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THE PRODUCT OF MULTIPLICATION SUBMODULES

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Abstract. Let R be a commutative ring with non-zero identity. This paper is devoted to the study some of properties of the product of submodules of a multiplication module. Suppose N is a submodule of a multiplication R-module M. We give a condition which allows us to determine whether N is finitely generated when we assume some power of N is finitely generated.

1. Introduction

Throughout this paper all rings will be commutative with identity and all modules will be unitary. The idea to extend the concepts of the product of two ideals from the category of rings to the category of modules has stimulated several authors to show that many, but not all, of the results in the theory of rings are also valid for modules (see, for example [2], [4], [5]). In [6], the following question was investigated: Suppose I ia an ideal of a ring R such that R/I is noetherian. If some power of I is finitely generated, under what condition is I finitely generated. The purpose of this paper is to explore some basic facts of the product of submodules of a multiplication R-module. In section 2, we give a condition giving an affirmative answer to the following question: Suppose N is a submodule of a multiplication R-module M such that M/N is noetherian. If some power of N is finitely generated, under

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what conditions is N finitely generated (see Theorem 2.12). Also, we show that if N is any weakly primary submodule of a multiplication R-module that is not primary, then $N^2 = 0$ (see Theorem 2.15).

Now we define the concepts that we will use. If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. An R-module M is called a multiplication module if for each submodule N of M, $N = IM$ for some ideal I of R. In this case we can take $I = (N : M)$. Clearly, M is a multiplication module if and only if for each $m \in M$, $Rm = (Rm : M)M$ (see [3]).

A proper submodule N of a module M over a ring R is said to be prime submodule if for each $r \in R$ the R-endomorphism of M/N produced by multiplication by r is either injective or zero, so $(0 : M/N) = P$ is a prime ideal of R, and N is said to be P-prime submodule. So N is prime in M if and only if whenever $rm \in N$, for some $r \in R$, $m \in M$, we have $m \in N$ or $rM \subseteq N$. The radical of a submodule, denoted $\mathrm{Rad}(N)$, is defined to be the intersection of all prime submodules containing N. Similarly, if I is an ideal of R, we denote the radical of I by $\mathrm{Rad}(I)$.

Let M be an R-module and N be a submodule of M such that $N = IM$ for some ideal I of R. Then we say that I is a presentation ideal of N. If M is a vector space over an arbitrary field F with $\dim_F(M) = k \geq 2$, then M has finite length, so M is notherian and artinian, but any proper subspace N $(\neq 0)$ of M does not have any presentation. Hence, M is not multiplication. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [2, theorem 3.4], the product of N and K is independent of presentation of N and K. Moreover, for $a, b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$.

2. Multiplication modules

If M is a faithful multiplication R-module, then M is noetherian if and only if R is notherian (see [3, p. 7 67]), so we have the following lemma:

Lemma 2.1. *Let R be a commutative ring, M a multiplication R-module, and N an R-submodule of M. Then M/N is a noetherian R-module if and only if $R/(N : M)$ is a noetherian ring.*

Lemma 2.2. *Let M be a finitely generated faithful multiplication R-module. Then*

(i) A submodule N of M is noetherian if and only if there exists a noetherian ideal I of R such that $N = IM$.

(ii) A submodule N of M is artinian if and only if there exists a artinian ideal I of R such that $N = IM$.

PROOF. Suppose first that N is a noetherian submodule of M. There exists an ideal I of R such that $N = IM$. Consider a chain of subideals of I; $I_1 \subseteq I_2 \subseteq ....$ So we can have a chain of submodules of N: $I_1M \subseteq I_2M \subseteq ...$ Since N is noetherian, there exists a positive integer n such that $I_nM = I_iM$ for all $i \geq n$. Therefore $I_n = I_i$ $(i \geq n)$ by [8, p. 231 Corollary], and hence I is noetherian. Similarly, if I is noetherian, then N is noethrian.

(ii) This proof is similar to that in case (i) and we omit it. □

Lemma 2.3. *Let N be an R-submodule of a finitely generated multiplication module M. Then $(N^k : M) = (N : M)^k$ for each positive integer k.*

PROOF. We shall prove by induction on k. The result is trivial for $k = 1$. Assume that $k \geq 1$ and that $(N^k : M) = (N : M)^k$. It then follows from [2, Lemma 3.6] that $(N^{k+1} : M) = (N^kN : M) = (N^k : M)(N : M) = (N : M)^{k+1}$, as required. □

Lemma 2.4. *Let M be a finitely generated faithful multiplication R-module, and let N be a submodule of M such that $R/(N : M)$ is noetherian. If N is finitely generated, then every submodule K of M containing N is finitely generated.*

PROOF. By Lemma 2.1, M/N is a noetherian R-module, hence so is K/N. Since N is finitely generated so is K. □

Proposition 2.5. *Let N be a submodule of a finitely generated faithful multiplication R-module M. If N^n is finitely generated for some $n \geq 1$, then there exists a finitely generated submodule $K \subseteq N$ such that $N^n = K^n$.*

PROOF. We have $N^n = (N^n : M)M = (N : M)^n M$ by Lemma 2.3. Since N^n is finitely generated we obtain from [7, Lemma 1.4] that $(N : M)^n$ is a finitely generated ideal of R, and hence there exists a finitely genenrared ideal $J \subseteq (N : M)$ of R such that $J^n = (N : M)^n$ by [6, Lemma 1.2]. Set $K = JM$. Clearly, $J = (K : M)$. Moreover, $J \subseteq (N : M)$ and $J^n = (N : M)^n$ implies $K \subseteq N$ and $K^n = N^n$. □

Proposition 2.6. *Let N, K be submodules of a finitely generated faithful multiplication R-module M such that* $\mathrm{Rad}N = \mathrm{Rad}K$. *Then the following two conditions are equivalent:*

(i) M/N is noetherian and N is finitely generated.

(ii) M/K is noetherian and K is finitely generated.

PROOF. There are ideals $I = (N : M)$ and $J = (K : M)$ of R such that $N = IM$ and $K = JM$, so $\mathrm{Rad}N = \mathrm{Rad}(I)M = \mathrm{Rad}K = \mathrm{Rad}(J)M$ by [3, Theorem 2.12]. Therefore, $\mathrm{Rad}(I) = \mathrm{Rad}(J)$ by [8, p. 231 Corollary]. It is enough to show that $(i) \Longrightarrow (ii)$. By Lemma 2.1 and [7, Lemma 1.4], R/I is noetherian and I is finitely generated respectively, and hence R/J is noethrian and J is finitely generated by [6, Lemma 1.5]. Now the asserition follows from Lemma 2.1 and [7, Lemma 1.4]. □

Theorem 2.7. *Let N be an R-submodule of a finitely generated faithful multiplication module M. Assume that M/N is noetherian and N^n is finitely generated for some $n \geq 1$. Then N is finitely generated if and only if M/N^n is noetherian.*

PROOF. There exists a finitely generated submodule $K \subseteq N$ of M such that $N^n = K^n$ for some $n \geq 1$ by Proposition 2.5. Then $(K : M) \subseteq (N : M)$ and $(N : M)^n = (K : M)^n$ is finitely generated by [7, Lemma 1.4] and Lemma 2.3. It then follows from [6, Lemma 1.1] that $(N : M)^m = (N : M)^i (K : M)^{m-i}$ for all $m \geq n$ and $0 \leq i \leq n$. Clearly, $\text{Rad}(N : M)^n \subseteq \text{Rad}(N : M)$. If $x \in \text{Rad}(N : M)$, then $x^s \in (N : M)^s (K : M)^{n-s} = (N : M)^n = (K : M)^n$ for some s, and hence $x \in \text{Rad}(K : M)^n$. Therefore, $\text{Rad}(N : M) = \text{Rad}(N : M)^n = (K : M)^n$. Thus $\text{Rad}N = \text{Rad}(N : M)M = \text{Rad}(N : M)^n M = \text{Rad}N^n$. Now the asserition follows from Proposition 2.6. □

Lemma 2.8. *Let N be an R-submodule of a finitely generated faithful multiplication module M. Assume that M/N is noetherian and that some power of N is finitely generated. Then $M/\text{Rad}N$ is noetherian and that some power of $\text{Rad}N$ is finitely generated.*

PROOF. Clearly, $M/\text{Rad}N$ is noetherian. Assume that N^n is finitely generated. There exists an ideal I of R such that $N = IM$, so $N^n = I^n M$, and hence I^n is finitely generated by [7, Lemma 1.4]. Since M/N is noetherian it follows from Lemma 2.1 that R/I is noetherian. Then there is a positive integer k such that $(\text{Rad}I)^{kn}$ is finitely generated by [6, Lemma 1.7]. Therefore $(\text{Rad}N)^{kn} = (\text{Rad}I)^{kn} M$ is finitely generated by [7, Lemma 1.4], as required. □

Theorem 2.9. *Let N be an R-submodule of a finitely generated faithful multiplication module M. Assume that M/N is noetherian and that some power of N is finitely generated. Then N is finitely generated if and only if $\text{Rad}(N)$ is finitely generated.*

PROOF. This follows from Proposition 2.6 and Lemma 2.8. □

Lemma 2.10. *Let N be an R-submodule of a multiplication module M. Then the set $(0:_M N) = \{m \in M : Rm.N = 0\}$ is a submodule of M.*

PROOF. Let $x, y \in (0:_M N)$, and let I_1, I_2, and I be presentation ideals of x, y, and N respectively, so $I_1IM = 0 = I_2IM$. As $I_1 + I_2$ is the presentation ideal of $x + y$ by [2. Proposition 3.8], it follows that $R(x+y).N = (I_1+I_2)IM = I_1IM+I_2IM = 0$, that is, $x+y \in (0:_M N)$. Also, for $x \in (0:_M N)$ and $r \in R$, we have $(Rrx).N = (rI_1)IM = r(I_1IM) = 0$. Thus $(0:_M N)$ is a submodule of M. □

Lemma 2.11. *Let $N = IM$ be an R-submodule of a multiplication module M. Then*

(i) $(0:_M N) = (0:_M I) = (0:_R I)M$.

(ii) If M is faithful, then $(0:_N N) = 0$ if and only if $(0:_I I) = 0$.

PROOF. (i) Let $x \in (0:_M I)$, and let J be a presentation ideal of x. Then $Rx.N = (JM)(IM) = I(Rx) = Ix = 0$, so $x \in (0:_M N)$, and hence $(0:_M I \subseteq (0:_M N)$. Similarly, $(0:_M N \subseteq (0:_M I)$. The last equality is immediate by [9, Lemma 2.6].

(ii) This follows from (i). □

Theorem 2.12. *Let N be an R-submodule of a finitely generated faithful multiplication module M. Assume M/N is noetherian and that some power of N is finitely generated. If $(0:_N N) = 0$, and if* $\mathrm{Rad}(N) = \mathrm{Rad}(Ra)$ *for some $a \in M$, then N is finitely generated.*

PROOF. Assume that N^n is finitely generated, and let I be a presentation ideal of N. Then I^n is finitely generated [5, Lemma 1.4]. Since M is a multiplication R-module, $Ra = (Ra : M)M$. Hence there exist $a_1, ..., a_s \in (Ra : M)$, $m_1, ..., m_s \in M$ such that $a = a_1m_1 + ... + a_sm_s$. Let J be an ideal of R generated by $\{a_1, a_2, .., a_s\}$. Then clearly,

$Ra = JM$. Therefore, $\text{Rad}(N) = \text{Rad}(I)M = \text{Rad}Ra = \text{Rad}(J)M$, so $\text{Rad}(I) = \text{Rad}(J)$. Moreover, by Lemma 2.11, $(0 :_I I) = 0$. It follows from [6, Theorem 1.10] that I is finitely generated, and hence $N = IM$ is finitely generated by [7, Lemma 1.4]. □

Definition 2.13. *A proper submodule N of a module M over a commutative ring R is said to be weakly primary submodule if whenever $0 \neq rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r^n M \subseteq N$ for some n.*

Clearly, every primary submodule of a module is a weakly primary submodule. However, since 0 is always weakly primary (by definition), a weakly primary submodule need not be primary.

Proposition 2.14. *Let R be a commutative ring, M an R-module, and N a weakly primary submodule of M. If N is not primary, then $(N : M)N = 0$.*

PROOF. Suppose that $(N : M)N \neq 0$, we will show that N is a primary submodule of M. Let $rm \in N$ where $r \in R$ and $m \in M$. If $rm \neq 0$, then either $m \in N$ or $r^m M \subseteq N$ for some m since N is weakly primary. So assume that $rm = 0$. First suppose that $rN \neq 0$, say $rn \neq 0$ where $n \in N$. Then $0 \neq rn = r(n+m) \in N$, so $r \in \text{Rad}(N : M)$ or $n + m \in N$. Hence $r \in \text{Rad}(N : M)$ or $m \in N$. so we can assume that $rN = 0$. Second suppose that $(N : M)m \neq 0$, say $sm \neq 0$ where $s \in (N : M)$. Then $0 \neq sm = (r + s)m \in N$, so $r \in \text{Rad}(N : M)$ or $m \in N$. So we can assume that $(N : M)m = 0$. Since we assumed $(N : M)N \neq 0$, there exist $t \in (N : M)$ and $n' \in N$ such that $tn' \neq 0$. Then $0 \neq tn' = (r+t)(n'+m) \in N$, so $r+t \in \text{Rad}(N : M)$ or $n'+m \in N$. Hence $r \in \text{Rad}(N : M)$ or $m \in N$. Thus N is a primary submodule of M. □

Compare the following theorem with Theorem 2.12

Theorem 2.15. *Let R be a commutative ring, M a multiplication R-module, and N a weakly primary submodule of M. If N is not primary, then $N^2 = 0$.*

PROOF. Since M is multiplication it follows from Proposition 2.14 that $N^2 = (N : M)^2 M = (N : M)N = 0$, as needed. □

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