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Extension of Exponentially Convex Function on the Heisenberg Group

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Abstract. The main purpose of this paper is to extend the exponentially convex functions which are defined and exponentially convex on a cylinderical neighborhood in the Heisenberg group. They are expanded in terms of an integral transform associated to the sub-Laplacian operator. Extension of such functions on abelian Lie group are studied in [15].

1. Introduction

Let G be a group, and F be a function defined on G. We say that F is exponentially convex if the kernel $K(x,y) \in C(G \times G)$ such that $K(x,y) = F(x,y)$ is positive definite, i.e., for all $n; x_1, x_2, \cdots, x_n \in G$, the matrix

$$(K(x_i, x_j))_{i,j=1}^{n} \tag{1.1}$$

is positive definite in the sense

$$\sum_{i,j=1}^{n} K(x_i, x_j)\rho_i\rho_j \geq 0$$

where $\rho_1, \rho_2, \cdots, \rho_n$ are arbitrary in $\mathbb{R}$. (see [7] and [3], p. 702).

If the function F is only defined in a subset C of G, then we say that F is exponentially convex on C if the matrix (1.1) is positive for all n, and all choices of points $x_1, x_2, \cdots, x_n \in C$.

Suppose G is a locally compact Lie group with a specified left invariant Haar measure dx, and B is an open subset of G such that $C = B.B = \{xy : x, y \in B\}$. If

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F is assumed to be continuous on C, then it is easy to show that F is exponentially convex on C, if and only if

$$\int_B \int_b F(xy)\varphi(x)\varphi(y)dxdy \geq 0 \tag{1.2}$$

for all compactly supported continuous functions φ on B. Let's define the convolution product $\times$ for $\varphi, \psi \in C_c^\infty(B)$ by

$$(\varphi \times \psi)(y) = \int_B \varphi(x)\psi(x^{-1}y)dx \tag{1.3}$$

and consider the identity involution $C_c^\infty(B) \ni \varphi \to \varphi^* \in C_c^\infty(B)$ by $\varphi^*(x) = \varphi(x),\ x \in G$. Now condition (1.2) on B can be rewritten by using (1.3) in the form

$$\int_B F(x)(\varphi \times \varphi)(x)dx \geq 0, \quad \varphi \in C_c^\infty(B). \tag{1.4}$$

If F is such a function defined as above on B, then (1.4) defines an inner product on $C_c^\infty(B)$ defined by

$$\int_B \int_B F(xy)\varphi(x)\psi(y)dxdy, \qquad \varphi, \psi \in C_c^\infty(B).$$

Theory of special functions is the base of known results for $G = \mathbb{R}^n$, but for general nonabelian Lie groups, this theory is less developed.

In [11], [12], special functions and transform theory were studied for a class of nilpotent Lie groups which includes the Heisenberg group.

A Lie algebra with generators $X_i, Y_i, W, \quad i = 1, 2, \cdots, n$ satisfying the commutation relation.

$$[X_i, X_j] = \delta_{ij}W, \qquad [X_i, W] = 0 \quad \text{and} \quad [Y_i, W] = 0$$

is said to be $2n+1$ dimensional Heisenberg algebra $\mathcal{G}_n$. The $2n+1$ Heisenberg group H_n is a unique simply connected Lie group having $\mathcal{G}_n$ as its Lie algebra.

The following set of variables is frequently used on three dimensional Heisenberg group. It can easily be generalized to $2n+1$ real dimensions for $n > 1$. However we will restrict the discussion, and the statement of our results to the case $n = 1$; the generalization to $n > 1$ is relatively trivial.

We shall use the variables used in [12] for the Heisenberg group which are more convenient for our discussion. In [12] the Hesienberg group G is given by upper triangular real 3×3 matrices

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

with $a, b, c \in \mathbb{R}$ and the binary operation is given by matrix multiplication

$$(a, b, c).(a', b', c') = (a + a', b + b', c + c', ab').$$

For the present problem, there is an alternative set of coordinates on G which is more convenient. Let a, b, c describe the matrix coordinates, and set

$$x_1 = \frac{1}{2}a, \qquad x_2 = \frac{1}{2}b, \qquad x_3 = c - \frac{1}{2}ab.$$

Further set $z = x_1 + ix_2$, then we have the system, (x_1, x_2, x_3) with three real coordinates and (z, x_3) with one complex and one real coordinates.

When the last mentioned coordinates system is used, the group operation of G takes the form

$$(z, x_3)(z', x_3') = (z + z', x_3 + x_3' + 2Imzz').$$

2. Operators on the Heisenberg group

The Lie algebra $\mathcal{G}$ of right invariant vector fields on G is spanned by the following familiar three, first-order, partial differential operators

$$X_1 = \frac{\partial}{\partial x_1} - 2x_2\frac{\partial}{\partial x_3}, \quad X_2 = \frac{\partial}{\partial x_2} + 2x_1\frac{\partial}{\partial x_3}, \quad X_3 = \frac{\partial}{\partial x_3}$$

where $[X_1, X_2] = 4X_3$, and the other two commutators vanish.

We introduce the vector fields

$$Z = \frac{1}{2}(X_1 - iX_2), \qquad \overline{Z} = \frac{1}{2}(X_1 + iX_2),$$
$$\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}), \qquad \bar{\partial} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}),$$

and

$$D = \frac{\partial}{\partial x_3}.$$

It follows that

$$\begin{aligned} Z &= \frac{\partial}{\partial z} - i\bar{z}\frac{\partial}{\partial x_3} = \partial - i\bar{z}D \\ \overline{Z} &= \frac{\partial}{\partial \bar{z}} + iz\frac{\partial}{\partial x_3} = \bar{\partial} + izD. \end{aligned} \tag{2.1}$$

The second order right invariant differential operator

$$L = -\frac{1}{4}(X_1^2 + X_2^2) = -\frac{1}{2}(Z\overline{Z} + \overline{Z}Z)$$

is said to be sub-Laplacian operator on the Heisenberg group.

When the variables (z, x_3) are introduced, we have

$$L = -\partial\bar{\partial} + iD(\bar{z}\bar{\partial} - z\partial) - |z|^2 D^2.$$

In terms of (x_1, x_2, x_3), we have

$$L = -\frac{1}{4}(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}) + D(x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2}) - (x_1^2 + x_2^2)D^2$$

and using polar coordinates, $z = re^{it}$, we get

$$L = -\frac{1}{4}[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial t^2}] - r^2D^2 - \frac{\partial}{\partial t}D \tag{2.2}$$

For more details about the operator L, see [6], [8].

It was proved in [9], [11], that L in (2.2) is positive self-adjoint in $\mathcal{L}(G)$ with absolutely continuous spectrum, and a spectral transform was established. Moreover the functions

$$h(r, t, x_3) = \sqrt{\frac{(j-|m|)!2^{(2|m|+1)}}{\Gamma(j+|m|+1)}}(\sqrt{|k|}r)^{2|m|}e^{-|k|r^2}L_{j-|m|}^{2|m|}(2|k|r^2)e^{i(mt+kx_3)} \tag{2.3}$$

are nonmalized eigenfunctions for the sub-Laplacian L with eigenvalues

$$\lambda_{j,m}^{(k)} = (2k+1)|k| + 2mk, \qquad j = 0, \frac{1}{2}, 1, \cdots; \qquad m = -j, \cdots, j,$$

where L_u^v are the familiar Laguerre polynomials of order v ([1]).

3. Unitary representation of the Lie algebra

Let the S^{-1}-action on G be denoted by $x \to e^{i\theta}x$, defined by

$$(z, x_3) \to (ze^{i\theta}, x_3), \qquad e^{i\theta} \in S^1, \qquad x = (z, x_3) \in G. \tag{3.1}$$

Let $B = \{(z, x_3) \in G : |z| < 1,\ x_3 \in \mathbb{R}\}$ denoted the unit cylinder in G. A kernel $k(x, y)$ defined on $B \times B$ is said to be bi-invariant in B if for $x, y \in B$

$$k(e^{i\theta}x, e^{i\theta}y) = k(xy) \tag{3.2}$$

and

$$k(O(s)x, O(s)y) = k(x, y) \tag{3.3}$$

for all $e^{i\theta} \in S^1$, $\theta \in \mathbb{R}$ and $x, y \in B$, where $O(s)$ denotes the translation group in the x_3 variable. Specifically $O(s)(z, x_3) = (z, x_3 + s)$.

Let F be continuous and defined as above in the cylinder B. Consider $S \subset B$, where

$$S = 2^{-1}B = \{(z, x_3) : |z| < \frac{1}{2}, \quad x_3 \in \mathbb{R}\}. \tag{3.4}$$

We restrict the positive condition (1.2) for all smooth functions φ with compact support in the cylinder S.

The two cylinders B and S are invariant under S^1-action (3.1). In the following, we shall work with the inner product, given by

$$\langle \varphi, \psi \rangle = \int_S \int_S F(xy)\varphi(x)\psi(y)dxdy, \; \varphi, \psi \in C_c^\infty(S), \tag{3.5}$$

i.e., for compactly supported smooth function on the cylinder S. Denote $C_c^\infty(S)$ with the inner product (3.5) by D. General speaking, the fact $\langle \varphi, \psi \rangle = 0$ for $\varphi \in C_c^\infty(S)$ does not imply $\varphi = 0$. We shall suppose that there is no degeneracy; i.e., that $\langle \varphi, \psi \rangle = 0$ for $\varphi \in C_c^\infty(S)$ implies $\varphi = 0$. For this consider the quotient $\mathcal{D}/\mathcal{K}$, where

$$\mathcal{K} = \{\varphi \in C_c^\infty(S) : \langle \varphi, \psi \rangle = 0, \, \psi \in C_c^\infty(S)\}.$$

The norm is then defined by

$$\|\varphi^2\| = \langle \varphi, \varphi \rangle, \; \varphi \in \mathcal{D}/\mathcal{K}. \tag{3.6}$$

The completion of $\mathcal{D}/\mathcal{K}$ with respect to the norm (3.6) gives a complete Hilbert space denoted by $\mathcal{H}_F$ with inner product and norm

$$\begin{aligned} \langle \varphi,\psi \rangle &= \int \int F(x,y)\varphi(x)\psi(y)dxdy \\ \|\varphi\| &= \langle \varphi, \varphi \rangle_F^{\frac{1}{2}}, \quad \varphi, \psi \in \mathcal{D}/\mathcal{K} \end{aligned} \tag{3.7}$$

Let G is a Heisenberg group and $\mathcal{G}$ be its Lie algebra. If $a \in \mathcal{G}$, then the mapping $D(a) : \varphi \to a\psi$, given by

$$(a\varphi)(x) = \frac{d}{dt}|_{t=0}(\exp ta.x), \; \varphi \in \mathcal{D} \tag{3.8}$$

defines a representation of $\mathcal{G}$ acting on $\mathcal{D}$. Consider $\mathcal{U}$ to be the universal enveloping algebra of G with the involution $\mathcal{U} \ni a \to a^* \in \mathcal{U}$ determined by

$$a^* = -a, \qquad a \in \mathcal{G}. \tag{3.9}$$

Recall that $\mathcal{U}$ may be identified with the algebra of all right invariant partial differential operators on G, and moreover, that $\mathcal{U}$ is a^*-algebra when $*$ operation is given by (3.9), see [4], [5].

Lemma 3.1.

(i) *The representation $D(a)$ defined by* (3.9) *is Hermitian relative to the inner product of the Hilbert space $\mathcal{H}_F$ i.e.,*

$$\langle \mathcal{D}(a)\varphi, \psi\rangle = -\langle \varphi, D(a)\psi\rangle, \quad \varphi, \psi \in \mathcal{D}/\mathcal{K},\ a \in \mathcal{G} \tag{3.10}$$

(ii) *If the action $R(\theta)$, defined by*

$$(R(\theta)\psi)(x) = \varphi(e^{i\theta}x), \quad \varphi \in \mathcal{D},\, x \in G \tag{3.11}$$

where $e^{i\theta}x = (e^{i\theta}z, x_3)$, then R is a unitary representation of S^1 on $\mathcal{H}_F$.

Proof. Since S in (3.4) is an open cylinder and $D(a)$ is chosen such that $\varphi(x), \psi(y)$ have compact support, then $\varphi(ax), \psi(a^{-1}y), \varphi(e^{i\theta x})$ and $\psi(e^{i\theta}y)$ have the same compact support.

The invariance conditions (3.2) and (3.3) of the kernel $K(x,y) = F(xy)$ and the unimodularity of the group G are useful in proving the lemma.

To prove the first part, consider $\varphi,\ \psi \in \mathcal{D}$, then

$$\begin{aligned}
\langle D(a)\varphi, \psi\rangle &= \int\int F(xy)(a\varphi)(x)\psi(y)dxdy \\
&= \int\int k(x,y)\frac{d}{dt}|_{t=0}\varphi(\exp ta.x)\psi(y)dxdy \\
&= \frac{d}{dt}|_{t=0}\int\int k(x,y)\varphi(\exp ta.x)\psi(y)dxdy \\
&= \frac{d}{dt}|_{t=0}\int\int k(\exp -ta.x, y)\varphi(x)\psi(y)dxdy \\
&= \frac{d}{dt}|_{t=0}\int\int k(x, \exp tay)\varphi(x)\psi(y)dxdy \\
&= \frac{d}{dt}|_{t=0}\int\int k(x,y)\varphi(x)\psi(\exp -ta.y)dxdy \\
&= \int\int k(x,y)\varphi(x)\frac{d}{dt}|_{t=0}\psi(\exp -ta.y)dxdy \\
&= -\int\int F(xy)\varphi(x)(a\psi)(y)dxdy \\
&= -\langle \varphi, D(a)\psi\rangle.
\end{aligned}$$

To prove the second part, for $\varphi \in \mathcal{D}$ and with (3.11)

$$\begin{aligned}
\langle R(\theta)\varphi, R(\theta)\psi\rangle &= \int\int F(xy)\varphi(e^{i\theta}x)\psi(e^{i\theta}y)dxdy \\
&= \int\int k(e^{-i\theta}x, e^{-i\theta}y)\varphi(x)\psi(y)dxdy \\
&= \int\int F(xy)\varphi(x)\psi(y)dxdy \\
&= \langle \varphi, \psi\rangle.
\end{aligned}$$

$D(a)$ and $R(\theta)$ pass to the quotient $\mathcal{D}/\mathcal{K}$. Indeed, for $a \in \mathcal{G}$ and $\varphi \in \mathcal{K}$, then for all $\psi \in \mathcal{D}, \langle \mathcal{D}(a)\varphi, \psi\rangle = -\langle \varphi, \mathcal{D}(a)\psi\rangle = 0$ implies $D(a)\varphi \in \mathcal{K}$ also $\langle R(\theta)\varphi, R(\theta)\psi\rangle = \langle \varphi, \psi\rangle = 0$ implies $R(\theta)\varphi \in \mathcal{K}$, this proves that $D(a)$ and $R(\theta)$ pass to the quotient $\mathcal{D}/\mathcal{K}$ which is a dense linear subspaces of the Hilbert space $\mathcal{H}_F$.

Since every representation A in $\mathcal{G}$ can be extended to a representation (denoted also by A) of the universal enveloping algebra $\mathcal{U}$ of G ([2], [14]), noting that the operator $D(a)$, $a \in \mathcal{G}$ is obtained by differentiating the left multiplication, it follows that $D(a)$ extends from $\mathcal{G}$ to $\mathcal{U}$ and (3.10) can be rewritten as

$$\langle D(a)\varphi, \psi\rangle = \langle \varphi, D(a^*)\psi\rangle, \quad \varphi, \psi \in \mathcal{D}/\mathcal{K},\ a \in \mathcal{G}$$

which is equivalent to

$$\langle a\varphi, \psi\rangle = \langle \varphi, a^*\psi\rangle,\ \varphi, \psi \in \mathcal{D}/\mathcal{K},\ a \in \mathcal{U}. \tag{3.12}$$

More generally, we get

$$\langle D(A)\varphi, \psi\rangle = \langle \varphi, D(A^*)\psi\rangle \quad \text{for} \quad A \in \mathcal{U}, \quad \varphi, \psi \in \mathcal{D},$$

i.e., (3.12) is valid for arbitrary right invariant partial differential operator $A \in \mathcal{U}$.

In particular, we have $\langle D(z)\varphi, \psi\rangle = -\langle \varphi, D(\bar{z})\psi\rangle$. It follows that $\langle D(L)\varphi, \psi\rangle = \langle \varphi, D(L)\psi\rangle$ where $z, \bar{z}$ and L are defined by (2.1) and (2.2).

Consider the sub-Laplacian operator L on G. The kernel $k(x, y) = F(xy)$ is said to be strongly exponentially convex if for $\varphi, \psi \in C_c^\infty(S)$, we have

$$\int_S \int_S k(x, y)(L\varphi)(x)\varphi(y)dxdy \geq 0.$$

If follows that $\langle L\varphi, \varphi\rangle \geq 0$. Let $\mathcal{F}$ denote the completion of $\mathcal{D}/\mathcal{K}$ with respect to the norm $\langle \varphi, \varphi\rangle_{\mathcal{F}} = \langle L\varphi, \varphi\rangle + \langle \varphi, \varphi\rangle$, and let $D(L^*)$ be the domain of the adjoint operator L^* (the adjoint of the sub-Laplacian L). Let $\mathcal{L}$ be the Friedrichs extension of L with the domain $\mathcal{D}(\mathcal{L}) = \mathcal{F} \cap \mathcal{D}(L^*)$. Since $O(s)\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L})$ and $R(\theta)\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathcal{L})$, then $\mathcal{D}(\mathcal{L})$ is invariant under both parameter group $\{O(s) : s \in \mathbb{R}\}$ and S^1. Also $\mathcal{O}(s)\mathcal{L}\nu = \mathcal{L}O(s)\nu$ and $R(\theta)\mathcal{L}\nu = \mathcal{L}R(\theta)\nu$, for $s \in \mathbb{R}$, $e^{i\theta} \in S^1$ and $\nu \in \mathcal{D}(\mathcal{L})$. Operators family consists of $R(\theta), D$ and $\mathcal{L}$ are commutative, i.e., the spectral projections of the respective operators generates an abelian Von Neuman algebra [14]. □

4. Eigen function expansion of sub-Laplacian operator

If E is spectral measure with support $\Gamma \subset 2^{-1}\mathbb{Z} \times \mathbb{R}^2$, (the dual of $S^1 \times \mathbb{R}^2$), then the representation

$$\{R(\theta)\exp(ip\mathcal{L})O(s),\, \theta \in S^1,\ \ p, s \in \mathbb{R}\}$$

is a unitary representation of G, and

$$R(\theta)\exp ip\mathcal{L}O(s) = \int_{2^{-1}\mathbb{Z}\times\mathbb{R}^2} e^{i\theta m}e^{ip\lambda}e^{is\xi}dE(m, \lambda, \xi),$$

where $dE(m,\lambda,\xi)$ is a projection valued measure with multiplicity one ([9]).

Consider

$$U(\theta,p,s)=R(\theta)\exp(ip\mathcal{L})O(s) \tag{4.1}$$

and

$$\mathcal{H}_F=\int^{\oplus}\mathcal{H}(\omega)d\omega \tag{4.2}$$

to be the direct integral decomposition associated to the representation (4.1) with norm

$$\|\nu\|^2_{\mathcal{H}_F}=\int_\gamma \|\nu\omega\|^2_\omega d\omega, \quad \nu\in\mathcal{H}_F,\ \nu(\omega)\in\mathcal{H}(\omega). \tag{4.3}$$

The family $(\mathcal{H}(\omega))_{\omega\in\Gamma}$ is a class of Hilbert spaces and $\|\nu(\omega)\|_\omega$ is the norm on $\mathcal{H}(\omega)$ corresponding to $\omega\in\Gamma$, ([10], [14]).

It follows from the spectral theorem that

$$(U(\theta,p,s)\nu)(\omega)=e^{i\theta m}e^{ip\lambda}e^{is\xi}\nu(\omega)$$

where $\omega=\omega(m,\lambda,\xi)\in 2^{-1}\mathbb{Z}\times\mathbb{R}^2$, see [13].

Let $H_+=C^\infty(U,\mathcal{H}_F)$, be the space of C^∞-vectors for the unitary representation U with $\|.\|_+\geq\|.\|_{\mathcal{H}_F}$ for all functions in H_+. If H_- is the dual space of H_+ with respect to the Hilbert space $\mathcal{H}_F$, then $H_+\subseteq\mathcal{H}_F\subseteq H_-$, and $\|.\|_{\mathcal{H}_F}\geq\|.\|_-$, for all functions in $\mathcal{H}_F$. Here, H_-, the space of generalized functions on S, is identified with measurable field of distribution on S^1, see [3].

Let $J:D\to\mathcal{H}_F$, be the inclusion mapping. Since $D\subseteq\mathcal{H}_F$ is a topological inclusion, then J is a continuous mapping. By the direct integral (4.2), there is a continuous mapping $J_\omega:D\to\mathcal{H}(\omega)$, for each $\omega\in\Gamma$, such that

$$J_\omega\varphi=J_\varphi \qquad \text{for all} \qquad \varphi\in D=C^\infty_c(S)$$

and $\omega\in 2^{-1}\mathbb{Z}\times\mathbb{R}^2$, see [10].

Since $D(S)$ is equipped with the inductive topology it follows that J_ω is a vector valued distribution and for $\varphi\in D(S)$

$$\begin{aligned} J_\omega\varphi &= \langle\nu(\omega),\varphi\rangle, \quad \omega\in\Gamma \\ &= \int_B \nu(\omega,x)\varphi(x)dx \end{aligned} \tag{4.4}$$

where dx is the Haar measure on G.

For $\varphi,\psi\in D(S)$, by (4.2) and (4.3)

$$\langle\varphi,\psi\rangle_F=\int_\Gamma J_\omega\varphi J_\omega\psi d\omega. \tag{4.5}$$

Since the sub-Laplacian operator L, given by (2.2) is known to be analytic hypoelliptic ([12]), then $(\nu(\omega))_{\omega\in\gamma}$ is a field of C^{∞}-functions on B.

Consider $\nu(\omega, r, t, x_3)$ to be a C^{∞}-function on B which defines the distribution $\nu(\omega)$. Then

$$\langle \nu(\omega), \varphi \rangle = \int_{\mathbb{R}} \int_0^{2\pi} \int_{\mathbb{R}_+} \nu(\omega, r, t, x_3)\varphi(r, t, x_3) r dr dt dx_3,$$

and, we obtain the three partial differential equations

$$\frac{\partial}{\partial t}\nu(\omega, r, t, x_3) = -im\nu(\omega, r, t, x_3) \tag{4.6}$$

$$L\nu(\omega, r, t, x_3) = \lambda\nu(\omega, r, t, x_3) \tag{4.7}$$

$$\frac{\partial}{\partial x_3}\nu(\omega, r, t, x_3) = -i\xi\nu(\omega, r, t, x_3) \tag{4.8}$$

for $\omega = \omega(m, \lambda, \xi) \in \Gamma$ and $x = (r, t, x_3) \in B$.

Theorem 4.1. *Let K be a continuous bi-invariant and strongly positive definite kernel on $S \times S$, where $S = 2^{-1}B$, and B is a unit cylinder on the Heisenberg group G. Then, there is a finite positive Radon measure $\rho(\omega)$ on Γ such that*

$$K(x, y) = \int_{\Gamma} \nu(\omega, x)\nu(\omega, y) d\rho(\omega)$$

where $K(x, y)$ is a positive definite extension of K on $S \times S$ into $G \times G$, and $\nu(\omega, x)$ are C^{∞}-functions on S which define distributions $\nu(\omega)$ satisfying (4.6)-(4.8).

Proof. The last two variables t, x_3 of the solutions $\nu(\omega, r, t, x_3)$ of equations (4.6)-(4.8), can be separated, say

$$\nu(\omega, r, t, x_3) = f(\omega, r)e^{-imt}e^{-i\xi x_3} \tag{4.9}$$

where $f(\omega, r)$ is C^{∞}-functions for $r \in [0, 1)$, which satisfy, for fixed ω

$$[-\frac{1}{4}(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} - \frac{m^2}{r^2}) + |\xi|^2 r^2 - m\xi]f(\omega, r) = \lambda f(\omega, r) \tag{4.10}$$

for

$$\lambda = (2j + 1)|\xi| + 2m\xi, \quad j \in 2^{-1}\mathbb{Z}, \quad j \geq 0.$$

By (2.3), for $r \in (0, \infty)$, the functions

$$f_{\omega}(r) = \left(\frac{(j - |m|)! 2^{2|m|+1}}{\Gamma\,(j + |m| + 1)}\right)^{\frac{1}{2}} (|\xi|r^2)^{|m|} L^{2|M|}_{j-|m|}(2|\xi|r^2)e^{-|\xi|r^2})$$

where L^o_o denotes the Laguerre function. Then $f_{\omega}(r)$ satisfies (4.10), and $\int_0^{\infty} |f_{\omega}(r)|^2 r dr = 1$, when $j, m \in 2^{-1}\mathbb{Z}$, $j \geq 0$, $|m| \leq j$, and $\lambda = (2j + 1)|\xi| + m\xi$.

Other solutions of (4.10) must be excluded by virtue of the hypoellioticity of the sub-Laplacian L, and does not define functions on the Heisenberg group $G.f(\omega, r)$ in (4.9) can be written as

$$f(\omega, r) = A(j, m, \xi) f_\omega(r)$$

and then

$$\nu(\omega, r, t, x_3) = A(j, m, \xi) f_\omega(r) e^{-imt} e^{-i\xi x_3}. \tag{4.11}$$

Since $\langle \varphi, \psi \rangle_F$ has the integral forms (4.5) and (3.7), using (4.4) and Fubini's Theorem, we get

$$K(x, y) = \int_\gamma \nu(\omega, x) \nu(\omega, y) d\omega. \tag{4.12}$$

Since $\nu(\omega, r, t, x_3)$, given by (4.11), defines a measure ρ in

$$\{\omega(j, m, \xi) \mid j,\ m \in \mathbb{Z}, \quad j \geq 0, \quad |m| \leq j, \quad \xi \in \mathbb{R}\}$$

for some finite Radon measure ρ on Γ and measurable field $\{\nu(\omega, .)|\omega \in \Gamma\}$. $\nu(\omega, .)$ may be taken to be C^∞-functions on B with C^∞ extensions to G.

For $\varphi \in D$, by virtue of Fubini's theorem, and (4.12)

$$\begin{aligned} \int_B \int_B K(x, y) \varphi(x) \varphi(y) dx dy &= \int_\Gamma \int_B \nu(\omega, x) \varphi(x) dx \int \nu(\omega, y) \varphi(y) dy d\rho(\omega) \\ &= \int_\Gamma \left| \int \nu(\omega, x) \varphi(x) dx \right|^2 d\rho(\omega) \geq 0. \end{aligned}$$

This is also valid when the integration is extended over $G \times G$, rather than $B \times B$, for functions $\varphi \in C_c^\infty(G)$. This proves that the kernel $K(x, y)$ is positive definite, when extended to all $G \times G$, for arbitrary positive, finite Radon measure ρ. □

5. Extension of exponentially convex functions of a single variable

Theorem 5.1. *Let F be a continuous exponentially convex function in the cylinder S of the Heisenberg group G, and K bi-invariant analytic kernel function on $G \times G$ such that*

$$F(x, y) = K(x, y) \quad in \quad S \times S. \tag{5.1}$$

Then there is an exponentially convex function $\widetilde{F}$ on G such that

$$\widetilde{F}(x.y) = K(x, y) \quad in \quad G \times G$$

and $\widetilde{F}$ extends F to all G.

Proof. Let the kernel function be defined on $S \times S$ by (5.1). Apply theorem 4.1 to the considered kernel function K which yields an extension K defined on $G \times G$.

For $X \in \mathcal{G}$, the Lie algebra of G, defines vector fields on $G \times G$ by the formula

$$\rho(x)K(x,y) = \frac{d}{dx}|_{t=0}K(x.\exp tx, \exp(tx).y). \tag{5.2}$$

Let $(x,y) \in S \times S$, and let $a \in G$ be chosen such that $(x.a, a^{-1}y) \in S \times S$. We then have

$$K(x.a, a^{-1}y) = F(x.a.a^{-1}y) = F(x.y) = k(x,y)$$

Take $a = \exp tx$, we note that by virtue of (5.2), $\rho(x)K = 0$ in $S \times S$. But K and its derivatives, $\rho(x)K$, are analytic on $G \times G$. It follows that $\rho(x)K$ vanishes on $G \times G$ for all x in the Lie algebra. This implies that

$$K(x.a, a^{-1}y) = K(x,y) \quad \text{for all points} \quad a, x, y \in G$$

Define the extension $\widetilde{F}$ of F on G, by $F(x) = K(x,e)$.

Since

$$\widetilde{F}(xy) = K(xy, e) = K(xyy^{-1}, yy^{-1}y) = K(x,y)$$

Therefore

$$\int_G \int_G \widetilde{F}(xy)\varphi(x)\varphi(y)dxdy = \int_G \int_G K(x,y)\varphi(x)\varphi(y)dxdy.$$

By Theorem 4.1, this proves that $\widetilde{F}$ is exponentially convex. Since G is connected, we have

$$\widetilde{F}(x) = F(X) \text{ for } x \in S$$

which completes the proof. □

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