KYUNGPOOK Math. J. 45(2005), 45-53

On the Basis Number of the Semi-Strong Product of Bipartite Graphs with Cycles

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Abstract. A basis of the cycle space $\mathcal{C}(G)$ is d-fold if each edge occurs in at most d cycles of $\mathcal{C}(G)$. The basis number, $b(G)$, of a graph G is defined to be the least integer d such that G has a d-fold basis for its cycle space. MacLane proved that a graph G is planar if and only if $b(G) \leq 2$. Schmeichel showed that for $n \geq 5$, $b(K_n \bullet P_2) \leq 1 + b(K_n)$. Ali proved that for $n, m \geq 5$, $b(K_n \bullet K_m) \leq 3 + b(K_n) + b(K_m)$. In this paper, we give an upper bound for the basis number of the semi-strong product of a bipartite graph with a cycle.

1. Introduction

Throughout this paper, we consider only finite simple connected graphs. Our terminology and notation will be standard except as indicated.

Let G be a graph and $e_1, e_2, \cdots, e_{|E(G)|}$ be an enumeration of its edges. Then any subset S of $E(G)$ corresponds to a $(0,1)$-vector $(\zeta_1, \zeta_2, \cdots, \zeta_{|E(G)|}) \in (Z_2)^{|E(G)|}$ with $\zeta_i = 1$ if $e_i \in S$ and $\zeta_i = 0$ if $e_i \notin S$. Let $\mathcal{C}(G)$, called the cycle space, be the subspace of $(Z_2)^{|E(G)|}$ generated by the vectors corresponding to the cycles in G. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is well known that if r is the number of components of G, then $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + r$.

A basis of $\mathcal{C}(G)$ is called d-fold if each edge of G occurs in at most d of the cycles in the basis. The basis number of G, $b(G)$, is the smallest non-negative integer number d such that $\mathcal{C}(G)$ has a d-fold basis. The first important result concerning the basis number of a graph was the theorem of MacLane when he proved that a graph G is planar if and only if $b(G) \leq 2$.

Schmeichel proved that there are graphs with arbitrary large basis numbers. Moreover, Schmeichel proved that $b(K_n) \leq 3$.

The required basis of $\mathcal{C}(G)$ is a basis with $b(G)$-fold. Let G and H be two graphs, $\varphi : G \longrightarrow H$ be an isomorphism and $\mathcal{B}$ be a (required) basis of $\mathcal{C}(G)$. Then $\mathcal{B}' = \{\varphi(c) | c \in \mathcal{B}\}$ is called the corresponding (required) basis of $\mathcal{B}$ in H.

Let G_1 and G_2 be two graphs. The direct product $G = G_1 \wedge G_2$ is the

Received December 8, 2003, and, in revised form, July 6, 2004.
2000 Mathematics Subject Classification: 05C38, 05C75.
Key words and phrases: fold, basis number, cycle space.

graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E_1 \text{ and } u_2v_2 \in E_2\}$. The semi-strong product $G = G_1 \bullet G_2$ is the graph with the vertex set $V(G) = V(G_1) \times V(G_2)$ and the edge set $E(G) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E_1 \text{ and } u_2v_2 \in E_2 \text{ or } u_1 = v_1 \text{ and } u_2v_2 \in E_2\}$. Note that, $|E(G_1 \wedge G_2)| = 2|E(G_1)||E(G_2)|$ and $|E(G_1 \bullet G_2)| = 2|E(G_1)||E(G_2)| + |V(G_1)||E(G_2)|$.

In this paper, we are interested in establishing an upper bound of the basis number of the semi-strong product of a bipartite graph with a cycle. In the following results of Schmeichel and Ali in which they give an upper bound for the basis number of the semi-strong product of a complete graph K_n with a path P_2 and a complete graph K_m.

Theorem 1.1. (Schmeichel) *For each* $n \geq 5, b(K_n \bullet P_2) \leq 1 + b(K_n)$.

Theorem 1.2. (Ali) *For each* $n, m \geq 5, b(K_n \bullet K_m) \leq 3 + b(K_n) + b(K_m)$.

A tree T consisting of n equal order paths $\{P^{(1)}, P^{(2)}, \cdots, P^{(n)}\}$ is called an n-special star if there is a vertex, say v_1, such that v_1 is an end vertex for each path in $\{P^{(1)}, P^{(2)}, \cdots, P^{(n)}\}$ and $V(P^{(i)}) \cap V(P^{(j)}) = \{v_1\}$ for each $i \neq j$ (see [5]). Jaradat proved the following result ([5]).

Theorem 1.3. (Jaradat) *For each bipartite graph* G, $b(G \wedge C_n) \leq 3 + b(G)$. *Moreover,* $b(G \wedge C_n) \leq 2 + b(G)$ *if* G *has a spanning tree which contains no subgraph isomorphic to a 3-special star of order 7.*

It is well known (see Harary [4]) that the direct product of a bipartite graph G with a path of order 2, P_2, is disconnected, the following result ([5]) generalize this result.

Proposition 1.4. (Jaradat) *Let* G *be a bipartite graph and* P_2 *be a path of order 2. Then* $G \wedge P_2$ *consists of two components* G_1 *and* G_2 *each of which is isomorphic to* G.

In view of the above results, a natural question arises: does there exist an upper bound of the basis number of the semi-strong product of graphs?

Our main purpose in this paper is to give a positive answer to the above question by considering the semi-strong product of a bipartite graph with a cycle.

2. Main results

In this section, we give an upper bound of the basis number of the semi-strong product of a bipartite graph with a cycle. Throughout this section we consider $C_n = v_1v_2 \cdots v_{n-1}v_nv_1$ and the fold of an edge e in a set $B \subseteq \mathcal{C}(G)$, $f_B(e)$, is the number of cycles in B containing e.

Lemma 2.1. *For each cycle* C_n *with* $n \geq 4$ *and path* $P_2 = uw$, *we have* $b(P_2 \bullet C_n) \geq 3$.

Proof. Let $A = \{(u,v_1),(w,v_1),(w,v_3)\}$ and $B = \{(u,v_2),(w,v_2),(u,v_n)\}$. Consider the subgraph H of $P_2 \bullet C_n$ whose vertex set $V(H) = A \cup B \cup \{(w,v_4),(w,v_5),\cdots,(w,v_{n-1})\}$ and edge set consists of the following nine paths: $P_1 = (u,v_1)(w,v_2)$, $P_2 = (w,v_1)(u,v_2)$, $P_3 = (u,v_1)(u,v_n)$, $P_4 = (w,v_1)(u,v_n)$, $P_5 = (u,v_1)(u,v_2)$, $P_6 = (u,v_2)(w,v_3)$, $P_7 = (w,v_1)(w,v_2)$, $P_8 = (w,v_2)(w,v_3)$, and $P_9 = (w,v_3)(w,v_4)\cdots(w,v_{n-1})(u,v_n)$. Then H is homeomorphic to $K_{3,3}$. Therefore, $b(P_2 \bullet C_n) \geq 3$. □

Theorem 2.2. *For each cycle C_n with $n \geq 4$ and path $P_2 = uw$, we have $b(P_2 \bullet C_n) = 3$.*

Proof. To prove this Lemma it suffices to exhibit a 3-fold basis for $\mathcal{C}(P_2 \bullet C_n)$. Set

$$\begin{aligned}
\mathcal{B}_{P_2u} &= \left\{\mathcal{B}_{P_2u}^{(j)} = (u,v_j)(u,v_{j+1})(u,v_{j+2})(w,v_{j+1})(u,v_j) \mid j = 1,2,\cdots,n-2\right\} \\
&\quad \cup \left\{\mathcal{B}_{P_2u}^{(n-1)} = (u,v_{n-1})(u,v_n)(u,v_1)(w,v_n)(u,v_{n-1})\right\}, \text{ and} \\
\mathcal{B}_{P_2w} &= \left\{\mathcal{B}_{P_2w}^{(j)} = (w,v_j)(w,v_{j+1})(w,v_{j+2})(u,v_{j+1})(w,v_j) \mid j = 1,2,\cdots,n-2\right\} \\
&\quad \cup \left\{\mathcal{B}_{P_2w}^{(n-1)} = (w,v_{n-1})(w,v_n)(w,v_1)(u,v_n)(w,v_{n-1})\right\}.
\end{aligned}$$

It is an easy matter to see that each of $\mathcal{B}_{P_2u}$ and $\mathcal{B}_{P_2w}$ is linearly independent. Note that every linear combination of cycles of $\mathcal{B}_{P_2u}$ contains at least one edge of the form $(u,v_j)(u,v_{j+1})$ and $(u,v_1)(u,v_n)$ for some j which is not in any cycle of $\mathcal{B}_{P_2w}$. Thus $\mathcal{B}_{P_2u} \cup \mathcal{B}_{P_2w}$ is linearly independent set. Now, consider the following two cycles:

$$C_u = (u,v_1)(u,v_2)\cdots(u,v_n)(u,v_1) \text{ and } C_w = (w,v_1)(w,v_2)\cdots(w,v_n)(w,v_1).$$

We now prove that C_u is independent from the cycles of $\mathcal{B}_{P_2u} \cup \mathcal{B}_{P_2w}$. Let $F = \sum_{k=1}^{\gamma_2} \mathcal{B}_{P_2w}^{(j_k)} \pmod 2$. Then F is an edge disjoint union of cycles and each of which contains at least one edge of the form $(w,v_j)(w,v_{j+1})$ and $(w,v_1)(w,v_n)$ for some j. Thus, if $C_u = \sum_{k=1}^{\gamma_1} \mathcal{B}_{P_2u}^{(j_k)} + \sum_{k=1}^{\gamma_2} \mathcal{B}_{P_2w}^{(j_k)} \pmod 2$, then γ_2 must be equal to 0. Hence $C_u = \sum_{k=1}^{\gamma_1} \mathcal{B}_{P_2u}^{(j_k)} \pmod 2$. To this end, we consider two cases:

Case 1. n is odd.

Since $(u,v_1)(u,v_2), (u,v_2)(u,v_3) \in E(C_u)$ and the only cycle in $\mathcal{B}_{P_2u}$ containing $(u,v_1)(u,v_2)$ is $\mathcal{B}_{P_2u}^{(1)}$, we get $\mathcal{B}_{P_2u}^{(1)} \in \left\{\mathcal{B}_{P_2u}^{(j_1)}, \mathcal{B}_{P_2u}^{(j_2)}, \cdots, \mathcal{B}_{P_2u}^{(j_{\gamma_1})}\right\}$ and $\mathcal{B}_{P_2u}^{(2)} \notin \left\{\mathcal{B}_{P_2u}^{(j_1)}, \mathcal{B}_{P_2u}^{(j_2)}, \cdots, \mathcal{B}_{P_2u}^{(j_{\gamma_1})}\right\}$. Also since $(u,v_3)(u,v_4), (u,v_4)(u,v_5) \in E(C_u)$ and the only two cycles in $\mathcal{B}_{P_2u}$ containing $(u,v_3)(u,v_4)$ are $\mathcal{B}_{P_2u}^{(2)}$ and $\mathcal{B}_{P_2u}^{(3)}$, we have $\mathcal{B}_{P_2u}^{(3)} \in \left\{\mathcal{B}_{P_2u}^{(j_1)}, \mathcal{B}_{P_2u}^{(j_2)}, \cdots, \mathcal{B}_{P_2u}^{(j_{\gamma_1})}\right\}$ and $\mathcal{B}_{P_2u}^{(4)} \notin \left\{\mathcal{B}_{P_2u}^{(j_1)}, \mathcal{B}_{P_2u}^{(j_2)}, \cdots, \mathcal{B}_{P_2u}^{(j_{\gamma_1})}\right\}$. Continuing in this way implies that $\mathcal{B}_{P_2u}^{(n-2)} \in \left\{\mathcal{B}_{P_2u}^{(j_1)}, \mathcal{B}_{P_2u}^{(j_2)}, \cdots, \mathcal{B}_{P_2u}^{(j_{\gamma_1})}\right\}$. It is easy to see that $(u,v_1)(u,v_n) \in E(C_u)$, and the only cycle in $\mathcal{B}_{P_2u}$ contains this edge is $\mathcal{B}^{(n-1)}_{P_2u}$. Then $\mathcal{B}^{(n-1)}_{P_2u} \in \left\{\mathcal{B}^{(j_1)}_{P_2u}, \mathcal{B}^{(j_2)}_{P_2u}, \cdots, \mathcal{B}^{(j_{\gamma_1})}_{P_2u}\right\}$. One can see easily that $(u, v_n)(u, v_{n-1})$ belongs only to $\mathcal{B}^{(n-2)}_{P_2u}$, $\mathcal{B}^{(n-1)}_{P_2u}$ and C_u. Therefore, it is not in $\sum_{k=1}^{\gamma_1} \mathcal{B}^{(j_k)}_{P_2u}$ (mod 2). This is a contradiction.

Case 2. n is even.
Then by the same arguments as in Case 1 we have that each of $\mathcal{B}^{(1)}_{P_2u}, \mathcal{B}^{(3)}_{P_2u}, \cdots, \mathcal{B}^{(n-3)}_{P_2u}, \mathcal{B}^{(n-1)}_{P_2u} \in \left\{\mathcal{B}^{(j_1)}_{P_2u}, \mathcal{B}^{(j_2)}_{P_2u}, \cdots, \mathcal{B}^{(j_{\gamma_1})}_{P_2u}\right\}$ and each of $\mathcal{B}^{(2)}_{P_2u}, \mathcal{B}^{(4)}_{P_2u}, \cdots, \mathcal{B}^{(n-2)}_{P_2u} \notin \left\{\mathcal{B}^{(j_1)}_{P_2u}, \mathcal{B}^{(j_2)}_{P_2u}, \cdots, \mathcal{B}^{(j_{\gamma_1})}_{P_2u}\right\}$. Therefore, $C_u + \sum_{k=1}^{\gamma_1} \mathcal{B}^{(j_k)}_{P_2u}$ (mod 2) contains $(u, v_{n-1})(w, v_n)$. This is a contradiction.

Using the same arguments as above one can prove that C_w is independent from the cycles of $\mathcal{B}_{P_2u} \cup \mathcal{B}_{P_2w} \cup \{C_u\}$. Therefore, $\mathcal{B}_{P_2u} \cup \mathcal{B}_{P_2w} \cup \{C_u\} \cup \{C_w\}$ is linearly independent. Now, set

$$D = (u, v_1)(u, v_2)(w, v_1)(w, v_2)(u, v_1).$$

To this end, we show that D is linearly independent from the cycles of $\mathcal{B}_{P_2u} \cup \mathcal{B}_{P_2w} \cup \{C_u\} \cup \{C_w\}$. Let $\mathcal{F} = \left\{\mathcal{B}^{(j_1)}_{P_2u}, \mathcal{B}^{(j_2)}_{P_2u}, \cdots, \mathcal{B}^{(j_{\gamma_1})}_{P_2u}\right\} \cup \left\{\mathcal{B}^{(j_1)}_{P_2w}, \mathcal{B}^{(j_2)}_{P_2w}, \cdots, \mathcal{B}^{(j_{\gamma_2})}_{P_2w}\right\} \cup \{C_f\}_{f \in A}$ where $A \subseteq \{u, w\}$. Assume $D = \sum_{k=1}^{\gamma_1} \mathcal{B}^{(j_k)}_{P_2u} + \sum_{k=1}^{\gamma_2} \mathcal{B}^{(j_k)}_{P_2w} + \sum_{f \in S \subseteq A} C_f$ (mod 2). Since $(u, v_1)(w, v_2)$ and $(w, v_1)(u, v_2)$ are two edges of $E(D)$ and the only cycles in $\mathcal{B}_{P_2u} \cup \mathcal{B}_{P_2w} \cup \{C_u\} \cup \{C_w\}$ containing these two edges are $\mathcal{B}^{(1)}_{P_2u}$ and $\mathcal{B}^{(1)}_{P_2w}$, respectively, as a result $\left\{\mathcal{B}^{(1)}_{P_2u}, \mathcal{B}^{(1)}_{P_2w}\right\} \subseteq \mathcal{F}$. Also since $(u, v_2)(w, v_3)$ and $(w, v_2)(u, v_3)$ are two edges of $E(\mathcal{B}^{(1)}_{P_2u} \oplus \mathcal{B}^{(1)}_{P_2w})$ where $\oplus$ is the ring sum, and are not in $E(D)$ and the only two cycles containing these edges are $\mathcal{B}^{(2)}_{P_2u}$ and $\mathcal{B}^{(2)}_{P_2w}$, we have $\left\{\mathcal{B}^{(2)}_{P_2u}, \mathcal{B}^{(2)}_{P_2w}\right\} \subseteq \mathcal{F}$. Continuing in this way it implies that $\left\{\mathcal{B}^{(n-1)}_{P_2u}, \mathcal{B}^{(n-1)}_{P_2w}\right\} \subseteq \mathcal{F}$. Note that $\mathcal{B}^{(n-1)}_{P_2u}$ is the only cycle which contains only one of the following two edges $(u, v_n)(w, v_1)$ and $(w, v_n)(u, v_2)$ and $\mathcal{B}^{(n-1)}_{P_2w}$ is the only cycle which contains the other. Hence, these two edges belong to D, a contradiction. Therefore, $\mathcal{B}_{P_2} = \mathcal{B}_{P_2u} \cup \mathcal{B}_{P_2w} \{C_u\} \cup \{C_w\} \cup \{D\}$ is linearly independent. Since $|\mathcal{B}_{P_2}| = 2n + 1 = \dim \mathcal{C}(P_2 \bullet C)$, $\mathcal{B}_{P_2}$ is a basis for $\mathcal{C}(P_2 \bullet C)$. To complete the proof of the Theorem, we show that $\mathcal{B}$ is a 3-fold basis. Let $e \in E(P_2 \bullet C_n)$. (1) If $e \in E(P_2 \wedge C_n)$, then $f_{\mathcal{B}_{P_2u}}(e) \leq 1, f_{\mathcal{B}_{P_2w}}(e) \leq 1, f_{\{C_u\} \cup \{C_w\}}(e) = 0$, and $f_{\{D\}}(e) \leq 1$. (2) If $e \in E(P_2 \bullet C_n) - E(P_2 \wedge C_n)$, then $f_{\mathcal{B}_{P_2u}}(e) = 0, f_{\mathcal{B}_{P_2w}}(e) = 0, f_{\{C_u\} \cup \{C_w\}}(e) \leq 2$, and $f_{\{D\}}(e) \leq 1$ (see figure 1 which illustrates the case $P_2 \bullet C_4$).

□

In order to achieve our goal we find it is useful to give the following definition. Let G be a graph and $e_1, e_2, \cdots e_{|E(G)|-1}, e_{|E(G)|}$ be an ordering of the edge set of G. For each e_i assign 1 to one of its two vertices and 0 to the other. Let u be a vertex which is incident to $e_{n_1}, e_{n_2}, \cdots, e_{n_r}$ where $n_1 < n_2 < \cdots < n_r$. Then u corresponds to a (0,1)-vector $(\xi_1, \xi_2, \cdots, \xi_r)$ where $\xi_i = 0$ if 0 is assigned to u in

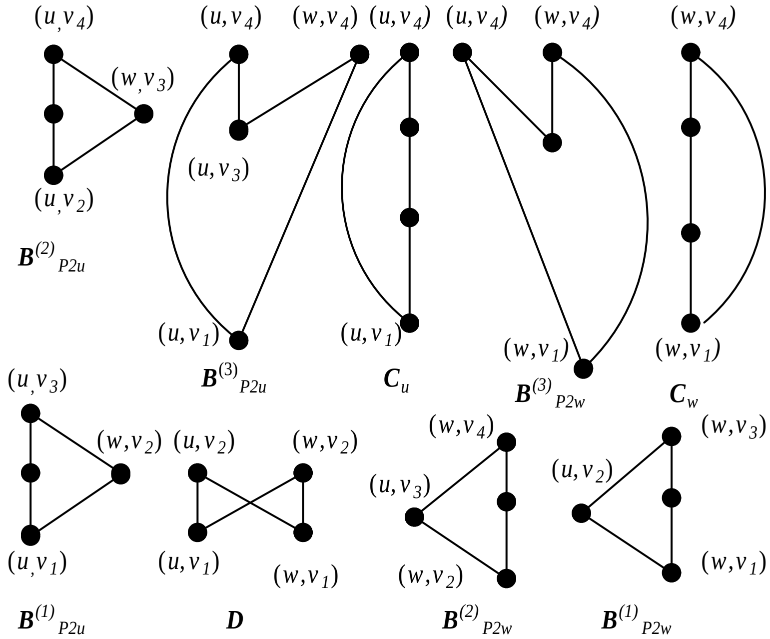

Figure 1:

e_{n_i} and $\xi_i = 1$ if 1 is assigned to u in e_{n_i}. We call this vector a degree vector of u and denote it by $DV_G(u)$. The set of all degree vectors of G will be denoted by $DVS(G)$. Note that $DVS(G)$ is not unique because the values of the components in each vector depend not only on the way we assign the 0's and 1's for the vertices of edges of G but also on the way we label the edges of G.

Proposition 2.3. *For each tree T of order ≥ 2, there is a degree vector set $DVS(G)$ such that the degree vector of any vertex contains exactly one entry of value 1, except one end vertex has degree vector* (0).

Proof. Label the edge of T. Pick any end vertex of T, say v^*, and let $v^*v \in E(T)$. Assign the value 0 to the vertex v^*, so the vertex v has to take the value 1 in the edge v^*v. Now, let $\{v_1, v_2, \cdots, v_r, v^*\}$ be the set of all vertices which are adjacent to v. For each $1 \leq j \leq r$ assign the value 0 to v and 1 to each v_i in the edge vv_1, $vv_1, \cdots, vv_r$. For each $1 \leq j \leq r$ assume $\left\{v_{j_1}, v_{j_2}, \cdots, v_{j_{r_j}}, v\right\}$ is the set of all vertices which are adjacent to v_j. For each $1 \leq j \leq r$ and $1 \leq s \leq r_j$ assign the value 0 to v_j and 1 to v_{j_s} in each edge $v_j v_{j_s}$. By continuing in this process, we get that every degree vector of every vertex has exactly one of its components the value 1 except the degree vector of v^* is (0) (see figure 2).

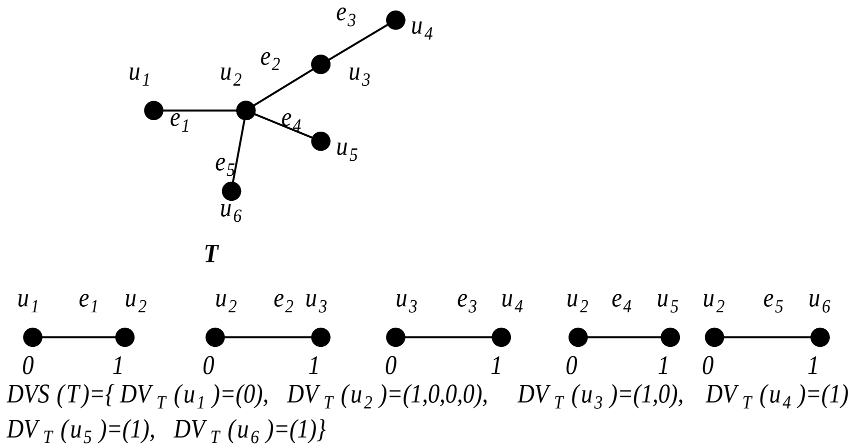

Figure 2:

□

The following lemma of Jaradat will play a useful role in the coming results:

Proposition 2.4. (Jaradat) *For each tree T of order ≥ 3, there is a set of paths $S(T) = \left\{P_3^{(1)}, P_3^{(2)}, \cdots, P_3^{(m)}\right\}$, called a path-sequence, such that*

(i) *each $P_3^{(i)}$ is a path of length 2,*

(ii) $\bigcup_{i=1}^{m} E(P_3^{(i)}) = E(T)$

(iii) *every edge $uv \in E(T)$ appears in at most three paths of $S(T)$,*

(iv) *each $P_3^{(j)}$ contains one edge which is not in $\bigcup_{i=1}^{j-1} P_3^{(i)}$,*

(v) *if uv appears in three paths of $S(T)$, then the paths have forms of either uva, uvb and cuv or auv, buv or uvc,*

(vi) *for each end point v the edge vv^* occurs in at most two paths of $S(T)$.*

(vii) $m = |V(T)| - 2 = |E(T)| - 1$.

One can easily see from the proof of Proposition 2.4 (see [5]) that each $S(T)$, which satisfies the conditions in Proposition 2.4, there is an edge whose one of its vertices is an end vertex of T and appears only in one path of $S(T)$. Moreover, from the proof of Proposition 2.3 we can assume that edge is the edge which contains the vertex of degree vector (0).

Let $e = uw$. In the following results we consider $\mathcal{B}^{(e)} = \mathcal{B}_{eu} \cup C_u$ if 1 is assigned to u and 0 to w, and $\mathcal{B}^{(e)} = \mathcal{B}_{ew} \cup C_w$ if 1 is assigned to w and 0 to u where $\mathcal{B}_{eu} = \mathcal{B}_{P_2 u}$ and $\mathcal{B}_{ew} = \mathcal{B}_{P_2 w}$ as in Lemma 2.2.

Lemma 2.5. *For each tree T of order ≥ 2 and cycle C_n with $n \geq 4$, we have $3 \leq b(T \bullet C_n) \leq 4$. Moreover, $b(T \bullet C_n) = 3$ if T contains no subgraph isomorphic to a 3-special star of order 7.*

Proof. Let $e \in E(T)$. Then $e \bullet C_n$ is a subgraph of $T \bullet C_n$. Since, by Lemma 2.2, $e \bullet C_n$ is non planar, we get that $T \bullet C_n$ is non planar and so $b(T \bullet C_n) \geq 3$. Now, let $S(T) = \left\{ P_3^{(1)} = a_1 b_1 c_1,\ P_3^{(2)} = a_2 b_2 c_2, \cdots,\ P_3^{(|V(T)|-2)} = a_{|V(T)|-2} b_{|V(T)|-2} c_{|V(T)|-2} \right\}$ be a path sequence as in Proposition 2.4. Let DVS(T) be the set of all degree vectors of G as in Proposition 2.3. Set

$$\begin{aligned} \mathcal{B}_{P_3^{(i)}} &= \{(a_i, v_{j+1})(b_i, v_j)(c_i, v_{j+1})(b_i, v_{j+2})(a_i, v_{j+1}) \mid j = 1, 2, \cdots, n-2\} \\ &\quad \cup \{(a_i, v_n)(b_i, v_{n-1})(c_i, v_n)(b_i, v_1)(a_i, v_n)\} \\ &\quad \cup \{(a_i, v_1)(b_i, v_2)(c_i, v_1)(b_i, v_n)(a_i, v_1)\}. \end{aligned}$$

Let $\mathcal{B}^* = \bigcup_{i=1}^{|V(T)|-2} \mathcal{B}_{P_3^{(i)}}$. Then $\mathcal{B}^*$ is linearly independent (see [5]). We may assume that P_2 is the edge which contains the vertex with degree vector (0). Let $\mathcal{B}' = \mathcal{B}_{P_2} \cup (\bigcup_{e \in E(T)-P_2} \mathcal{B}^{(e)})$. Since the degree vector of each vertex contains exactly one entry of value 1 except one of the vertices of P_2 which is an end vertex, we get that $E(\mathcal{B}^{(e)}) \cap E(\mathcal{B}^{(e')}) = \phi$ and $E(\mathcal{B}^{(e)}) \cap E(\mathcal{B}_{P_2}) = \phi$ whenever $e' \neq e$ and $e \neq P_2$. Hence, $\mathcal{B}'$ is linearly independent. To this end, one can easily see that each cycle of $\mathcal{B}^*$ consists of four edges. Moreover, each of these cycles either contains an edge which is not in any cycle of $\mathcal{B}'$ or has exactly two edge belong to $\mathcal{B}^{(e)}$ and the other two belong to $\mathcal{B}^{(e')}$ or to $\mathcal{B}_{P_2}$ for some $e' \neq e$. Therefore, $\mathcal{B} = \mathcal{B}^* \cup \mathcal{B}'$ is linearly independent. Since

$$\begin{aligned} |\mathcal{B}| &= |\mathcal{B}^*| + \left|\mathcal{B}'\right| \\ &= \sum_{i=1}^{|V(T)|-2} \left|\mathcal{B}_{P_3^{(i)}}\right| + \sum_{e \in E(T)-P_2} \left|\mathcal{B}^{(e)}\right| + |\mathcal{B}_{P_2}| \\ &= \sum_{i=1}^{|V(T)|-2} n + \sum_{e \in E(T)-P_2} n + (2n+1) \\ &= \dim \mathcal{C}\left(T \bullet C_n\right), \end{aligned}$$

$\mathcal{B}$ is a basis for $\mathcal{C}\left(T \bullet C_n\right)$. To conclude the proof of this Theorem, we show that $\mathcal{B}$ satisfied the fold stated in the theorem. Let $e \in E(T \bullet C_n)$. (1) If $e \in E((T - P_2) \wedge C_n)$, then $f_{\mathcal{B}^*}(e) \leq 3$ (see [5]) and $f_{\mathcal{B}'}(e) \leq 1$. Moreover, $f_{\mathcal{B}^*}(e) \leq 2$ (see [5]) and $f_{\mathcal{B}'}(e) \leq 1$ if T contains no subgraph isomorphic to a 3-special star of order 7, (2) if

$e \in E(P_2 \wedge C_n)$, then $f_{\mathcal{B}^*}(e) \leq 1$ and $f_{\mathcal{B}'}(e) \leq 2$ (3) if $e \in E(P_2 \bullet C_n) - E(P_2 \wedge C_n)$, then $f_{\mathcal{B}^*}(e) = 0$ and $f_{\mathcal{B}'}(e) \leq 3$. □

Theorem 2.6. *Let G be a bipartite graph and C_n be a cycle. Then $b(G \bullet C_n) \leq 4 + b(G)$. Moreover, $b(G \bullet C_n) \leq 3 + b(G)$ if G has a spanning tree contains no subgraph isomorphic to a 3-special star of order 7.*

Proof. Let T_G be a spanning tree of G. Let $\mathcal{B}_T$ be the basis of $\mathcal{C}(T_G \bullet C_n)$ as in Lemma 2.5. Let $\mathcal{B}_{v_iv_{i+1}} = \mathcal{B}^{(1)}_{v_iv_{i+1}} \cup \mathcal{B}^{(2)}_{v_iv_{i+1}}$ and $\mathcal{B}_{v_1v_n} = \mathcal{B}^{(1)}_{v_1v_n} \cup \mathcal{B}^{(2)}_{v_1v_n}$ where $\mathcal{B}^{(1)}_{v_iv_{i+1}}$ and $\mathcal{B}^{(2)}_{v_iv_{i+1}}$, and $\mathcal{B}^{(1)}_{v_1v_n}$ and $\mathcal{B}^{(2)}_{v_1v_n}$ are the corresponding basis of the required basis of the two copies of $G \wedge v_iv_{i+1}$ and $G \wedge v_1v_n$, respectively. It is an easy matter to see that $E(\mathcal{B}^{(1)}_{v_iv_{i+1}}) \cap E(\mathcal{B}^{(2)}_{v_iv_{i+1}}) = \phi$ and $E(\mathcal{B}^{(1)}_{v_1v_n}) \cap E(\mathcal{B}^{(2)}_{v_1v_n}) = \phi$. Moreover, $E(\mathcal{B}_{v_iv_{i+1}}) \cap E(\mathcal{B}_{v_jv_{j+1}}) = \phi$ and $E(\mathcal{B}_{v_1v_n}) \cap E(\mathcal{B}_{v_jv_{j+1}}) = \phi$ if $i \neq j$. Hence, $\mathcal{B}_G = \left(\bigcup_{i=1}^{n-1} \mathcal{B}_{v_iv_{i+1}}\right) \cup \mathcal{B}_{v_1v_n}$ is linearly independent set. Note that each cycle of $\mathcal{B}_G$ contains at least one edge of $E((G - T_G) \wedge C_n)$ which is not in any cycle of $\mathcal{B}_T$. Therefore $\mathcal{B} = \mathcal{B}_G \cup \mathcal{B}_T$ is linearly independent set. Now

$$\begin{aligned} |\mathcal{B}| &= |\mathcal{B}_G| + |\mathcal{B}_T| \\ &= 2n \dim\ \mathcal{C}(G) + 2\,|E(T_G)|\,|E(C)| + 1 \\ &= \dim \mathcal{C}(G \bullet C_n), \end{aligned}$$

Therefore, $\mathcal{B}$ is a basis. To this end, if $e \in E(G \bullet C_n)$ then $f_{\mathcal{B}_G}(e) \leq b(G)$ and $f_{\mathcal{B}_T}(e) \leq 4$. Moreover, $f_{\mathcal{B}_G}(e) \leq b(G)$ and $f_{\mathcal{B}_T}(e) \leq 3$ if G contains no subgraph isomorphic to a 3-special star of order 7. □

Acknowledgement. The author would like to thank the referee for his valuable comments.

References

[1] A. A. Ali, *The basis number of complete multipartite graphs*, Ars Combin., **28**(1989), 41-49.

[2] A. A. Ali and G. T. Marougi, *The basis number of cartesian product of some graphs*, J. Indian Math. Soc. (NS), **58(1-4)**(1992), 123-134.

[3] J. A. Bondy and U. S. Murty, Graph theory with applications, American Elsevier Publishing Co. Inc., New York, 1976.

[4] F. Harary, Graph theory, Addison-Wesley Publishing Co., Reading, Massachusetts, 1971.

[5] M. M. Jaradat, *On the basis number of the direct product of graphs*, Australasian Journal of Combinatorics, **27**(2003), 293-306.

[6] S. MacLane, *A combinatorial condition for planar graphs*, Fundamenta Math., **28**(1937), 22-32.

[7] E. F. Schmeichel, *The basis number of a graph*, J. Combin. Theory Ser. B, **30(2)**(1981), 123-129.