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On (Φ, Ψ)-intuitionistic Fuzzy Subgroups

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Abstract. Using the *belongs to* relation ($\in$) and *quasi-coincidence with* relation (q) between intuitionistic fuzzy points and intuitionistic fuzzy sets, the concept of (Φ, Ψ)-intuitionistic fuzzy subgroup where Φ, Ψ are any two of $\{\in, \mathrm{q}, \in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q}\}$ with $\Phi \neq \in\wedge\,\mathrm{q}$ is introduced, and related properties are investigated.

1. Introduction

After the introduction of the concept of fuzzy sets by L. A. Zadeh ([7]), several researchers were conducted on the generalizations of the notion of fuzzy sets. The idea of *intuitionistic fuzzy set* was first published by K. T. Atanassov ([1], [2]) as a generalization of the notion of fuzzy sets. A. Rosenfeld ([6]) studied fuzzy subgroups of a group. K. Hur et al. ([5]) introduced the notion of intuitionistic fuzzy subgroup of a group by using the notion of intuitionistic fuzzy sets.

In this paper, we introduce the concept of (Φ, Ψ)-intuitionistic fuzzy subgroup, where Φ and Ψ are any two of $\{\in, \mathrm{q}, \in\vee\,\mathrm{q}, \in\wedge\,\mathrm{q}\}$ with $\Phi \neq \in\wedge\,\mathrm{q}$, by using the *belongs to* relation ($\in$) and *quasi-coincidence with* relation (q) between intuitionistic fuzzy points and intuitionistic fuzzy sets, and investigate related properties.

2. Preliminaries

Let X be a nonempty set. An *intuitionistic fuzzy set* (IFS for short) A is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$ (see [1], [2]). For the sake of simplicity, we shall use the symbol

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$A = \langle x, \mu_A, \gamma_A \rangle$ for the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$. Let c be a point in a nonempty set X. If $\alpha \in (0,1]$ and $\beta \in [0,1)$ are two real numbers such that $\alpha + \beta \leq 1$, then the IFS

$$c(\alpha, \beta) = \langle x, c_\alpha, 1 - c_{1-\beta} \rangle$$

is called an *intuitionistic fuzzy point* (IFP for short) in X (see [4]) where α (resp., β) is the degree of membership (resp., nonmembership) of $c(\alpha, \beta)$ and $c \in X$ is the support of $c(\alpha, \beta)$. Let $c(\alpha, \beta)$ be an IFP in X and let $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFS in X. Then $c(\alpha, \beta)$ is said to *belong* to A, written $c(\alpha, \beta) \in A$, if $\mu_A(c) \geq \alpha$ and $\gamma_A(c) \leq \beta$. We say that $c(\alpha, \beta)$ is *quasi-coincident with* A, written $c(\alpha, \beta) \, \mathrm{q} \, A$, if $\mu_A(c) + \alpha > 1$ and $\gamma_A(c) + \beta < 1$. To say that $c(\alpha, \beta) \in\!\vee\, \mathrm{q} \, A$ (resp., $c(\alpha, \beta) \in\!\wedge\, \mathrm{q} \, A$) means that $c(\alpha, \beta) \in A$ or $c(\alpha, \beta) \, \mathrm{q} \, A$ (resp., $c(\alpha, \beta) \in A$ and $c(\alpha, \beta) \, \mathrm{q} \, A$).

3. (Φ, Ψ)-intuitionistic fuzzy subgroups

In what follows let G denote a group, and Φ and Ψ denote any one of $\in$, q, $\in\!\vee\,\mathrm{q}$, or $\in\!\wedge\,\mathrm{q}$ unless otherwise specified. To say that $x(\alpha, \beta) \, \overline{\Phi} \, A$ means that $x(\alpha, \beta) \, \Phi \, A$ does not hold. For all $\alpha_1, \alpha_2 \in [0,1]$, $\min\{\alpha_1, \alpha_2\}$ (resp., $\max\{\alpha_1, \alpha_2\}$) will be denoted by $m(\alpha_1, \alpha_2)$ (resp., $M(\alpha_1, \alpha_2)$).

Definition 3.1 ([5]). An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in G is called an *intuitionistic fuzzy subgroup* of G if it satisfies

(i) $\mu_A(xy) \geq m(\mu_A(x), \mu_A(y))$ and $\gamma_A(xy) \leq M(\gamma_A(x), \gamma_A(y))$

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\gamma_A(x^{-1}) \leq \gamma_A(x)$

for all $x, y \in G$.

Theorem 3.2. *For any IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in G, the following are equivalent for every $x, y \in G$,*

(i) $\mu_A(xy) \geq m(\mu_A(x), \mu_A(y))$ *and* $\gamma_A(xy) \leq M(\gamma_A(x), \gamma_A(y))$

(ii) *for every $\alpha_1, \alpha_2 \in (0,1]$ and $\beta_1, \beta_2 \in [0,1)$, if $x(\alpha_1, \beta_1) \in A$ and $y(\alpha_2, \beta_2) \in A$ then $(xy)(m(\alpha_1, \alpha_2), M(\beta_1, \beta_2)) \in A$.*

Proof. (i) $\Rightarrow$ (ii). Assume that $x(\alpha_1, \beta_1) \in A$ and $y(\alpha_2, \beta_2) \in A$, where $\alpha_1, \alpha_2 \in (0,1]$ and $\beta_1, \beta_2 \in [0,1)$. Then $\mu_A(x) \geq \alpha_1$, $\gamma_A(x) \leq \beta_1$, $\mu_A(y) \geq \alpha_2$, and $\gamma_A(y) \leq \beta_2$. It follows from (i) that

$$\mu_A(xy) \geq m(\mu_A(x), \mu_A(y)) \geq m(\alpha_1, \alpha_2)$$

and

$$\gamma_A(xy) \leq M(\gamma_A(x), \gamma_A(y)) \leq M(\beta_1, \beta_2)$$

so that $(xy)(m(\alpha_1, \alpha_2), M(\beta_1, \beta_2)) \in A$.

(ii) $\Rightarrow$ (i). Suppose that (ii) is valid. Note that $x(\mu_A(x), \gamma_A(x)) \in A$ and $y(\mu_A(y), \gamma_A(y)) \in A$. Thus, by (ii), we have

$$(xy)(m(\mu_A(x), \mu_A(y)), M(\gamma_A(x), \gamma_A(y))) \in A,$$

and so $\mu_A(xy) \geq m(\mu_A(x), \mu_A(y))$ and $\gamma_A(xy) \leq M(\gamma_A(x), \gamma_A(y))$. This completes the proof. □

It follows from Theorem 3.2 that the condition (i) of Definition 3.1 may be replaced by the condition (ii) of Theorem 3.2. If we take notice of the idea of quasi-coincidence of an IFP with an IFS, it is natural to inquire what happens if the $\in$'s in the left and right hand side of the condition (ii) of Theorem 3.2 are replaced by q, $\in \vee$ q or $\in \wedge$ q. With this in mind, we introduce the concept of a (Φ, Ψ)-intuitionistic fuzzy subgroup.

Definition 3.3. An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in G is said to be a (Φ, Ψ)-*intuitionistic fuzzy subgroup* of G, where $\Phi \neq \in \wedge \mathrm{q}$, if for every $x, y \in G$, $\alpha_1, \alpha_2 \in (0, 1]$, and $\beta_1, \beta_2 \in [0, 1)$, the following holds:

(i) $x(\alpha_1, \beta_1)\Phi A$ and $y(\alpha_2, \beta_2)\Phi A$ imply $(xy)(m(\alpha_1, \alpha_2), M(\beta_1, \beta_2))\Psi A$,

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\gamma_A(x^{-1}) \leq \gamma_A(x)$.

Combining Theorem 3.2 and Definitions 3.1 and 3.3, we know that every $(\in, \in)$-intuitionistic fuzzy subgroup is an intuitionistic fuzzy subgroup. An IFS $A = \langle x, \mu_A, \gamma_A \rangle$ in G is said to be *nonzero-nonunit* if there exists $x \in G$ such that $\mu_A(x) \neq 0$ and $\gamma_A(x) \neq 1$.

Theorem 3.4. *Let $A = \langle x, \mu_A, \gamma_A \rangle$ be a nonzero-nonunit (Φ, Ψ)-intuitionistic fuzzy subgroup of G. Then*

(i) $\mu_A(e) > 0$ *and* $\gamma_A(e) < 1$.

(ii) $[\mu_A > 0] := \{x \in G \mid \mu_A(x) > 0\}$ *and* $[\gamma_A < 1] := \{x \in G \mid \gamma_A(x) < 1\}$ *are subgroups of G.*

Proof. (i) Since $A = \langle x, \mu_A, \gamma_A \rangle$ is nonzero-nonunit, there exists $x \in G$ such that $\mu_A(x) = \alpha_1 > 0$ and $\gamma_A(x) = \beta_1 < 1$. We know that

- $x(\alpha_1, \beta_1) \in A$ and $x^{-1}(\alpha_1, \beta_1) \in A$,
- $x(\alpha_1, \beta_1) \in \vee\, \mathrm{q}\, A$ and $x^{-1}(\alpha_1, \beta_1) \in \vee\, \mathrm{q}\, A$,
- $x(1, 0)\mathrm{q}A$ and $x^{-1}(1, 0)\mathrm{q}A$.

We consider the first and second cases. If $\mu_A(e) = 0$, then $\mu_A(e) = 0 \ngeq \alpha_1$ and $\mu_A(e) + \alpha_1 = \alpha_1 \not> 1$. If $\gamma_A(e) = 1$, then $\gamma_A(e) = 1 \nleq \beta_1$ and $\gamma_A(e) + \beta_1 = 1 + \beta_1 \not< 1$. Hence we have

$$(xx^{-1})(m(\alpha_1, \alpha_1), M(\beta_1, \beta_1)) = e(\alpha_1, \beta_1)\overline{\Psi}A,$$

a contradiction. For the last case, we get

$$(xx^{-1})(m(1,1), M(0,0)) = e(1,0)\overline{\Psi}A$$

whenever $\mu_A(e) = 0$ or $\gamma_A(e) = 1$. This is a contradiction. Therefore $\mu_A(e) > 0$ and $\gamma_A(e) < 1$.

(ii) Let $x, y \in [\mu_A > 0]$ and $u, v \in [\gamma_A < 1]$. Then $\mu_A(x) > 0$, $\mu_A(y) > 0$, $\gamma_A(u) < 1$, and $\gamma_A(v) < 1$. If $\Phi = \in$ or $\Phi = \in \vee\, \mathrm{q}$, then $x(\mu_A(x), \gamma_A(x))\Phi A$, $y(\mu_A(y), \gamma_A(y))\Phi A$, $u(\mu_A(u), \gamma_A(u))\Phi A$, and $v(\mu_A(v), \gamma_A(v))\Phi A$. But

$$(xy)(m(\mu_A(x), \mu_A(y)), M(\gamma_A(x), \gamma_A(y)))\overline{\Psi}A$$

whenever $\mu_A(xy) = 0$; and

$$(uv)(m(\mu_A(u), \mu_A(v)), M(\gamma_A(u), \gamma_A(v)))\overline{\Psi}A$$

whenever $\gamma_A(uv) = 1$. These are contradictions. Now if $\Phi = \mathrm{q}$, then $x(1,0)\Phi A$, $y(1,0)\Phi A$, $u(1,0)\Phi A$, and $v(1,0)\Phi A$. But

$$(xy)(m(1,1), M(0,0)) = (xy)(1,0)\overline{\Psi}A$$

whenever $\mu_A(xy) = 0$, a contradiction; and

$$(uv)(m(1,1), M(0,0)) = (uv)(1,0)\overline{\Psi}A$$

whenever $\gamma_A(uv) = 1$, a contradiction. Hence $\mu_A(xy) > 0$ and $\gamma_A(uv) < 1$, that is, $xy \in [\mu_A > 0]$ and $uv \in [\gamma_A < 1]$. Finally if $x \in [\mu_A > 0]$ and $y \in [\gamma_A < 1]$, then $\mu_A(x^{-1}) \geq \mu_A(x) > 0$ and $\gamma_A(y^{-1}) \leq \gamma_A(y) < 1$. Thus $x^{-1} \in [\mu_A > 0]$ and $y^{-1} \in [\gamma_A < 1]$. Consequently, $[\mu_A > 0]$ and $[\gamma_A < 1]$ are subgroups of G. □

Theorem 3.5. *If an IFS $A = \langle x, \mu_A, \gamma_A \rangle$ is a nonzero-nonunit* (q, q)*-intuitionistic fuzzy subgroup of G, then μ_A* (resp., γ_A) *is constant on* $[\mu_A > 0]$ (resp., $[\gamma_A < 1]$).

Proof. Assume that there exist $b \in [\mu_A > 0]$ and $c \in [\gamma_A < 1]$ such that $\alpha_1 = \mu_A(b) \neq \mu_A(e) = \alpha$ and $\beta_1 = \gamma_A(c) \neq \gamma_A(e) = \beta$. Then either $\alpha_1 < \alpha$ or $\alpha_1 > \alpha$, and either $\beta_1 < \beta$ or $\beta_1 > \beta$. Let $\alpha_1 < \alpha$ and choose $\alpha_2, \alpha_3 \in (0, 1]$ such that $1 - \alpha < \alpha_2 < 1 - \alpha_1 < \alpha_3$. For any $\eta_1 < 1 - \beta$ and $\eta_2 < 1 - \gamma_A(b)$, we get $e(\alpha_2, \eta_1)\, \mathrm{q}\, A$ and $b(\alpha_3, \eta_2)\, \mathrm{q}\, A$ but

$$(eb)(m(\alpha_2, \alpha_3), M(\eta_1, \eta_2)) = b(\alpha_2, M(\eta_1, \eta_2))\, \overline{\mathrm{q}}\, A.$$

This is a contradiction. Let $\alpha_1 > \alpha$. Then for every $\eta < 1 - \gamma_A(b)$, we obtain $b(1 - \alpha, \eta)\, \mathrm{q}\, A$ and $b^{-1}(1 - \alpha, \eta)\, \mathrm{q}\, A$ but

$$(bb^{-1})(m(1 - \alpha, 1 - \alpha), M(\eta, \eta)) = e(1 - \alpha, \eta)\, \overline{\mathrm{q}}\, A,$$

a contradiction. Let $\beta_1 > \beta$ and take $\beta_2, \beta_3 \in [0, 1)$ such that $\beta_2 < 1 - \beta_1 < \beta_3 < 1 - \beta$. Then $e(\delta_1, \beta_3)\, \mathrm{q}\, A$ and $c(\delta_2, \beta_2)\, \mathrm{q}\, A$ for every $\delta_1 > 1 - \alpha$ and $\delta_2 > 1 - \mu_A(c)$, but

$$(ec)(m(\delta_1, \delta_2), M(\beta_2, \beta_3)) = c(m(\delta_1, \delta_2), \beta_3)\, \overline{\mathrm{q}}\, A.$$

This is a contradiction. Finally let $\beta_1 < \beta$. Then for every $\delta > 1-\mu_A(c)$, we get $c(\delta, 1-\beta)\,\mathrm{q}\,A$ and $c^{-1}(\delta, 1-\beta)\,\mathrm{q}\,A$, but

$$(cc^{-1})(m(\delta,\delta), M(1-\beta, 1-\beta)) = e(\delta, 1-\beta)\,\overline{\mathrm{q}}\,A,$$

a contradiction. Therefore $\mu_A(b) = \mu_A(e)$ and $\gamma_A(c) = \gamma_A(e)$ for all $b \in [\mu_A > 0]$ and $c \in [\gamma_A < 1]$. Consequently, μ_A and γ_A are constant on $[\mu_A > 0]$ and $[\gamma_A < 1]$ respectively. □

We provide conditions for an IFS to be a $(\mathrm{q}, \in\vee\,\mathrm{q})$-intuitionistic fuzzy subgroup.

Theorem 3.6. *Let H be a subgroup of G and let $A = \langle x, \mu_A, \gamma_A\rangle$ be an IFS in G such that*

(i) $\mu_A(x) = 0$ *and* $\gamma_A(x) = 1$ *for all* $x \in G \setminus H$,

(ii) $\mu_A(x^{-1}) \geq \mu_A(x) \geq 0.5$ *for all* $x \in H$,

(iii) $\gamma_A(x^{-1}) \leq \gamma_A(x) \leq 0.5$ *for all* $x \in H$.

Then $A = \langle x, \mu_A, \gamma_A\rangle$ is a $(\mathrm{q}, \in\vee\,\mathrm{q})$-intuitionistic fuzzy subgroup of G.

Proof. Let $x, y \in G$, $\alpha_1, \alpha_2 \in (0,1]$ and $\beta_1, \beta_2 \in [0,1)$ be such that

$$x(\alpha_1, \beta_1)\,\mathrm{q}\,A \ \text{ and } \ y(\alpha_2, \beta_2)\,\mathrm{q}\,A.$$

Then $x, y \in H$, and so $xy \in H$. If $m(\alpha_1,\alpha_2) > 0.5$ and $M(\beta_1,\beta_2) < 0.5$, then $(xy)(m(\alpha_1,\alpha_2), M(\beta_1,\beta_2))\,\mathrm{q}\,A$. If $m(\alpha_1,\alpha_2) \leq 0.5$ and $M(\beta_1,\beta_2) \geq 0.5$, then $(xy)(m(\alpha_1,\alpha_2), M(\beta_1,\beta_2)) \in A$. Since $\alpha_1+\beta_1 \leq 1$ and $\alpha_2+\beta_2 \leq 1$, the case $\begin{cases} m(\alpha_1,\alpha_2) > 0.5, \\ M(\beta_1,\beta_2) \geq 0.5 \end{cases}$ does not occur. From the fact that $x, y \in H$, $x(\alpha_1,\beta_1)\,\mathrm{q}\,A$ and $y(\alpha_2,\beta_2)\,\mathrm{q}\,A$, it follows that the case $\begin{cases} m(\alpha_1,\alpha_2) \leq 0.5, \\ M(\beta_1,\beta_2) < 0.5 \end{cases}$ does not occur. Thus in any case we get $(xy)(m(\alpha_1,\alpha_2), M(\beta_1,\beta_2)) \in\vee\,\mathrm{q}\,A$. Hence $A = \langle x, \mu_A, \gamma_A\rangle$ is a $(\mathrm{q}, \in\vee\,\mathrm{q})$-intuitionistic fuzzy subgroup of G. □

Theorem 3.7. *Let $A = \langle x, \mu_A, \gamma_A\rangle$ be a $(\mathrm{q}, \in\vee\,\mathrm{q})$-intuitionistic fuzzy subgroup of G such that μ_A and γ_A are not constant on $[\mu_A > 0]$ and $[\gamma_A < 1]$ respectively. Then*

(i) *there exist $x, y \in G$ such that $\mu_A(x) \geq 0.5$ and $\gamma_A(y) \leq 0.5$.*

(ii) $\mu_A(e) \geq 0.5 \geq \gamma_A(e)$.

(iii) $\mu_A(x) \geq 0.5$ *and* $\gamma_A(y) \leq 0.5$ *for all* $x \in [\mu_A > 0]$ *and* $y \in [\gamma_A < 1]$ *respectively.*

Proof. Since μ_A and γ_A are not constant on $[\mu_A > 0]$ and $[\gamma_A < 1]$ respectively, there exist $x \in [\mu_A > 0]$ and $y \in [\gamma_A < 1]$ such that $\alpha_1 = \mu_A(x) \neq \mu_A(e) = \alpha$ and

$\beta_1 = \gamma_A(y) \neq \gamma_A(e) = \beta$. Then either $\alpha_1 < \alpha$ or $\alpha_1 > \alpha$, and either $\beta_1 < \beta$ or $\beta_1 > \beta$. Assume that $\mu_A(x) < 0.5$ for all $x \in G$. If $\alpha_1 > \alpha$, choose $\delta > 0.5$ such that $\alpha + \delta < 1 < \alpha_1 + \delta$. Then $x(\delta, \eta) \, \mathrm{q} \, A$ and $x^{-1}(\delta, \eta) \, \mathrm{q} \, A$ for every $\eta < 1 - \gamma_A(x)$, but

$$(xx^{-1})(m(\delta, \delta), M(\eta, \eta)) = e(\delta, \eta) \, \overline{\in \vee \, \mathrm{q}} \, A,$$

a contradiction. If $\alpha_1 < \alpha$, we can choose $\delta > 0.5$ such that $\alpha_1 + \delta < 1 < \alpha + \delta$. Then $e(\delta, \eta) \, \mathrm{q} \, A$ and $x(1, 0) \, \mathrm{q} \, A$ for every $\eta < 1 - \beta$, but

$$(ex)(m(\delta, 1), M(\eta, 0)) = x(\delta, \eta) \, \overline{\in \vee \, \mathrm{q}} \, A,$$

a contradiction. Now suppose that $\gamma_A(y) > 0.5$ for all $y \in G$. If $\beta_1 < \beta$, take $\delta < 0.5$ such that $\beta_1 + \delta < 1 < \beta + \delta$. Then $y(\eta, \delta) \, \mathrm{q} \, A$ and $y^{-1}(\eta, \delta) \, \mathrm{q} \, A$ for every $\eta > 1 - \mu_A(y)$, but

$$(yy^{-1})(m(\eta, \eta), M(\delta, \delta)) = e(\eta, \delta) \, \overline{\in \vee \, \mathrm{q}} \, A.$$

This is a contradiction. If $\beta_1 > \beta$, we can choose $\delta < 0.5$ such that $\beta + \delta < 1 < \beta_1 + \delta$. Then $e(\eta, \delta) \, \mathrm{q} \, A$ and $y(1, 0) \, \mathrm{q} \, A$ for every $\eta > 1 - \alpha$, but

$$(ey)(m(\eta, 1), M(\delta, 0)) = y(\eta, \delta) \, \overline{\in \vee \, \mathrm{q}} \, A,$$

a contradiction. Therefore $\mu_A(x) \geq 0.5$ and $\gamma_A(y) \leq 0.5$ for some $x, y \in G$. To show that (ii) is valid, we first let $\alpha = \mu_A(e) < 0.5$. By (i), there exists $x \in G$ such that $\alpha_x = \mu_A(x) \geq 0.5$. Thus $\alpha < \alpha_x$. Choose $\alpha_1 > \alpha$ such that $\alpha + \alpha_1 < 1 < \alpha_x + \alpha_1$. Then $x(\alpha_1, \delta) \, \mathrm{q} \, A$ and $x^{-1}(\alpha_1, \delta) \, \mathrm{q} \, A$ for every $\delta < 1 - \gamma_A(x)$, but

$$(xx^{-1})(m(\alpha_1, \alpha_1), \, M(\delta, \delta)) = e(\alpha_1, \delta) \, \overline{\in \vee \, \mathrm{q}} \, A.$$

This is a contradiction. We now let $\beta = \gamma_A(e) > 0.5$. The condition (i) implies that $\beta_y = \gamma_A(y) \leq 0.5$ for some $y \in G$. Hence $\beta_y < \beta$, and so we can take $\beta_1 < \beta$ such that $\beta_y + \beta_1 < 1 < \beta + \beta_1$. Then $y(\eta, \beta_1) \, \mathrm{q} \, A$ and $y^{-1}(\eta, \beta_1) \, \mathrm{q} \, A$ for every $\eta > 1 - \mu_A(y)$, but

$$(yy^{-1})(m(\eta, \eta), \, M(\beta_1, \beta_1)) = e(\eta, \beta_1) \, \overline{\in \vee \, \mathrm{q}} \, A,$$

which is impossible. Therefore $\gamma_A(e) \leq 0.5 \leq \mu_A(e)$. To verify that (iii) holds, let $\alpha_x = \mu_A(x) < 0.5$ for some $x \in [\mu_A > 0]$. Take $\alpha_1 > 0$ such that $\alpha_x + \alpha_1 < 0.5$. Then $x(1, 0) \, \mathrm{q} \, A$ and $e(0.5 + \alpha_1, 0) \, \mathrm{q} \, A$. But

$$(ex)(m(1, 0.5 + \alpha_1), \, M(0, 0)) = x(0.5 + \alpha_1, 0) \, \overline{\in \vee \, \mathrm{q}} \, A.$$

This is a contradiction. Finally assume that $\beta_y = \gamma_A(y) > 0.5$ for some $y \in [\gamma_A < 1]$. Let us take $0 < \delta < 0.5$ such that $0.5 + \delta < 1 < \beta_y + \delta$. Then $e(\eta, \delta) \, \mathrm{q} \, A$ for every $\eta > 1 - \mu_A(e)$, and $y(\zeta, 0) \, \mathrm{q} \, A$ for every $\zeta > 1 - \mu_A(y)$. But

$$(ey)(m(\eta, \zeta), \, M(\delta, 0)) = y(m(\eta, \zeta), \delta) \, \overline{\in \vee \, \mathrm{q}} \, A,$$

which is a contradiction. Consequently, $\mu_A(x) \geq 0.5$ and $\gamma_A(y) \leq 0.5$ for all $x \in [\mu_A > 0]$ and $y \in [\gamma_A < 1]$ respectively. This completes the proof. □

Example 3.8. Consider the Klein's 4-group $G = \{e, a, b, c\}$ with the multiplication table:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $A = \langle x, \mu_A, \gamma_A \rangle$ be an IFS in G defined by $\mu_A(e) = 0.6$, $\mu_A(a) = 0.7$, $\mu_A(b) = \mu_A(c) = 0.4$, $\gamma_A(e) = 0.33$, $\gamma_A(a) = 0.22$, and $\gamma_A(b) = \gamma_A(c) = 0.25$. Then $A = \langle x, \mu_A, \gamma_A \rangle$ is an $(\in, \in \vee \mathrm{q})$-intuitionistic fuzzy subgroup of G. But (1) $A = \langle x, \mu_A, \gamma_A \rangle$ is not an $(\in, \in)$-intuitionistic fuzzy subgroup of G since $a(0.62, 0.32) \in A$ and $a(0.66, 0.31) \in A$, but

$$(aa)(m(0.62, 0.66),\ M(0.32, 0.31)) = e(0.62, 0.32) \overline{\in} A.$$

(2) $A = \langle x, \mu_A, \gamma_A \rangle$ is not a $(\mathrm{q}, \in \vee \mathrm{q})$-intuitionistic fuzzy subgroup of G since $a(0.41, 0.57)\, \mathrm{q}\, A$ and $b(0.67, 0.32)\, \mathrm{q}\, A$, but

$$(ab)(m(0.41, 0.67),\ M(0.57, 0.32)) = c(0.41, 0.57)\, \overline{\in \vee \mathrm{q}}\, A.$$

(3) $A = \langle x, \mu_A, \gamma_A \rangle$ is not an $(\in \vee \mathrm{q}, \in \vee \mathrm{q})$-intuitionistic fuzzy subgroup of G since $a(0.5, 0.3) \in \vee \mathrm{q}\, A$ and $c(0.63, 0.33) \in \vee \mathrm{q}\, A$, but

$$(ac)(m(0.5, 0.63),\ M(0.3, 0.33)) = b(0.5, 0.33)\, \overline{\in \vee \mathrm{q}}\, A.$$

Lemma 3.9. (1) *Every* $(\in \vee \mathrm{q}, \in \vee \mathrm{q})$*-intuitionistic fuzzy subgroup is an* $(\in, \in \vee \mathrm{q})$*-intuitionistic fuzzy subgroup.*

(2) *Every* $(\in, \in)$*-intuitionistic fuzzy subgroup is an* $(\in, \in \vee \mathrm{q})$*-intuitionistic fuzzy subgroup.*

Proof. Straightforward. □

Example 3.8 shows that the converse of Lemma 3.9 is not true in general.

For any subset H of G, the *intuitionistic fuzzy characteristic function* of H is defined to be the IFS $\chi_H = \langle x, \mu_{\chi_H}, \nu_{\chi_H} \rangle$ in G given by

$$\mu_{\chi_H}(x) := \begin{cases} 1 & \text{if } x \in H, \\ 0 & \text{if } x \notin H, \end{cases} \qquad \nu_{\chi_H}(x) := \begin{cases} 0 & \text{if } x \in H, \\ 1 & \text{if } x \notin H. \end{cases}$$

Theorem 3.10. *For any subset* H *of* G*, the intuitionistic fuzzy characteristic function* $\chi_H = \langle x, \mu_{\chi_H}, \nu_{\chi_H} \rangle$ *of* H *is an* $(\in, \in \vee \mathrm{q})$*-intuitionistic fuzzy subgroup of* G *if* *and only if H is a subgroup of G.*

Proof. Assume that $\chi_H = \langle x, \mu_{\chi_H}, \nu_{\chi_H} \rangle$ is an $(\in, \in\vee\mathrm{q})$-intuitionistic fuzzy subgroup of G. Let $x, y \in H$. Then $\mu_{\chi_H}(x) = 1 = \mu_{\chi_H}(y)$ and $\nu_{\chi_H}(x) = 0 = \nu_{\chi_H}(y)$, and so $x(1,0) \in \chi_H$ and $y(1,0) \in \chi_H$. Since χ_H is an $(\in, \in\vee\mathrm{q})$-intuitionistic fuzzy subgroup of G, it follows that

$$(xy)(m(1,1),\, M(0,0)) = (xy)(1,0) \in\vee\mathrm{q}\, \chi_H$$

so that $\mu_{\chi_H}(xy) = 1$ and $\nu_{\chi_H}(xy) = 0$. Thus $xy \in H$. Now if $x \in H$, then

$$\mu_{\chi_H}(x^{-1}) \geq \mu_{\chi_H}(x) = 1 \text{ and } \nu_{\chi_H}(x^{-1}) \leq \nu_{\chi_H}(x) = 0.$$

Hence $\mu_{\chi_H}(x^{-1}) = 1$ and $\nu_{\chi_H}(x^{-1}) = 0$, and so $x^{-1} \in H$. Therefore H is a subgroup of G.

Conversely suppose that H is a subgroup of G. Let $x, y \in G$, $\alpha_1, \alpha_2 \in (0,1]$, and $\beta_1, \beta_2 \in [0,1)$ be such that $x(\alpha_1, \beta_1) \in \chi_H$ and $y(\alpha_2, \beta_2) \in \chi_H$. Then $\mu_{\chi_H}(x) \geq \alpha_1 > 0$, $\nu_{\chi_H}(x) \leq \beta_1 < 1$, $\mu_{\chi_H}(y) \geq \alpha_2 > 0$, and $\nu_{\chi_H}(y) \leq \beta_2 < 1$. It follows that $\mu_{\chi_H}(x) = 1 = \mu_{\chi_H}(y)$ and $\nu_{\chi_H}(x) = 0 = \nu_{\chi_H}(y)$ so that $x, y \in H$. Since H is a subgroup of G, we get $xy \in H$, which yields $\mu_{\chi_H}(xy) = 1 \geq m(\alpha_1, \alpha_2)$ and $\nu_{\chi_H}(xy) = 0 \leq M(\beta_1, \beta_2)$. This shows that

$$(xy)(m(\alpha_1, \alpha_2),\, M(\beta_1, \beta_2)) \in \chi_H.$$

Obviously, we have $\mu_{\chi_H}(x^{-1}) \geq \mu_{\chi_H}(x)$ and $\nu_{\chi_H}(x^{-1}) \leq \nu_{\chi_H}(x)$. Hence χ_H is an $(\in, \in\vee\mathrm{q})$-intuitionistic fuzzy subgroup of G, and consequently it is an $(\in, \in\vee\mathrm{q})$-intuitionistic fuzzy subgroup of G by Lemma 3.9(2). □

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