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A SOLUTION OF EGGERT'S CONJECTURE IN SPECIAL CASES

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Abstract. Let M be a finite commutative nilpotent algebra over a perfect field k of prime characteristic p and let M^p be the subalgebra of M generated by $x^p, x \in M$. Eggert[3] conjectures that $\dim_k M \geq p\dim_k M^p$.

In this paper, we show that the conjecture holds for $M = R^+/I$, where $R = k[X_1, X_2, \cdots, X_t]$ is a polynomial ring with indeterminates $X_1, X_2, \cdots, X_t$ over k and R^+ is the maximal ideal of R generated by $X_1, X_2, \cdots, X_t$ and I is a monomial ideal of R containing $X_1^{n_1+1}, X_2^{n_2+1}, \cdots, X_t^{n_t+1}$ ($n_i \geq 0$ for all i).

1. Introduction

Let k be a perfect field of prime characteristic p and let M be a finite nilpotent algebra over k and let M^p be the subalgebra of M generated by $x^p, x \in M$.

In [3] , Eggert investigated between the algebra structure of M and its quasi algebra group and he conjectured the following inequality

$$(1-1) \qquad \dim_k M \geq p\dim_k M^p$$

He proved the inequality (1-1) of the cases $\dim_k M^p \leq 2$ in [3] and Stark also proved it of the same cases in [4]. In case $\dim_k M^p = 3$, Bautista proved in [1].

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In this paper, we show that the inequality (1-1) holds for a finite commutative nilpotent algebra $M = R^+/I$, where $R = k[X_1, X_2, \cdots, X_t]$ is a polynomial ring with indeterminates $X_1, X_2, \cdots, X_t$ over k and R^+ is the maximal ideal of R generated by $X_1, X_2, \cdots, X_t$ and I is a monomial ideal of R containing $X_1^{n_1+1}, X_2^{n_2+1}, \cdots, X_t^{n_t+1}$ ($n_i \geq 0$ for all i).

Now, we consider a polynomial ring $R = k[X]$ with an indeterminate X over a perfect field k of prime characteristic p and let R^+ be the maximal ideal of R generated by X and let I be an ideal of R generated by X^{n+1}. Then we have a finite commutative nilpotent algebra $M = R^+/I$ and the inequality (1-1) holds for this finite commutative nilpotent algebra M.

THEOREM 1.1. *Under the same notation as above, we have* $dim_k M \geq p dim_k M^p$.

proof. If $p > n$, since $M^p = 0$, we are done. Now, assume $p \leq n$ and $n = pn' + r (0 \leq r < p)$. As $k-$vector spaces, $x, x^2, \cdots, x^n$ is a basis of M, where x is the image of X under the canonical projection $R^+ \longrightarrow M$. Since the characteristic of k is p , $x^p, x^{2p}, \cdots, x^{pn'}$ is a basis of the $k-$vector space M^p. Hence $\dim_k M = n \geq pn' = p\dim_k M^p$. □

2. main theorem

Let k be a perfect field of prime characteristic p and let $R = k[X_1, X_2, \cdots, X_t]$ be a polynomial ring with indeterminates $X_1, X_2, \cdots, X_t$ over k and let R^+ be the maximal ideal of R generated by $X_1, X_2, \cdots, X_t$. For any monomial ideals I of R containing $X_1^{n_1+1}, X_2^{n_2+1}, \cdots, X_t^{n_t+1}$ ($n_i \geq 0$ for all i), we have a finite commutative nilpotent algebra $M = R^+/I$. In this section, we will show that the Eggert's Conjecture is true for those M. For the proof, we need a proposition and a lemma.

PROPOSITION 2.1. [2: Ch2.4, Lemma3] *Let I be a monomial ideal of a polynomial ring $k[X_1, X_2, \cdots, X_t]$ and let $f \in k[X_1, X_2, \cdots, X_t]$.*

Then $f \in I$ if and only if f is a $k-$linear combination of monomials in I.

Let $\mathbb{Z}^t_{\geq 0} = \{(a_1, a_2, \cdots, a_t) \mid$ for all i, a_i is a nonnegative integer $\}$. We say $(a_1, a_2, \cdots, a_t) > (b_1, b_2, \cdots, b_t)$ for $(a_1, a_2, \cdots, a_t)$, $(b_1, b_2, \cdots, b_t) \in \mathbb{Z}^t_{\geq 0}$ if, in $(a_1 - b_1, a_2 - b_2, \cdots, a_t - b_t)$, the left-most nonzero entry is positive. By Proposition 4 of [2, Ch2.2] , $>$ is a total order on $\mathbb{Z}^t_{\geq 0}$. Let $(n_1, n_2, \cdots, n_t) > (0, 0, \cdots, 1)$ in $\mathbb{Z}^t_{\geq 0}$ and let

$$D = \{(a_1, a_2, \cdots, a_t) \in \mathbb{Z}^t_{\geq 0} \mid (0, 0, \cdots, 1) \leq (a_1, a_2, \cdots, a_t) \leq (n_1, n_2, \cdots, n_t)\} ,$$

$$D_{i_1 i_2 \cdots i_{t-1}} = \{(i_1, i_2, \cdots, i_{t-1}, j) \mid j = 1, 2, \cdots, n_t\} \quad (0 \leq i_j \leq n_j, \quad j = 1, 2, \cdots, t-1),$$

$$D^0 = \{(i_1, i_2, \cdots, i_{t-1}, 0) \mid (i_1, i_2, \cdots, i_{t-1}, 0) \in D\},$$

$$D' = \{(i_1, i_2, \cdots, i_t) \in D \mid i_t > n_t\},$$

and let

$$D_I = \{(a_1, a_2, \cdots, a_t) \in \mathbb{Z}^t_{\geq 0} \mid X_1^{a_1} X_2^{a_2} \cdots X_t^{a_t} \notin I\}.$$

Then we have a partition of D as the following by the Lemma2 of [2: Ch2.4,] .

LEMMA 2.2. *Under the same notations as above, we have the following;*

i) $(i_1, i_2, \cdots, i_{t-1}) \neq (j_1, j_2, \cdots, j_{t-1})$ *with* $0 \leq i_s, j_s \leq n_s$ *for all* $s = 1, 2, \cdots, t-1$ *if and only if* $D_{i_1 i_2 \cdots i_{t-1}} \cap D_{j_1 j_2 \cdots j_{t-1}} = \varnothing$.

ii) $D_{i_1 i_2 \cdots i_{t-1}} \cap D^0 = \varnothing$ *and* $D_{i_1 i_2 \cdots i_{t-1}} \cap D' = \varnothing$ *with* $0 \leq i_s \leq n_s$ *for all* $s = 1, 2, \cdots, t-1$.

iii) $D^0 \cap D' = \varnothing$.

iv)

$$D = \{ \bigcup_{0 \leq i_s \leq n_s (s=1,2,cdots,t-1)} D_{i_1 i_2 \cdots i_{t-1}} \} \bigcup D^0 \bigcup D'.$$

v) $X^{i_1}X^{i_2}\cdots X^{i_t} \in I$ *for all* $(i_1 i_2 \cdots i_t) \in D'$.

vi)

$$D_I \subseteq \{ \bigcup_{0 \le i_s \le n_s (s=1,2,cdots,t-1)} D_{i_1 i_2 \cdots i_{t-1}} \} \bigcup D^0.$$

Now we are ready to prove our main theorem.

THEOREM 2.3. *Let k be a perfect field of prime characteristic p and let $R = k[X_1, X_2, \cdots, X_t]$ be a polynomial ring with indeterminates $X_1, X_2, \cdots, X_t$ over k and let R^+ be the maximal ideal of R generated by $X_1, X_2, \cdots, X_t$. For any monomial ideal I of R containing $X_1^{n_1+1}, X_2^{n_2+1}, \cdots, X_t^{n_t+1}$ ($n_i \ge 0$ for all i), $M = R^+/I$ is a finite commutative nilpotent algebra over k and $\dim_k M \ge p \cdot \dim_k M^p$.*

proof. We will do induction on the number t of variables $X_1, X_2, \cdots, X_t$. Theorem 1.1 implies the initial step of this induction. Assume that the assertion is true for all less number of variables than t and assume $t > 1$. Let I be a monomial ideal of R containing $X_1^{n_1+1}, X_2^{n_2+1}, \cdots, X_t^{n_t+1}$ ($n_i \ge 0$ for all i). We may assume that for all i, $X_i^{n_i} \notin I$ and $X_i^{n_i+1} \in I$. Let

$$D_I = \{(a_1, a_2, \cdots, a_t) \in \mathbb{Z}^t_{\ge 0} \mid X_1^{a_1} X_2^{a_2} \cdots X_t^{a_t} \notin I\}.$$

By Proposition 2.1, $\{x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t} \mid (a_1, a_2, \cdots, a_t) \in D_I\}$ is a basis of M as a $k-$vector space, where x_i is the image of X_i under the canonical projection $R^+ \longrightarrow M$. Since $D_I \subset D$, by Lemma 2.2,

$$D_I = \{ \bigcup_{0 \le i_s \le n_s (s=1,2,\cdots,t-1)} (D_I \cap D_{i_1 i_2 \cdots i_{t-1}}) \} \bigcup (D^0 \cap D_I).$$

Since M^p can be generated by

$$\{x_1^{pa_1} x_2^{pa_2} \cdots x_t^{pa_t} \mid (a_1, a_2, \cdots, a_t) \in D_I\}$$

$$\mathcal{B} = \{x_1^{pa_1} x_2^{pa_2} \cdots x_t^{pa_t} \mid (a_1, a_2, \cdots, a_t) \in D_I,\ (pa_1, pa_2, \cdots, pa_t) \in D_I\}$$

forms a basis of M^p as a $k-$vector space. Put

$$D_I^p = \{(a_1, a_2, \cdots, a_t) \mid x_1^{a_1} x_2^{a_2} \cdots x_t^{a_t} \in \mathcal{B}\}.$$

Since $D_I^p \subset D$, again by Lemma 2.2,

$$D_I^p = \{ \bigcup_{0 \leq i_s \leq n_s (s=1,2,\cdots,t-1)} (D_I^p \cap D_{i_1 i_2 \cdots i_{t-1}})\} \bigcup (D_I^p \cup D^0).$$

Let $R_1 = k[X_1, X_2, \cdots, X_{t-1}]$ be a polynomial ring with indeterminates X_1, X_2, $\cdots, X_{t-1}$ over k and let R_1^+ be the maximal ideal of R_1 generated by X_1, X_2, $\cdots, X_{t-1}$ and let J be the inverse image of I under the canonical inclusion $R_1^+ \longrightarrow R^+$. Then J is a monomial ideal of R_1^+ containing $X_1^{n_1+1}, X_2^{n_2+1}, \cdots, X_{t-1}^{n_{t-1}+1}$ and hence, $M_1 = R_1^+/J$ becomes a finite commutative nilpotent algebra over k. By induction, we have $\dim_k M_1 \geq p\dim_k M_1^p$ and by our construction of D_0, $\dim_k M_1 = \mathrm{Card}(D_0 \cap D_I)$ and $\dim_k M_1^p = \mathrm{Card}(D_0 \cap D_I^p)$. Now, it is enough to show that

$$\mathrm{Card}(D_I \cap D_{i_1 i_2 \cdots i_{t-1}}) \geq p \cdot \mathrm{Card}(D_I^p \cap D_{i_1 i_2 \cdots i_{t-1}})$$

for all $(i_1, i_2, \cdots, i_{t-1})$ between $(0, 0, \cdots, 0)$ and $(n_1, n_2, \cdots, n_{t-1})$. Suppose $(i_1, i_2, \cdots, i_{t-1}, j) \in D_I$ and $(i_1, i_2, \cdots, i_{t-1}, j+1) \notin D_I$. Then by the choice of D_I, $(i_1, i_2, \cdots, i_{t-1}, i) \in D_I$ for $0 \leq i \leq j$ and $(i_1, i_2, \cdots, i_{t-1}, i) \notin D_I$ for $j < i \leq n_t$. So we have

$$\mathrm{Card}(D_I \cap D_{i_1 i_2 \cdots i_{t-1}}) \geq p \cdot \mathrm{Card}(D_I^p \cap D_{i_1 i_2 \cdots i_{t-1}}).$$

In case $D_{i_1 i_2 \cdots i_{t-1}} \cap D_I = \varnothing$ we also have the same inequality. This completes the proofs. $\square$

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