 0$.

Theorem 2.1. *Assume that* (1.2) *holds. If every solution of the delay difference equation*

$$\Delta y_n + \frac{q_n}{p_{n-\sigma}} \left(\frac{n-\sigma}{2}\right)^{\gamma} y_{n-\sigma} = 0, \quad n \geq n_1 \geq n_0, \tag{2.1}$$

oscillates, then every solution of Eq.(1.1) *oscillates for all* $\gamma > 0$.

Proof. Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1) such that $x_n > 0$ and $x_{n-\sigma} > 0$ for all $n \geq n_1 \geq n_0$. We shall consider only this case, since the substitution $y_n = -x_n$ transforms Eq.(1.1) into an equation of the same form. From Eq.(1.1) we have

$$\Delta(p_n\,(\Delta x_n)^\gamma) = -q_n x_{n-\sigma}^\gamma \leq 0, \qquad n \geq n_1, \tag{2.2}$$

and so $p_n\,(\Delta x_n)^\gamma$ is an eventually nonincreasing sequence. We first show that $p_n(\Delta x_n)^\gamma$ is eventually positive. Indeed, since $\{q_n\}_{n_0}^\infty$ has a positive subsequence, the nondecreasing sequence $\{p_n(\Delta x_n)^\gamma\}$ is either eventually positive or eventually negative. Suppose there exists an integer $n_2 \geq n_1$ such that $p_{n_2}(\Delta x_{n_2})^\gamma = c < 0$ for $n \geq n_2$, then (2.2) implies that $p_n\,(\Delta x_n)^\gamma \leq p_{n_2}(\Delta x_{n_2})^\gamma = c$, hence

$$\Delta x_n \leq c^{1/\gamma}\left(\frac{1}{p_n}\right)^{1/\gamma},$$

which implies that

$$x_n \leq x_{n_2} + c^{1/\gamma}\sum_{i=n_2}^{n-1}\left(\frac{1}{p_i}\right)^{1/\gamma} \to -\infty \text{ as } n \to \infty, \tag{2.3}$$

which contradicts the fact that $x_n > 0$ for all large n. Hence $p_n(\Delta x_n)^\gamma$ is eventually positive. Therefore, we see that there is some $n_1 \geq n_0$ such that

$$x_n > 0,\ \Delta x_n \geq 0,\ \Delta(p_n\,(\Delta x_n)^\gamma) \leq 0,\ n \geq n_1. \tag{2.4}$$

From (2.4), since $\Delta(p_n\,(\Delta x_n)^\gamma) \leq 0$, then we have $\Delta^2 x_n \leq 0$ for $n \geq n_0$. If not there exists $n_2 \geq n_1$ such that $\Delta^2 x_n > 0$ and this implies that $\Delta x_{n+1} > \Delta x_n$, so that since $\Delta p_n \geq 0$, $p_{n+1}\,(\Delta x_{n+1})^\gamma > p_{n+1}\,(\Delta x_n)^\gamma \geq p_n\,(\Delta x_n)^\gamma$ and this contradicts the fact that $\{p_n\,(\Delta x_n)^\gamma\}$ is nonincreasing sequence, Then $\Delta^2 x_n \leq 0$ and then $\{\Delta x_n\}$ is nonincreasing sequence, and this implies that $x_n - x_{n_1} = \sum_{k=n_1}^{n-1} \Delta x_k \geq (n-n_1)\Delta x_n$ which leads to $x_n \geq \frac{n}{2}\Delta x_n$ for $n \geq n_2 \geq 2n_1 + 1$. Then

$$x_{n-\sigma} \geq \frac{n-\sigma}{2}\Delta x_{n-\sigma}, \qquad n \geq n_3 = n_2 + \sigma. \tag{2.5}$$

Hence, from (2.5) and (1.1), we have

$$\Delta(p_n\,(\Delta x_n)^\gamma) + q_n\left(\frac{n-\sigma}{2}\right)^\gamma(\Delta x_{n-\sigma})^\gamma \leq 0, \quad n \geq n_3. \tag{2.6}$$

Set $y_n = p_n\,(\Delta x_n)^\gamma$, then $y_n > 0$ and satisfies

$$\Delta y_n + \frac{q_n}{p_{n-\sigma}}\left(\frac{n-\sigma}{2}\right)^\gamma y_{n-\sigma} \leq 0, \quad n \geq n_3. \tag{2.7}$$

But, then by Lemma 1 in [38] the delay difference equation (2.1) have an eventually positive solution also, which contradicts the assumption that every solution of Eq.(2.1) oscillates. Then every solution of (1.1) is oscillatory. □

Theorem 2.1 shows that the oscillation of problem (1.1) is equivalent to the oscillation of the delay difference equation (2.1). Thus, we can use the results of oscillation of first order difference equations in [38] and the references cited therein to obtain several oscillation criteria for Eq.(1.1). The details are left to the reader.

Note that Theorem 2.1 can not be applied to Eq.(1.1) when $\sigma = 0$, since the first-order difference equation $\Delta y_n + \frac{q_n}{p_n}\left(\frac{n}{2}\right)^{\gamma} y_n = 0$ can not have oscillatory solutions, since $\frac{q_n}{p_n}\left(\frac{n}{2}\right)^{\gamma} > 0$. Then the retarded arguments σ appearing in the nonlinear term plays important role in the generating qualitative behavior for equation (1.1) different from that for the corresponding equations with $\sigma = 0$. It is of interest to find some new oscillation criteria different from of the results in Theorem 2.1.

Next, we consider the case when $\gamma \geq 1$ and $\Delta p_n \geq 0$.

In the following by sing the Riccati transformation technique, we will give new oscillation result of (1.1) which is the discrete analogy Philos-type (1.10), and from it we derive the Kamenev-type oscillation condition which can be considered as the discrete analogy of (1.8).

Theorem 2.2. *Assume that* (1.2) *holds. Let* $\{\rho_n\}_{n=0}^{\infty}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *such that*:

(i) $H_{m,m} = 0$ *for* $m \geq 0$;

(ii) $H_{m,n} > 0$ *for* $m > n \geq 0$;

(iii) $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ *for* $m \geq n \geq 0$.

If

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} \left[H_{m,n}\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n} \left(h_{m,n}\sqrt{H_{m,n}} - \frac{\Delta\rho_n}{\rho_{n+1}} H_{m,n} \right)^2 \right] = \infty, \tag{2.8}$$

where

$$h_{m,n} = \frac{-\Delta_2 H_{m,n}}{\sqrt{H_{m,n}}}, \quad \bar{\rho}_n = \frac{\gamma\left(\frac{n-\sigma}{2}\right)^{\gamma-1}\rho_n}{p_{n-\sigma}}$$

then every solution of Eq.(1.1) *oscillates for all* $\gamma \geq 1$.

Proof. Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1) such that $x_n > 0$ and $x_{n-\sigma} > 0$ for all $n \geq n_1 \geq n_0$. We proceed as in the proof of Theorem 2.1 to prove that (2.4) holds for $n \geq n_1$. Define $\{w_n\}$ by

$$w_n = \rho_n \frac{p_n (\Delta x_n)^{\gamma}}{x_{n-\sigma}^{\gamma}}, \tag{2.9}$$

then $w_n > 0$, and

$$\Delta w_n = p_{n+1}\left(\Delta x_{n+1}\right)^{\gamma} \Delta\left[\frac{\rho_n}{x_{n-\sigma}^{\gamma}}\right] + \frac{\rho_n \Delta(p_n\left(\Delta x_n\right)^{\gamma})}{x_{n-\sigma}^{\gamma}}. \tag{2.10}$$

From (1.1) and (2.10), we have

$$\Delta w_n = -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n p_{n+1}\left(\Delta x_{n+1}\right)^{\gamma} \Delta(x_{n-\sigma}^{\gamma})}{x_{n+1-\sigma}^{\gamma} x_{n-\sigma}^{\gamma}}. \tag{2.11}$$

But, (2.4) implies that

$$p_{n-\sigma}\left(\Delta x_{n-\sigma}\right)^{\gamma} \geq p_{n+1}\left(\Delta x_{n+1}\right)^{\gamma}, \quad \textit{and} \quad x_{n+1-\sigma} \geq x_{n-\sigma}, \tag{2.12}$$

and then from (2.11) and (2.12), we have

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n p_{n+1}\left(\Delta x_{n+1}\right)^{\gamma} \Delta(x_{n-\sigma}^{\gamma})}{\left(x_{n+1-\sigma}^{\gamma}\right)^2}. \tag{2.13}$$

Now, by using the inequality (cf. [9, p. 39])

$$x^{\gamma} - y^{\gamma} \geqslant \gamma y^{\gamma-1}(x-y) \text{ for all } x \neq y > 0 \text{ and } \gamma \geqslant 1,$$

we obtain

$$\left.\begin{aligned} \Delta(x_{n-\sigma}^{\gamma}) = x_{n+1-\sigma}^{\gamma} - x_{n-\sigma}^{\gamma} &\geqslant \gamma\left(x_{n-\sigma}\right)^{\gamma-1}\left(x_{n+1-\sigma} - x_{n-\sigma}\right) \\ &= \gamma\left(x_{n-\sigma}\right)^{\gamma-1}\left(\Delta x_{n-\sigma}\right), \quad \gamma \geqslant 1. \end{aligned}\right\}, \tag{2.14}$$

since $\Delta x_n \geq 0$. Substituting from (2.14) in (2.13), we have

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \rho_n p_{n+1} \frac{\gamma\left(x_{n-\sigma}\right)^{\gamma-1}\left(\Delta x_{n-\sigma}\right)\left(\Delta x_{n+1}\right)^{\gamma}}{\left(x_{n+1-\sigma}^{\gamma}\right)^2}. \tag{2.15}$$

From (2.5), (2.12) and (2.15), we have for $n \geq n_3$

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \gamma\left(\frac{n-\sigma}{2}\right)^{\gamma-1} \frac{\rho_n\left(p_{n+1}\right)^2}{p_{n-\sigma}} \frac{\left(\Delta x_{n+1}\right)^{2\gamma}}{\left(x_{n+1-\sigma}^{\gamma}\right)^2}. \tag{2.16}$$

Hence,

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\gamma\left(\frac{n-\sigma}{2}\right)^{\gamma-1}\rho_n}{\left(\rho_{n+1}\right)^2 p_{n-\sigma}} \frac{\left(p_{n+1}\right)^2\left(\rho_{n+1}\right)^2\left(\Delta x_{n+1}\right)^{2\gamma}}{\left(x_{n+1-\sigma}^{\gamma}\right)^2}. \tag{2.17}$$

From (2.9) and (2.17), we have for $n \geq n_3$

$$\rho_n q_n \leq -\Delta w_n + \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \frac{\bar{\rho}_n}{\left(\rho_{n+1}\right)^2} w_{n+1}^2. \tag{2.18}$$

Therefore,

$$\sum_{n=n_3}^{m-1} H_{m,n}\rho_n q_n \le -\sum_{n=n_3}^{m-1} H_{m,n}\Delta w_n + \sum_{n=n_3}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_3}^{m-1} H_{m,n}\frac{\bar\rho_n}{\rho_{n+1}^2} w_{n+1}^2,$$

which yields, after summing by parts,

$$\begin{aligned}
&\sum_{n=n_3}^{m-1} H_{m,n}\rho_n q_{n+1} \le H_{m,n_3} w_{n_3}\\
&\quad + \sum_{n=n_3}^{m-1} w_{n+1}\Delta_2 H_{m,n} + \sum_{n=n_3}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_3}^{m-1} H_{m,n}\frac{\bar\rho_n}{(\rho_{n+1})^2} w_{n+1}^2\\
= \;& H_{m,n_3} w_{n_3} - \sum_{n=n_3}^{m-1} h_{m,n}\sqrt{H_{m,n}}\, w_{n+1}\\
&\quad + \sum_{n=n_3}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \sum_{n=n_3}^{m-1} H_{m,n}\frac{\bar\rho_n}{(\rho_{n+1})^2} w_{n+1}^2\\
= \;& H_{m,n_3} w_{n_3}\\
&\quad - \sum_{n=n_3}^{m-1}\left[\frac{\sqrt{H_{m,n}\bar\rho_n}}{\rho_{n+1}} w_{n+1} + \frac{\rho_{n+1}}{2\sqrt{H_{m,n}\bar\rho_n}}\left(h_{m,n}\sqrt{H_{m,n}} - \frac{\Delta\rho_n}{\rho_{n+1}} H_{m,n}\right)\right]^2\\
&\quad + \frac{1}{4}\sum_{n=n_3}^{m-1}\frac{(\rho_{n+1})^2}{\bar\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2
\end{aligned}$$

Then,

$$\sum_{n=n_3}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < H_{m,n_3} w_{n_3} \le H_{m,0} w_{n_3},$$

which implies that

$$\begin{aligned}
&\sum_{n=n_3}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right]\\
< \;& H_{m,0}\sum_{n=n_3}^{n_3-1}\rho_n q_{n+1} + H_{m,0} w_{n_3}.
\end{aligned}$$

Hence

$$\limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=n_3}^{m-1}\left[H_{m,n}\rho_n q_n - \frac{\rho_{n+1}^2}{4\bar\rho_n}\left(h_{m,n} - \frac{\Delta\rho_n}{\rho_{n+1}}\sqrt{H_{m,n}}\right)^2\right] < \infty,$$

and this contradicts (2.8). The proof is complete. □

From Theorem 2.2, by choosing the sequence $\{H_{m,n}\}$ in appropriate manners, we can derive several oscillation criteria for (1.1). For instance, let us consider the double sequence $\{H_{m,n}\}$ defined by $H_{m,m} = 0$ and $H_{m,n} = c$, where c is a constant. Then, we have the following result.

Corollary 2.1. *Assume that* (1.2) *holds. Furthermore, assume that there exists a positive sequence* $\rho_n\}_{n=0}^{\infty}$ *such that*

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[\rho_l q_l - \frac{p_{l-\sigma}(\Delta\rho_l)^2}{4\gamma\left(\frac{l-\sigma}{2}\right)^{\gamma-1}\rho_l} \right] = \infty, \tag{2.19}$$

then every solution of Eq.(1.1) *oscillates for all* $\gamma \geq 1$.

Note that from Corollary 2.1, if $\rho_n = n$ and $p_n = 1$, we have

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[l q_l - \frac{1}{4\gamma\left(\frac{l-\sigma}{2}\right)^{\gamma-1} l} \right] = \infty.$$

which improves the condition (1.6). Then Corollary 2.1 extend and improve the results of Thandapani [30].

When $\gamma = 1$, Eq.(1.1) reduces to the linear delay difference equation

$$\Delta(p_n \Delta x_n) + q_n x_{n-\sigma} = 0, \qquad n \geq n_0,$$

and the condition (2.19) in Corollary 2.1 reduces to

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[\rho_l q_l - \frac{(p_{l-\sigma})\,(\Delta\rho_l)^2}{4\rho_l} \right] = \infty, \tag{2.20}$$

which is the same condition in Corollary 1 in [26]. Then Theorem 2.2 is an extension of Theorem 1 and Corollary 1 in [26].

We can obtain different conditions for the oscillation of all solutions of (1.1) by different choices of $\{\rho_n\}$. For instance, let $\rho_n = n^{\lambda}$, $n \geq n_0$ and $\lambda > 1$. Then from Corollary 2.1, we have the following result.

Corollary 2.2. *Assume that* (1.2) *holds and*

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} \left[s^{\lambda} q_s - \frac{p_{s-\sigma}((s+1)^{\lambda} - s^{\lambda})^2}{4\gamma\left(\frac{s-\sigma}{2}\right)^{\gamma-1} s^{\lambda}} \right] = \infty, \tag{2.21}$$

then every solution of Eq.(1.1) *oscillates for all* $\gamma \geq 1$.

Next, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$H_{m,n} = (m-n)^{\lambda},\ \lambda \geq 1,\ m \geq n \geq 0, \tag{2.22}$$

Then $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. Then from Theorem 2.2, we have the following oscillation result of Kamenev-type.

Corollary 2.3. *Assume that* (1.2) *holds. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}_{n=0}^{\infty}$ *such that for all* $\lambda \geq 1$,

$$\lim_{m\to\infty} \sup \frac{1}{m^{\lambda}} \sum_{n=n_0}^{m-1} (m-n)^{\lambda} \left[\rho_n q_n - \frac{(\rho_{n+1})^2}{4\,\bar{\rho}_n} C_{m,n} \right] = \infty, \tag{2.23}$$

where

$$C_{m,n} = \left(\frac{\Delta \rho_n}{\rho_{n+1}} + \frac{\lambda (m-n-1)^{\lambda-1}}{(m-n)^{\lambda}} \right)^2,\ \bar{\rho}_n = \gamma \left(\frac{n-\sigma}{2} \right)^{\gamma-1} \rho_n / (p_{n-\sigma}), \tag{2.24}$$

then every solution of equation (1.1) *oscillates for all* $\gamma \geq 1$.

Corollary 2.4. *Assume that all the assumptions of Corollary 2.3 hold, except the condition* (2.23) *is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{m^{\lambda}} \sum_{n=n_0}^{m-1} (m-n)^{\lambda} \rho_n q_n = \infty, \tag{2.25}$$

and

$$\lim_{m\to\infty} \frac{1}{m^{\lambda}} \sum_{n=n_0}^{m-1} (m-n) \frac{(\rho_{n+1})^2}{\bar{\rho}_n} \left(\frac{\Delta \rho_n}{\rho_{n+1}} + \frac{\lambda (m-n-1)^{\lambda-1}}{(m-n)^{\lambda}} \right)^2 < \infty, \tag{2.26}$$

then every solution of Eq.(1.1) *oscillates for all* $\gamma \geq 1$.

Next, let us consider the double sequence $\{H_{m,n}\}$ defined by

$$H_{m,n} = \left(\ln \frac{m+1}{n+1} \right)^{\lambda},\ \lambda \geq 1,\ m \geq n \geq 0. \tag{2.27}$$

Then $H_{m,m} = 0$ for $m \geq 0$ and $H_{m,n} > 0$ and $\Delta_2 H_{m,n} \leq 0$ for $m > n \geq 0$. Then from Theorem 2.2, we have the following result.

Corollary 2.5. *Assume that all the assumptions of Theorem 2.2 hold, except the condition* (2.8) *is replaced by*

$$\lim_{m\to\infty} \sup \frac{1}{(\log(m+1))^{\lambda}} \sum_{n=0}^{m} \left[\left(\log \frac{m+1}{n+1} \right)^{\lambda} \rho_n q_n - \frac{\rho_{n+1}^2}{4\bar{\rho}_n} B_{m,n} \right] = \infty,$$

where

$$B_{m,n} = \left(\frac{\lambda}{n+1} \left(\ln \frac{m+1}{n+1} \right)^{\frac{\lambda-2}{2}} - \frac{\Delta \rho_n}{\rho_{n+1}} \sqrt{\left(\ln \frac{m+1}{n+1} \right)^{\lambda}} \right)^2,$$

then every solution of Eq.(1.1) *oscillates for all* $\gamma \geq 1$.

Another $H_{m,n}$ may be chosen as

$$H_{m,n} = \phi(m-n), \ m \geq n \geq 0, \tag{2.28}$$

$$H_{m,n} = (m-n)^{(\lambda)} \quad \lambda > 2, \ m \geq n \geq 0.$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuously differentiable function which satisfies $\phi(0) = 0$ and $\phi(u) > 0$, $\phi^{'}(u) \geq 0$ for $u > 0$, and $(m-n)^{(\lambda)} = (m-n)(m-n+1) \cdots (m-n+\lambda-1)$ and

$$\Delta_2 (m-n)^{(\lambda)} = (m-n-1)^{(\lambda)} - (m-n)^{(\lambda)} = -\lambda (m-n)^{(\lambda-1)}$$

A corresponding corollary can also be stated.

Next, we give some new oscillation criteria for Eq.(1.1) without the assumption that $\Delta p_n \geq 0$.

Theorem 2.3. *Assume that* (1.2) *holds. Furthermore, assume that there exists a positive sequence* $\{\rho_n\}$ *such that*

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[\rho_l q_l - \frac{(p_{l-\sigma}) (\Delta \rho_l)^2}{2^{3-\gamma} \rho_l} \right] = \infty, \tag{2.29}$$

then every solution of Eq.(1.1) *oscillates for all* $\gamma \geq 1$.

Proof. Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1) such that $x_n > 0$ and $x_{n-\sigma} > 0$ for all $n \geq n_0$. As in the proof of Theorem 2.1 we have that $\Delta x_n \geq 0$ for $n \geq n_1$. Defining $\{w_n\}$ by (2.9) then as in the proof of Theorem 2.2 we obtain

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\rho_n p_{n+1} (\Delta x_{n+1})^\gamma \Delta x_{n-\sigma}^\gamma}{\left(x_{n+1-\sigma}^\gamma \right)^2}. \tag{2.30}$$

Now, by using the inequality

$$x^\gamma - y^\gamma \geqslant 2^{1-\gamma} (x-y)^\gamma \ for\ all\ x \geqslant y > 0\ and\ \gamma \geqslant 1,$$

we have

$$\Delta x_{n-\sigma}^\gamma = x_{n+1-\sigma}^\gamma - x_{n-\sigma}^\gamma \geqslant 2^{1-\gamma} (x_{n+1-\sigma} - x_{n-\sigma})^\gamma = 2^{1-\gamma} (\Delta x_{n-\sigma})^\gamma. \tag{2.31}$$

Substituting from (2.31) in (2.30), we obtain

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - 2^{1-\gamma} \rho_n p_{n+1} \frac{(\Delta x_{n+1})^\gamma (\Delta x_{n-\sigma})^\gamma}{\left(x^\gamma_{n+1-\sigma}\right)^2}. \tag{2.32}$$

Again, from (2.12) in (2.32), we obtain

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - 2^{1-\gamma} \rho_n \frac{(p_{n+1})^2}{(p_{n-\sigma})} \frac{(\Delta x_{n+1})^{2\gamma}}{\left(x^\gamma_{n+1-\sigma}\right)^2}. \tag{2.33}$$

Using (2.9) in (2.33), we get

$$\Delta w_n \leq -\rho_n q_n + \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - 2^{1-\gamma} \frac{\rho_n}{(\rho_{n+1})^2} \frac{1}{(p_{n-\sigma})} w^2_{n+1}. \tag{2.34}$$

The remainder of the proof is by completing the square and similar to that of Theorem 2.2 and hence is omitted. □

From Theorem 2.3 if $\rho_n = n$ and $p_n = 1$, then Theorem 2.3 reduces to

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[l q_l - \frac{1}{2^{3-\gamma} l} \right] = \infty,$$

which also improves the condition (1.6). Then Theorem 2.3 also improve the results in [30].

Note that from Theorem 2.3, we can obtain different conditions for the oscillation of all solutions of Eq.(1.1) when (1.2) holds by different choices of $\{\rho_n\}$. Let $\rho_n = n^\lambda$, $n \geq n_0$ and $\lambda > 1$ is a constant, we have the following result.

Corollary 2.6. *Assume that* (1.2) *holds and*

$$\lim_{n\to\infty} \sup \sum_{s=n_0}^{n} \left[s^\lambda q_s - \frac{(p_{s-\sigma}) \left((s+1)^\lambda - s^\lambda\right)^2}{2^{3-\gamma} s^\lambda} \right] = \infty,$$

then every solution of Eq.(1.1) *oscillates for all* $\gamma \geq 1$.

The following example illustrates our main results.

Example 2.1. Consider the half-linear discrete Euler equation

$$\Delta\left((\Delta x_n)^\gamma\right) + \frac{\mu}{n^2} (x_{n-\sigma})^\gamma = 0,\ n \geq 1, \tag{2.35}$$

where $\mu > \frac{1}{2^{3-\gamma}}$. Then, $p_n = 1$, $\sigma > 0$. If we take $\rho_n = n$, then we have

$$\sum_{s=n_0}^{n} \left[s q_s - \frac{p_{s-\sigma}((s+1)-s)^2}{2^{3-\gamma} s} \right] = \sum_{s=1}^{n} \left[s \frac{\mu}{s^2} - \frac{1}{2^{3-\gamma} s} \right] = \sum_{s=1}^{n} \frac{2^{3-\gamma}\mu - 1}{s} \to \infty,$$

as $n \to \infty$ since $\mu > \frac{1}{2^{3-\gamma}}$. By Corollary 2.6, every solution of (2.35) oscillates. In the case when$\gamma = 1$, $\mu \leq \frac{1}{4}$, it is known that [37], (2.35) has a nonoscillatory solution. Hence, Theorem 2.3 and Corollary 2.6 are Sharp. Note that none of the results by Cheng [4], Dosly and Rehak [7], Liu and Cheng [15], Rehak [20], [21] and Thandapani et al. [29], [30] can be applied to Eq.(2.35).

Note that when $\gamma = 1$, Eq.(1.1) reduces to the linear delay difference and the condition (2.29) in Theorem 2.3 reduces to

$$\lim_{n\to\infty} \sup \sum_{l=n_0}^{n} \left[\rho_l q_l - \frac{(p_{l-\sigma})\,(\Delta\rho_l)^2}{4\rho_l}\right] = \infty,$$

which is the same condition in Corollary 1 in [26].

The proof of the following theorems are similar to that of the proof of Theorem 2.2 by using (2.34) and hence is omitted.

Theorem 2.4. *Assume that all the assumptions of Theorem 2.2 hold, except the sequence* $\{\bar{\rho}_n\}_{n=0}^{\infty}$ *is being replaced by* $P_n = 2^{3-\gamma}\rho_n/\,(p_{n-\sigma})$. *Then every solution of Eq.*(1.1) *oscillates for all* $\gamma \geq 1$.

3. Oscillation criteria when (1.3) holds

In this section, we consider the case when (1.3) holds and establish some sufficient conditions which guarantee that every solution of (1.1) oscillates or converges to zero.

First, we consider the case when $\gamma \geq 1$ and $\Delta p_n \geq 0$.

Theorem 3.1. *Assume that* (1.3) *holds and let* $\{\rho_n\}$ *be a positive sequence. Furthermore, assume that there exists a double sequence* $\{H_{m,n} : m \geq n \geq 0\}$ *be as defined in Theorem 2.2 such that* (2.8) *and*

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{p_n}\sum_{i=n_0}^{n-1} q_i\right)^{\frac{1}{\gamma}} = \infty, \tag{3.1}$$

then every solution of Eq.(1.1) *oscillates or* $\lim_{n\to\infty} x_n = 0$.

Proof. Suppose to the contrary that $\{x_n\}$ is an eventually positive solution of (1.1) such that $x_n > 0$ and $x_{n-\sigma} > 0$ for all $n \geq n_1$. We shall consider only this case, since the substitution $y_n = -x_n$ transforms Eq.(1.1) into an equation of the same form. From Eq.(1.1) we have

$$\Delta(p_n\,(\Delta x_n)^{\gamma}) = -q_n x_{n-\sigma}^{\gamma} \leq 0, \qquad n \geq n_0, \tag{3.2}$$

and so $\{p_n(\Delta x_n)^{\gamma}\}$ is an eventually nonincreasing sequence. Since $\{q_n\}$ has a positive subsequence, either $\{\Delta x_n\}$ is eventually negative or eventually positive.

If $\{\Delta x_n\}$ is eventually positive, we are then back to the case where (2.4) holds. Thus the proof of Theorem 2.2 goes through, and we obtain a contradiction. If $\{\Delta x_n\}$ is eventually negative. Then $\lim_{n\to\infty} x_n = b \geq 0$. We assert that $b = 0$. If not then $x_{n-\sigma}^{\gamma} \to b^{\gamma} > 0$ as $n \to \infty$, and hence there exists $n_2 \geq n_1$ such that $x_{n-\sigma}^{\gamma} \geq b^{\gamma}$. Therefore from (3.2) we have

$$\Delta(p_n\,(\Delta x_n)^{\gamma}) \leq -q_n b^{\gamma}.$$

Define the sequence $u_n = p_n\,(\Delta x_n)^{\gamma}$ for $n \geq n_2$. Then we have

$$\Delta u_n \leq -b^{\gamma} q_n.$$

Summing the last inequality from n_2 to $n-1$, we have

$$u_n \leq u_{n_2} - b^{\gamma} \sum_{s=n_2}^{n-1} q_s \leq -b^{\gamma} \sum_{s=n_2}^{n-1} q_s.$$

Summing the last inequality from n_3 to n we obtain

$$x_{n+1} \leq x_{n_3} - b \sum_{s=n_3}^{n} \left(\frac{1}{p_s} \sum_{i=n_2}^{s-1} q_i \right)^{\frac{1}{\gamma}}. \tag{3.3}$$

Condition (3.1) implies that $\{x_n\}$ is eventually negative, which is a contradiction. Thus $\{x_n\}$ converges to zero. The proof is complete. □

From Theorem 3.1, as in Section 2, we can provide several sufficient conditions which guarantee that every solution of Eq.(1.1) oscillates or converges to zero. Due to the limited space we state the following corollary and the remainder of the results are left to the reader.

Corollary 3.1. *Assume that* (1.3) *and* (3.1) *hold, and let* $\{\rho_n\}$ *be a positive sequence such that* (2.29) *holds. Then every solution of Eq.*(1.1) *oscillates or* $\lim_{n\to\infty} x_n = 0$.

The following examples illustrates the main results in this section.

Example 3.1. Consider the half-linear difference equation

$$\Delta((n+1)^2 \Delta x_n) + \mu x_{n-1} = 0, \quad n \geq 1, \tag{3.4}$$

where $\mu > 1/4$. Then, $p_n = (n+1)^2$, $\gamma = 1$ and $\sigma = 1$. If we take $\rho_n = n$, then one can easily see that (3.1) holds, and

$$\begin{aligned} \sum_{s=n_0}^{n} \left[s q_s - \frac{p_{s-1}((s+1)-(s))^2}{4s} \right] &= \sum_{s=1}^{n} \left[\mu s - \frac{s^2}{4s} \right] \\ &= \sum_{s=1}^{n} \frac{(4\mu - 1)}{4} s \to \infty, \end{aligned}$$

as $n \to \infty$. Thus, Corollary 3.1 asserts that every solution of (3.4) oscillates or $x_n \to 0$ as $n \to \infty$. Note that none of the above mentioned papers can be applied to (3.4).

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