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Study of Isotropic Immersions

Dedicated to Prof. D. E. Blair on his retirement from Michigan State University

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Abstract. In this expository paper we survey basic results on isotropic immersions.

1. Introduction

An n-dimensional real space form $M^n(c;\mathbb{R})$ is a Riemannian manifold of constant sectional curvature c, which is locally congruent to either a standard sphere $S^n(c)$, a Euclidean space $\mathbb{R}^n$ or a real hyperbolic space $\mathbb{R}H^n(c)$, according as c is positive, zero or negative.

We recall the notion of isotropic immersions introduced by O'Neill [18]. An isometric immersion $f : M \to \widetilde{M}$ of a Riemannian manifold M into an ambient Riemannian manifold $\widetilde{M}$ (with metric $\langle\ ,\ \rangle$) is said to be *isotropic* at $x \in M$ if $\|\sigma(X,X)\|/\|X\|^2 (= \lambda(x))$ does not depend on the choice of $X (\neq 0) \in T_xM$. If the immersion is isotropic at every point, then the immersion is said to be *isotropic*. When the function $\lambda = \lambda(x)$ is constant on M, M is called a constant (λ-)isotropic submanifold. We note that a totally umbilic immersion is isotropic, but not *vice versa*. The class of isotropic submanifolds gives *nice* examples in submanifold theory. For example, take a G-equivariant isometric immersion f of a rank one symmetric space $M(= G/K)$ into an arbitrary Riemannian homogeneous space $\widetilde{M}(= \widetilde{G}/\widetilde{K})$. Then this submanifold (M, f) is a (constant) isotropic submanifold. We here emphasize the fact that for each geodesic $\gamma = \gamma(s)$ on the submanifold (M, f) the curve

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$f \circ \gamma$ is a one-parameter subgroup of the isometry group of the ambient space $\widetilde{M}$. This fact tells us that study of isotropic immersions is one of the most interesting objects in submanifold theory.

The purpose of this paper is to survey fundamental results related to the following four problems:

Problem 1. Find a necessary and sufficient condition for an isotropic immersion of M to be totally umbilic in an arbitrary Riemannian manifold $\widetilde{M}$.

Problem 2. Give a geometric meaning of a constant isotropic immersion of M into an arbitrary Riemannian manifold $\widetilde{M}$.

Proclaim 3. Classify isotropic immersions of M^n with low codimension p into a real space form $\widetilde{M}^{n+p}(c;\mathbb{R})$.

Problem 4. Give a sufficient condition that an isotropic immersion of a rank one symmetric space M into a real space form $\widetilde{M}(c;\mathbb{R})$ has parallel second fundamental form.

In section 2, we give an answer to Problem 1 in terms of shape operators of M in the ambient space $\widetilde{M}$ (Theorem 1). In section 3, we provide an answer to Problem 2 by studying circles on the submanifold M (Theorem 2). In section 4, we solve Problem 3 in the case of $p \leqq (n+2)/2$ (Theorem 3). In section 5, we study Problem 4 one by one for each rank one symmetric space M by using an inequality related to the codimension p of M (Theorems 4, 5, 6 and 7). In section 6 we investigate Problem 4 from a different point of view. We solve this problem by using inequalities related to mean curvatures of these submanifolds (Theorems 8, 9, 10 and 11). In the last section we pose some open problems on isotropic immersions.

2. Characterization of totally umbilic immersions

In this section we investigate the second fundamental form at one point. Let V and W be Euclidean vector spaces with inner products $\langle\ ,\ \rangle$, whose dimensions are n and k, respectively. We abstract the second fundamental form at one point to a symmetric bilinear form $\sigma : V \times V \to W$. We adopt for σ the usual notation and terminology of isometric immersions. Let $S^2(V)$ be the space of all symmetric endomorphisms of V. Then we define the linear map $A : W \to S^2(V)$ by $\langle A_\xi x, y\rangle = \langle \sigma(x,y), \xi\rangle$ for $x, y \in V$ and $\xi \in W$. We say that σ is λ-isotropic if there exists a real constant λ such that $\|\sigma(x,x)\| = \lambda$ for every unit vector $x \in V$. We prepare the following without proof for the later use ([18]):

Lemma 1. *Let $\sigma : V \times V \to W$ be a symmetric bilinear form. Then the following are equivalent*:

(1) *σ is λ-isotropic,*

(2) $\langle \sigma(x,x), \sigma(x,y)\rangle = 0$ *for each orthogonal pair of vectors $x, y \in V$,*

(3) $\langle\sigma(x,y),\sigma(z,w)\rangle + \langle\sigma(x,z),\sigma(y,w)\rangle + \langle\sigma(x,w),\sigma(y,z)\rangle = \lambda^2(\langle x,y\rangle\langle z,w\rangle + \langle x,z\rangle\langle y,w\rangle + \langle x,w\rangle\langle y,z\rangle)$ *for any vectors* $x,y,z,w \in V$.

We provide a characterization of umbilic bilinear forms ([17]):

Proposition 1. *Let* $\sigma : V \times V \to W$ *be a symmetric bilinear form. Then the following are equivalent*:

(i) σ *is umbilic,*

(ii) σ *is isotropic and for any* $\xi,\eta \in W$ $A_\xi A_\eta = A_\eta A_\xi$.

Proof. (i) $\Longrightarrow$ (ii): By assumption we have $A_\xi = \langle\xi,\mathfrak{h}\rangle Id$ for any $\xi \in W$, where $\mathfrak{h}$ is the mean curvature vector. Hence A_ξ and A_η commute for any $\xi,\eta \in W$.

(ii) $\Longrightarrow$ (i): Suppose that σ is λ-isotropic. We take a unit vector $x \in V$ and put $\xi = \sigma(x,x) \in W$. It follows from the statement (2) in Lemma 1 that x is an eigenvector of A_ξ with eigenvalue λ^2. We choose an orthonormal basis $\{e_1,\cdots,e_n\}$ of V satisfying that $A_\xi e_i = \lambda_i e_i$ $(i = 1,\cdots,n)$, where $e_1 = x$ and $\lambda_1 = \lambda^2$. We fix $i(\geqq 2)$. By the statement (3) in Lemma 1 we find that

$$\langle\sigma(x,e_i),\sigma(x,e_j)\rangle = (\lambda^2 - \lambda_i)/2 \cdot \delta_{ij} \quad \text{for any } j,$$

and hence $A_{\sigma(x,e_i)}x = (\lambda^2 - \lambda_i)/2 \cdot e_i$. Since $A_\xi A_{\sigma(x,e_i)}x = A_{\sigma(x,e_i)}A_\xi x$, we obtain

$$(\lambda^2 - \lambda_i)\lambda_i/2 = (\lambda^2 - \lambda_i)\lambda^2/2$$

so that $\lambda_i = \lambda^2$ for $i \geqq 2$, that is, V is the eigenspace of A_ξ with eigenvalue λ^2. Therefore we find that σ is umbilic. □

As an immediate consequence of this proposition we have the following ([17]):

Theorem 1. *Let* M *be a Riemannian submanifold in a Riemannian manifold* $\widetilde{M}$. *Then the following are equivalent*:

(i) M *is totally umbilic in* $\widetilde{M}$,

(ii) M *is isotropic in* $\widetilde{M}$ *and for any normal vectors* ξ, η *on* M $A_\xi A_\eta = A_\eta A_\xi$ *holds.*

Let the ambient space $\widetilde{M}$ be a real space form $\widetilde{M}(c;\mathbb{R})$ of curvature c. Then it follows from Theorem 1 that

Corollary. *Let* M *be a Riemannian submanifold in a real space form* $\widetilde{M}(c;\mathbb{R})$. *Then the following are equivalent*:

(i) M *is totally umbilic in* $\widetilde{M}(c;\mathbb{R})$,

(ii) M *is an isotropic submanifold with flat normal connection in* $\widetilde{M}(c;\mathbb{R})$.

3. Geometric meaning of constant isotropic immersions

A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s in a Riemannian manifold M is called a *circle* if there exist a field $Y = Y_s$ of unit vectors along γ and a constant $\kappa(\geqq 0)$ satisfying

$$\text{(3.1)} \qquad \begin{cases} \nabla_{\dot\gamma}\dot\gamma = \kappa Y \\ \nabla_{\dot\gamma} Y = -\kappa\dot\gamma, \end{cases}$$

where $\dot\gamma$ denotes the unit tangent vector of γ and $\nabla_{\dot\gamma}$ the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. The constant κ is called the *curvature* of the circle. A circle of null curvature is nothing but a geodesic. For each point $x \in M$, each orthonormal pair (u, v) of vectors at x and each positive constant κ, there exists locally a unique circle $\gamma = \gamma(s)$ on M with initial condition that $\gamma(0) = x, \dot\gamma(0) = u$ and $\nabla_{\dot\gamma}\dot\gamma(0) = \kappa v$. The purpose of this section is to prove the following ([15]):

Theorem 2. *Let M be a connected Riemannian submanifold of a Riemannian manifold $\widetilde M$ through an isometric immersion f. Then the following are equivalent*:

(i) *M is a constant (λ-) isotropic submanifold of $\widetilde M$,*

(ii) *there exists $\kappa > 0$ satisfying that for each circle γ of curvature κ on the submanifold M the curve $f \circ \gamma$ in $\widetilde M$ has constant first curvature κ_1 along this curve.*

Proof. (i) $\Rightarrow$ (ii): Let $f : M \to \widetilde M$ be a constant λ-isotropic immersion. In the following, for simplicity we also denote $f \circ \gamma$ by γ. It follows from equation (3.1) and the formula of Gauss $\widetilde\nabla_X Z = \nabla_X Z + \sigma(X, Z)$ that

$$\text{(3.2)} \qquad \widetilde\nabla_{\dot\gamma(s)}\dot\gamma(s) = \kappa Y_s + \sigma(\dot\gamma(s), \dot\gamma(s)), \quad s \in I.$$

Here $\widetilde\nabla$ is the Riemannian connection of the ambient space $\widetilde M$ and I is some open interval on $\mathbb{R}$. Then from (3.2) we can see that the first curvature $\kappa_1 = \|\widetilde\nabla_{\dot\gamma}\dot\gamma\|$ of the curve $f \circ \gamma$ is equal to $\sqrt{\kappa^2 + \lambda^2}$, which is constant on I.

(ii) $\Rightarrow$ (i): Let $f : M \to \widetilde M$ be an isometric immersion satisfying the condition (ii). We take a point $x \in M$ and choose an arbitrary orthonormal pair of vectors $u, v \in T_x M$. Let $\gamma = \gamma(s), s \in I$ be a circle of curvature κ on the submanifold M with initial condition that $\gamma(0) = x, \dot\gamma(0) = u$ and $\nabla_{\dot\gamma}\dot\gamma(0) = \kappa v$. By condition (ii) the first curvature $\kappa_1 = \|\widetilde\nabla_{\dot\gamma}\dot\gamma\|$ of the curve $f \circ \gamma$ is constant, so that equation (3.2) implies $\|\sigma(\dot\gamma, \dot\gamma)\|$ is constant on I. Hence, denoting by D the connection of the normal bundle of M in $\widetilde M$, from (3.1) we obtain

$$\text{(3.3)} \quad \begin{aligned} 0 = \frac{d}{ds}\|\sigma(\dot\gamma, \dot\gamma)\|^2 &= 2\langle D_{\dot\gamma}(\sigma(\dot\gamma, \dot\gamma)), \sigma(\dot\gamma, \dot\gamma)\rangle \\ &= 2\langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma) + 2\sigma(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma), \sigma(\dot\gamma, \dot\gamma)\rangle \\ &= 2\langle(\bar\nabla_{\dot\gamma}\sigma)(\dot\gamma, \dot\gamma), \sigma(\dot\gamma, \dot\gamma)\rangle + 4\kappa\langle\sigma(\dot\gamma, \dot\gamma), \sigma(\dot\gamma, Y)\rangle. \end{aligned}$$

Evaluating equation (3.3) at $s = 0$, we get

$$\langle(\bar{\nabla}_u\sigma)(u,u), \sigma(u,u)\rangle + 2\kappa\langle\sigma(u,u), \sigma(u,v)\rangle = 0. \tag{3.4}$$

On the other hand, for another circle $\rho = \rho(s)$ of the same curvature κ on the submanifold M with initial condition that $\rho(0) = x, \dot{\rho}(0) = u$ and $\nabla_{\dot{\rho}}\dot{\rho}(0) = -\kappa v$, we have

$$\langle(\bar{\nabla}_u\sigma)(u,u), \sigma(u,u)\rangle - 2\kappa\langle\sigma(u,u), \sigma(u,v)\rangle = 0 \tag{3.5}$$

which corresponds to equation (3.4). Thus, from (3.4) and (3.5) we can see that $\langle\sigma(u,u),\ \sigma(u,v)\rangle = 0$ for any orthonormal pair of vectors u, v at each point x of M, so that the submanifold M is (λ-) isotropic in $\widetilde{M}$ through the isometric immersion f by Lemma 1.

Next, we shall show that $\lambda : M \to \mathbb{R}$ is constant. It follows from (3.4) and (3.5) that

$$\langle(\bar{\nabla}_u\sigma)(u,u), \sigma(u,u)\rangle = 0 \quad \text{for every unit vecor } u \text{ at each point } x \text{ of } M^n.$$

Then, for every geodesic $\tau = \tau(s)$ on the submanifold M we see that $\lambda = \lambda(s)$ is constant along τ. Therefore we can conclude that λ is constant on M. □

4. Isotropic submanifolds with low codimension in a real space form

We shall give a classification theorem of isotropic submanifolds with low codimension in a real space form ([17]). For this purpose we prepare the following algebraic lemma ([22]):

Lemma 2. *Let V and W be Euclidean vector spaces with inner products $\langle\ ,\ \rangle$, where $\dim V = n$. Let $\sigma : V \times V \to W$ be a λ-isotropic symmetric bilinear form. Suppose that $\dim W \leqq (n+2)/2$. Then σ satisfies one of the following.*

(i) *σ is umbilic, that is, there exists $\xi \in W$ satisfying $\sigma(x,y) = \langle x, y\rangle\xi$.*

(ii) *$n = 2, 4, 8, 16$ and $\dim W = (n+2)/2$.*
In addition, in case of (ii)*, the following statements* (*a*) *and* (*b*) *hold.*

  (a) *For each unit vector $x \in V$ there exists an orthonormal basis $\{e_1 = x,\ e_2,\ \cdots,\ e_n\}$ for V satisfying $\sigma(e_i, e_i) = \sigma(x,x)$ for $1 \leqq i \leqq n/2$ and $\sigma(e_j, e_j) = -\sigma(x,x)$ for $(n+2)/2 \leqq j \leqq n$.*

  (b) *For each unit vector $\xi \in W$ there exists an orthonormal basis $\{e_1, \cdots, e_n\}$ for V satisfying $A_\xi e_i = \lambda e_i$ for $1 \leqq i \leqq n/2$ and $A_\xi e_j = -\lambda e_j$ for $(n+2)/2 \leqq j \leqq n$.*

Proof. For $\xi \in W$ we set a symmetric endomorphism $A_\xi : V \to V$ defined by $\langle A_\xi x, y\rangle = \langle\sigma(x,y), \xi\rangle$. In order to prove our Lemma we shall check several facts.

Fact 1. For each unit vector $x \in V$ we have $-\lambda^2 \leqq$ (eigenvalues of $A_{\sigma(x,x)}$) $\leqq \lambda^2$. Moreover, if we put $A_{\sigma(x,x)}y = \lambda^2 y$ (resp. $A_{\sigma(x,x)}y = -\lambda^2 y$) for a unit vector y, then $\sigma(y,y) = \sigma(x,x)$ (resp. $\sigma(y,y) = -\sigma(x,x)$).

We check this fact. Let $m \leqq$ (eigenvalues of $A_{\sigma(x,x)}$) $\leqq M$. Then it is well-known that

$$\begin{aligned} m &= \min_{|y|=1}\langle A_{\sigma(x,x)}y, y\rangle = \min_{|y|=1}\langle \sigma(x,x), \sigma(y,y)\rangle \\ M &= \max_{|y|=1}\langle A_{\sigma(x,x)}y, y\rangle = \max_{|y|=1}\langle \sigma(x,x), \sigma(y,y)\rangle. \end{aligned}$$

So we have $\langle \sigma(x,x), \sigma(y,y)\rangle \geqq -\|\sigma(x,x)\|\|\sigma(y,y)\| = -\lambda^2$ and this equality sign holds if and only if $\sigma(y,y) = -\sigma(x,x)$. Similarly we see that $\langle \sigma(x,x), \sigma(y,y)\rangle \leqq \|\sigma(x,x)\|\|\sigma(y,y)\| = \lambda^2$ and this equality sign holds if and only if $\sigma(y,y) = \sigma(x,x)$.

Next, for a unit $x \in V$ we consider a linear subspace $V_x = \{y \in V | A_{\sigma(x,x)}y = \lambda^2 y\}$. Then as a corollary of Fact 1 we can see that

(i) If there exists a unit vector $x \in V$ with $V_x = V$, then σ is umbilic.

(ii) If $V_x \cap V_y \neq \{0\}$ for $\exists x, y \in V$, then $V_x = V_y$.

Fact 2. For each unit $x \in V$ we know that $\dim V_x \geqq 1$ and $\sigma(y,z) = \langle y,z\rangle\sigma(x,x)$ for $\forall y, z \in V_x$.

In fact, it follows from $x \in V_x$ that $\dim V_x \geqq 1$. Fact 1 implies that $\sigma(y,y) = \langle y,y\rangle\sigma(x,x)$ for $\forall y \in V_x$. Hence, symmetrizing this equation, we obtain Fact 2.

We consider a linear subspace $V'_x = \{y \in V | A_{\sigma(x,x)}y = -\lambda^2 y\}$ for a unit $x \in V$.

Fact 2′. If $V'_x \neq \{0\}$, then for any unit $y \in V'_x$ we see $V'_x = V_y$.

It follows from Fact 1 that for unit $y \in V'_x$ we see $\sigma(z,z) = \langle z,z\rangle\sigma(y,y)$ for $\forall z \in V'_x$, so that $V'_x = V_y$.

Fact 2″. We denote by μ the minimum eigenvalue of $A_{\sigma(x,x)}$ for a unit $x \in V$. Let y be a unit eigenvector of $A_{\sigma(x,x)}$ with eigenvalue μ. Suppose that $\mu \neq \lambda^2$. Then every unit $z \in V_y$ is an eigenvector of $A_{\sigma(x,x)}$ with eigenvalue μ, so that in particular $V_x \perp V_y$.

Note that $\sigma(z,z) = \sigma(y,y)$. So $\langle A_{\sigma(x,x)}z, z\rangle = \langle \sigma(x,x), \sigma(y,y)\rangle = \mu$. This, together with the assumption on μ, shows that z is an eigenvector with the same eigenvalue μ.

Fact 3. If $\dim V_x < \dim V$ for a unit $x \in V$, then $\dim W \geqq \max\{r+1, n-r+1\}$, where $\dim V_x = r$ and $\dim V = n$.

We consider a linear mapping $\sigma_x : V \to W$ defined by $\sigma_x(y) = \sigma(x,y)$ for $y \in V$. Then we find $\ker \sigma_x = \{y \in V_x | \langle x,y\rangle = 0\}$.

In fact, Lemma 1(3) tells us that $y \in \ker\sigma_x$ is equivalent to

$$2\lambda^2\langle x,y\rangle^2 + \{\lambda^2\langle y,y\rangle - \langle \sigma(x,x), \sigma(y,y)\rangle\} = 0.$$

Note that these two terms of the left-hand side of the above equation are nonnegative. Thus we have $\langle x, y\rangle = 0$ and $y \in V_x$.

Hence

$$\begin{aligned} \dim W \quad \geqq \quad \dim \mathrm{Im}\sigma_x &= \dim V - \dim \ker \sigma_x \\ &= n-(r-1) = n-r+1. \end{aligned}$$

On the other hand, by assumption there exists a unit $y \notin V_x$. Then Fact 1 yields that $V_x \cap V_y = \{0\}$. So $\dim V_y \leqq n-r$. Again by applying our argument to a linear mapping σ_y, we get $\dim W \geqq n-(n-r)+1 = r+1$. Therefore we obtain Fact 3.

We shall prove our Lemma from now on by using the above facts. Suppose that σ is not umbilic. Take a unit vector $x \in V$. Note that $\dim V_x = r < n$. It follows from our assumption $\dim W \leqq (n+2)/2$ and Fact 3 that $r+1 \leqq (n+2)/2$ and $n-r+1 \leqq (n+2)/2$. Thus we see that $r = n/2$ and $\dim W = (n+2)/2$. In the following, we put $n/2 = m$. We take an orthonormal basis $\{e_1 = x, e_2, \cdots, e_m\}$ of V_x. Fact 1 yields $\sigma(e_i, e_j) = \delta_{ij}\sigma(x,x)$ for $i, j \in \{1, \cdots, n\}$. Let u be a unit vector of $V_x^{\perp} (\neq \phi)$. We set $\langle \sigma(u,u), \sigma(x,x)\rangle = \mu$. We remark that $\mu < \lambda^2$. From Lemma 1(3) we find that

$$\begin{aligned} &\langle \sigma(u, e_i), \sigma(u, e_j)\rangle = \frac{1}{2}(\lambda^2 - \mu)\delta_{ij}, \\ &\langle \sigma(u, e_i), \sigma(x,x)\rangle = \langle \sigma(u, e_i), \sigma(e_i, e_i)\rangle = 0 \quad \text{for } i = 1, \cdots, m. \end{aligned}$$

These imply that $\{\sigma(x,x), \sigma(u, e_1), \cdots, \sigma(u, e_m)\}$ is an orthogonal basis of W. On the other hand, $\langle \sigma(u,u), \sigma(u, e_i)\rangle = 0$ for $i = 1, \cdots, m$. Then $\sigma(u,u) = -\sigma(x,x)$. So we obtain (a) in our Lemma.

We next prove (b). Our vector space V is decomposed as: $V = V_x \oplus V_x'$, where $V_x' = \{u \in V | A_{\sigma(x,x)}u = -\lambda^2 u\}$. We consider a mapping $\varphi : S^{n-1}(1)(\subset V) \to S^m(1)(\subset W)$ defined by $\varphi(x) = (1/\lambda)\sigma(x,x)$. We shall show that the mapping φ is surjective:

We easily see that the differential at x of φ is given by $d\varphi_x(u) = (2/\lambda)\sigma(x,u) = (2/\lambda)\sigma_x(u)$ for $u \in T_x S^{n-1} \cong \{u \in V | \langle u, x\rangle = 0\} (\supset V_x')$. We note that σ_x is injective on V_x', because $\ker \sigma_x = \{y \in V_x | \langle x, y\rangle = 0\}$. Hence rank $d\varphi_x = m$ for $\forall x \in S^{n-1}$, which implies that $\varphi(S^{n-1}(1))$ is open and closed in $S^m(1)$. Therefore φ is surjective. So we obtain (b) in our Lemma.

We finally prove that $n = 2, 4, 8, 16$. Let $W' = \{\xi \in W | \langle \xi, \sigma(x,x)\rangle = 0\}$ for a fixed unit $x \in V$. Remark that $\dim W' = m$. For each unit $y \in V_x$ we see that $\langle \sigma(y,v), \sigma(x,x)\rangle = \langle \sigma(y,v), \sigma(y,y)\rangle = 0$. So we get

$$\sigma(y,v) \in W' \quad \text{for } y \in V_x,\ v \in V_x'.$$

Moreover we have the following equations:

$$\langle \sigma(y,u), \sigma(y,v)\rangle = \lambda^2 \langle u, v\rangle \quad \text{for unit } y \in V_x \textit{ and } u, v \in V_x'$$

and

$$\langle \sigma(y,v), \sigma(z,v)\rangle = \lambda^2\langle y,z\rangle \quad \text{for } y,z \in V_x \text{ and unit } v \in V_x'.$$

By virtue of these equations we can continue our argument as follows.

Let $\{e_1 = x, e_2, \cdots, e_m\}$ be an orthonormal basis of V_x. We consider a linear isometry $\varphi : V_x' \to W'$ defined by $\varphi(v) = (1/\lambda)\sigma_x(v) = (1/\lambda)\sigma(x,v)$ for $v \in V_x'$. Then the mapping φ induces an isometry on the sphere $S^{m-1}(1)$. For the sake of simplicity we denote by the same letter φ this isometry. We define a global field of orthonormal frames $E_1, \cdots, E_{m-1}$ on $S^{m-1}(1)$ as:

$$E_j(\xi) = \frac{1}{\lambda}\sigma(e_{j+1}, \varphi^{-1}(\xi)) \quad \text{for } j = 1, \cdots, m-1.$$

Then our sphere $S^{m-1}(1)$ is parallelizable, so that $m-1 = 0, 1, 3, 7$ (see [1]). Therefore we can see that $n = 2m = 2, 4, 8, 16$. □

Our aim here is to prove the following ([17]):

Theorem 3. *Let f be an isotropic immersion of an $n(\geqq 3)$-dimensional connected Riemannian manifold M^n into an $(n+p)$-dimensional real space form $\widetilde{M}^{n+p}(c;\mathbb{R})$ of curvature c. Suppose that $p \leqq (n+2)/2$. Then M has parallel second fundamental form, so that either f is totally umbilic or f is locally congruent to one of the first standard minimal immersion of M^n into $\widetilde{M}^{n+p}(c;\mathbb{R})$:*

(1) $M^n = \mathbb{C}P^2(4c/3), \widetilde{M}^{n+p}(c;\mathbb{R}) = S^7(c)$,

(2) $M^n = \mathbb{H}P^2(4c/3), \widetilde{M}^{n+p}(c;\mathbb{R}) = S^{13}(c)$,

(3) $M^n = \mathbb{C}ayP^2(4c/3), \widetilde{M}^{n+p}(c;\mathbb{R}) = S^{25}(c)$,

where $\mathbb{C}P^2(4c/3)$, $\mathbb{H}P^2(4c/3)$ and $\mathbb{C}ayP^2(4c/3)$ denote projective planes of maximal sectional curvature $4c/3$ over the complex, quaternion and Cayley numbers, respectively.

Proof. First we note that our manifold is totally umbilic in $\widetilde{M}^{n+p}(c;\mathbb{R})$ in the case of $p < (n+2)/2$ (see Lemma 2). In the following, we consider the case that $p = (n+2)/2$ and f is not totally umbilic. Let $U = \{x \in M \mid x$ is an umbilic point of the isometric immersion $f\}$ and $U^c = M - U$. Needless to say $U^c \neq \phi$. We shall show that $U^c = M$ and the second fundamental form of f is parallel. We denote by R and S the curvature tensor and the Ricci tensor of M, respectively. Then it follows from the following Gauss equation

$$\begin{aligned} &\langle \sigma(X,Y), \sigma(Z,W)\rangle - \langle \sigma(Z,Y), \sigma(X,W)\rangle \\ = \; &\langle R(Z,X)Y, W\rangle - c(\langle X,Y\rangle\langle Z,W\rangle - \langle Z,Y\rangle\langle X,W\rangle) \end{aligned} \tag{4.1}$$

that $S(X,Y) = (n-1)(c+\lambda^2)\langle X,Y\rangle$ on U. On the other hand, by virtue of (4.1), Lemma 1(3) and the fact that the immersion f is minimal on the open set U^c (see the statement (ii) in Lemma 2) we know that $S(X,Y) = \{(n-1)c - (n+2)/2\cdot\lambda^2\}\langle X,Y\rangle$

on U^c. These, together with the hypothesis $\dim M \geqq 3$, imply that M is Einstein. Hence $M = U^c$. In addition, λ is constant so that

$$\langle (\bar{\nabla}_Y \sigma)(X, X), \sigma(X, X)\rangle = 0 \quad \text{for any } X, Y. \tag{4.2}$$

Here the covariant differentiation $\bar{\nabla}$ of the second fundamental form σ with respect to the connection in (tangent bundle)$\oplus$(normal bundle) is defined as follows:

$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

Similarly, from Lemma 1(3) we have

$$\langle (\bar{\nabla}_X \sigma)(X, X), \sigma(X, Y)\rangle + \langle \sigma(X, X), (\bar{\nabla}_X \sigma)(X, Y)\rangle = 0 \quad \text{for any } X, Y. \tag{4.3}$$

It follows from (4.2), (4.3) and the Codazzi equation $(\bar{\nabla}_X \sigma)(Y, Z) = (\bar{\nabla}_Y \sigma)(X, Z)$ that $\langle (\bar{\nabla}_X \sigma)(X, X), \sigma(X, Y)\rangle = 0$ for all X, Y. We here recall the fact that the first normal space $N^1 := \{\sigma(X, Y) |\ X, Y \in TM\}_{\mathbb{R}}$ of f spans the normal space of M at its each point (see the proof of Lemma 2). Hence, $(\bar{\nabla}_X \sigma)(X, X) = 0$ for any X, so that $\bar{\nabla}\sigma = 0$. Therefore, by virtue of the works [8], [21] we can get the conclusion. □

5. Characterization of parallel immersions of rank one symmetric spaces (I)

In this section, motivated by Theorem 3, we characterize parallel immersions of every rank one symmetric space M^n into a real space form $\widetilde{M}^{n+p}(\tilde{c}; \mathbb{R})$ by using inequalities related to the codimension p. For this purpose we prepare the following two lemmas:

Lemma 3 ([18]). *Let f be a $\lambda(>0)$-isotropic immersion of a Riemannian manifold M^n into a Riemannian manifold $\widetilde{M}^{n+p}$. The discriminant Δ_x at $x \in M$ is defined by $\Delta_x = K(X, Y) - \widetilde{K}(X, Y)$, where $K(X, Y)$ (resp. $\widetilde{K}(X, Y)$) represents the sectional curvature of the plane spanned by $X, Y \in T_xM$ for M (resp. $\widetilde{M}$). Suppose that the discriminant Δ_x at $x \in M$ is constant. Then the following inequalities hold at x:*

$$-\frac{n+2}{2(n-1)}\lambda(x)^2 \leqq \Delta_x \leqq \lambda(x)^2.$$

Moreover,

(i) $\Delta_x = \lambda(x)^2 \iff f$ *is umbilic at* $x \iff \dim N_x^1 = 1$,

(ii) $\Delta_x = -\{(n+2)/2(n-1)\}\lambda(x)^2 \iff f$ *is minimal at* x $\iff \dim N_x^1 = (n(n+1)/2) - 1$,

(iii) $-\{(n+2)/2(n-1)\}\lambda(x)^2 < \Delta_x < \lambda(x)^2 \iff \dim N_x^1 = n(n+1)/2$.

Here, we denote by N_x^1 *the first normal space at* x*, that is* $N_x^1 = Span_{\mathbb{R}}\{\sigma(X,Y) : X, Y \in T_xM\}$.

Lemma 4. *Let* M^n *be a constant* λ*-isotropic submanifold in a real space form* $\widetilde{M}^{n+p}(c;\mathbb{R})$. *Suppose that* M *is locally symmetric and the first normal space spans the normal space at any point of* M. *Then second fundamental form of* M^n *in* $\widetilde{M}^{n+p}(c;\mathbb{R})$ *is parallel.*

Proof. It follows from (3) in Lemma 1 and (4.1) that

$$\begin{aligned}(5.1)\qquad \langle\sigma(X,Y),\sigma(Z,W)\rangle &= (1/3)(\langle R(Z,X)Y,W\rangle + \langle R(Z,Y)X,W\rangle) \\ &\quad - (c/3)(2\langle X,Y\rangle\langle Z,W\rangle - \langle Y,Z\rangle\langle X,W\rangle \\ &\quad - \langle Z,X\rangle\langle Y,W\rangle) + (\lambda^2/3)(\langle X,Y\rangle\langle Z,W\rangle \\ &\quad + \langle X,Z\rangle\langle Y,W\rangle + \langle X,W\rangle\rangle\langle Y,Z\rangle).\end{aligned}$$

Since λ is constant and M is locally symmetric, differentiating (5.1) with respect to any tangent vector field T on M, we have

$$(5.2)\qquad \langle(\bar{\nabla}_T\sigma)(X,Y),\sigma(Z,W)\rangle = -\langle\sigma(X,Y),(\bar{\nabla}_T\sigma)(Z,W)\rangle.$$

By using (5.2) and the Codazzi equation $(\bar{\nabla}_X\sigma)(Y,Z) = (\bar{\nabla}_Y\sigma)(X,Z)$ repeatedly, we find

$$\begin{aligned} &\langle(\bar{\nabla}_T\sigma)(X,Y),\sigma(Z,W)\rangle = -\langle\sigma(X,Y),(\bar{\nabla}_Z\sigma)(T,W)\rangle \\ =\ &\langle(\bar{\nabla}_X\sigma)(Z,Y),\sigma(T,W)\rangle = -\langle\sigma(Z,Y),(\bar{\nabla}_W\sigma)(X,T)\rangle \\ =\ &\langle(\bar{\nabla}_Y\sigma)(Z,W),\sigma(X,T)\rangle = -\langle\sigma(Z,W),(\bar{\nabla}_T\sigma)(X,Y)\rangle.\end{aligned}$$

So we see that $\langle(\bar{\nabla}_T\sigma)(X,Y),\sigma(Z,W)\rangle = 0$, which, together with the hypothesis that the first normal space spans the normal space at any point of M, shows that the second fundamental form of our immersion is parallel. □

We first characterize parallel immersions of real space forms into real space forms with low codimension ([2], [10]).

Theorem 4. *Let* f *be a* λ*-isotropic immersion of an* $n(\geqq 2)$*-dimensional real space form* $M^n(c;\mathbb{R})$ *into an* $(n+p)$*-dimensional space form* $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$. *Suppose that*

$$p \leqq \frac{1}{2}n(n+1) - 1.$$

Then f *is a parallel immersion and locally equivalent to one of the following*:

(1) f *is a totally umbilic immersion of* $M^n(c;\mathbb{R})$ *into* $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$, *where* $c \geqq \tilde{c}$ *and* $p \leqq (n(n+1)/2) - 1$,

(2) f *is the second standard minimal immersion of* $M^n(c;\mathbb{R}) = S^n(c)$ *into* $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R}) = S^{n+p}(\tilde{c})$, *where* $\tilde{c} = 2(n+1)c/n$ *and* $p = (n(n+1)/2) - 1$.

Proof. First, we consider the case that f is a totally geodesic immersion. Then this case is included in (1) of our Theorem.

Next, we investigate the case that f is not a totally geodesic immersion. Since λ is a continuous function on $M^n(c;\mathbb{R})$, we have only to study on the open dense subset $U(=\{x \in M^n(c;\mathbb{R}) | \lambda(x) > 0\})$ from now on. It follows from Lemma 3, the assumption of our Theorem and the continuity of λ that the function λ is constant on U, so that $\lambda^2 = c - \tilde{c}$ or $\lambda^2 = 2(n-1)(\tilde{c}-c)/(n+2)$. In case of $\lambda^2 = c - \tilde{c}$, this case is included in (1) of our Theorem. In case of $\lambda^2 = 2(n-1)(\tilde{c}-c)/(n+2)$, from Lemma 3 we know that $\dim N_x^1 = (n(n+1)/2) - 1$ for all $x \in U$. Hence by virtue of Lemma 4 we can see that our immersion has parallel second fundamental form. Therefore we get the case (2) of our Theorem (cf. [8], [21]). □

A complex n-dimensional complex space form $M^n(c;\mathbb{C})$ is a Kähler manifold of constant holomorphic sectional curvature c, which is locally congruent to either a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n(=\mathbb{R}^{2n})$ or a complex hyperbolic space $\mathbb{C}H^n(c)$, according as c is positive, zero or negative.

The following lemma shows a necessary and sufficient condition that the isometric immersion of a complex space form into a real space form has parallel second fundamental form ([8], [9], [21]):

Lemma 5. *Let M be a complex n-dimensional connected Kähler manifold with complex structure J which is isometrically immersed into a real space form $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$. Then the following two conditions are equivalent*:

(i) *The second fundamental form σ of M in $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$ is parallel,*

(ii) *$\sigma(JX, JY) = \sigma(X, Y)$ for all $X, Y \in TM$.*

We characterize parallel immersions of complex space forms into real space forms with low codimension ([4], [12]).

Theorem 5. *Let f be a λ-isotropic immersion of a complex space form $M^n(4c;\mathbb{C})$ ($n \geqq 2$) of constant holomorphic sectional curvature $4c$ into a real space form $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$. If $p \leqq n^2+n-2$, then f is a parallel immersion and locally equivalent to one of the following*:

(1) *f is a totally geodesic immersion of $\mathbb{C}^n(=\mathbb{R}^{2n})$ into $\mathbb{R}^{2n+p}$, where $p \leqq n^2+n-2$,*

(2) *f is a totally umbilic immersion of $\mathbb{C}^n(=\mathbb{R}^{2n})$ into $\mathbb{R}H^{2n+p}(\tilde{c})$, where $p \leqq n^2+n-2$,*

(3) *is the first standard minimal immersion of $\mathbb{C}P^n(4c)$ into $S^{2n+p}(\tilde{c})$, where $p = n^2-1$ and $\tilde{c} = 2(n+1)c/n$,*

(4) *f is a parallel immersion defined by*

$$f = f_2 \circ f_1 : \mathbb{C}P^n(4c) \xrightarrow{f_1} S^{n^2+2n-1}(2(n+1)c/n) \xrightarrow{f_2} \widetilde{M}^{2n+p}(\tilde{c};\mathbb{R}),$$

where f_1 is the first standard minimal immersion, f_2 is a totally umbilic immersion, $n^2 \leqq p \leqq n^2+n-2$ and $2(n+1)c/n \geqq \tilde{c}$.

We need the following lemma in order to prove Theorem 5.

Lemma 6. *The value of λ in our Theorem 5 is the following*:
(1) $\lambda = 0$, (2) $\lambda^2 = -\tilde{c}$; (3), (4) $\lambda^2 = 4c - \tilde{c}$.

Proof. (1) and (2) are clear from Lemma 3. We shall consider the cases (3) and (4). Let ι be a totally real totally geodesic immersion of a real projective space $\mathbb{R}P^n(c)$ into $\mathbb{C}P^n(4c)$. We denote by f a λ-isotropic minimal immersion of $\mathbb{C}P^n(4c)$ into $S^{n^2+2n-1}(2(n+1)c/n)$ (with parallel second fundamental form σ) and by J the complex structure on $\mathbb{C}P^n(4c)$. We choose a local field of orthonormal frames $\{e_1, \cdots, e_n\}$ on $\mathbb{R}P^n(c)$. Then $\{e_1, \cdots, e_n, Je_1, \cdots, Je_n\}$ is a local field of orthonormal frames on $\mathbb{C}P^n(4c)$. Since $\mathbb{R}P^n(c)$ is totally geodesic in $\mathbb{C}P^n(4c)$, we can denote by the same letter σ the second fundamental form of $\mathbb{R}P^n(c)$ in the ambient space $S^{n^2+2n-1}(2(n+1)c/n)$ through $f \circ \iota$. We here remark that $\sigma(e_i, e_i) = \sigma(Je_i, Je_i)$ for $1 \leqq i \leqq n$ (see Lemma 5). This, together with the fact that $\mathbb{C}P^n(4c)$ is minimal in $S^{n^2+2n-1}(2(n+1)c/n)$, implies that our manifold $\mathbb{R}P^n(c)$ is minimal in $S^{n^2+2n-1}(2(n+1)c/n)$. Therefore, by virtue of our discussion, we know that $f \circ \iota$ is a λ-isotropic minimal immersion of $\mathbb{R}P^n(c)$ into $S^{n^2+2n-1}(2(n+1)c/n)$.

Using (ii) in Lemma 3, we can see that

$$\begin{aligned} \lambda^2 &= -\frac{2(n-1)}{n+2}\left(c - \frac{2(n+1)c}{n}\right) \\ &= 4c - \frac{2(n+1)c}{n} = 4c - \tilde{c}. \end{aligned}$$

Thus, we can check the case (3).

Let g be a totally umbilic immersion of $S^{n^2+2n-1}(2(n+1)c/n)$ into $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$. Then g is $\sqrt{2(n+1)c/n - \tilde{c}}$-isotropic. Hence the above computation yields that

$$\begin{aligned} \lambda^2 &= \left(4c - \frac{2(n+1)c}{n}\right) + \left(\frac{2(n+1)c}{n} - \tilde{c}\right) \\ &= 4c - \tilde{c}. \end{aligned}$$

So we can check the case (4). □

We are now in a position to prove Theorem 5:

Proof of Theorem 5. Let J be the complex structure on $M^n(4c;\mathbb{C})$. Then the curvature tensor R of $M^n(4c;\mathbb{C})$ is given by

$$(5.3) \qquad \begin{aligned} R(X,Y)Z &= c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX \\ &\quad - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ\} \end{aligned}$$

for all vector fields X, Y and Z tangent to $M^n(4c;\mathbb{C})$. It follows from (4.1) and (5.3) that

$$(5.4)\ \langle\sigma(X,Y),\sigma(Z,W)\rangle = \frac{\lambda^2+2(c-\tilde{c})}{3}\langle X,Y\rangle\langle Z,W\rangle + \frac{\lambda^2-(c-\tilde{c})}{3}\{\langle X,W\rangle\langle Y,Z\rangle+\langle X,Z\rangle\langle Y,W\rangle\} + c\{\langle JX,W\rangle\langle JY,Z\rangle+\langle JX,Z\rangle\langle JY,W\rangle\}$$

for all vector fields $X,\ Y,\ Z$ and W tangent to $M^n(4c;\mathbb{C})$.

We have only to consider the case that $M^n(4c;\mathbb{C})$ is not totally geodesic in $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$. We shall study on the open dense subset $U(=\{x\in M^n(4c;\mathbb{C})|\lambda(x)>0\})$ from now on. Our discussion is divided into the two cases:

(i) $\lambda^2(x)\neq c-\tilde{c}$,

(ii) $\lambda^2(x)=c-\tilde{c}$.

(i) In the following, we study at an arbitrary fixed point x of U. Note that $\lambda^2(x)\neq c-\tilde{c}$. Now we investigate the first normal space N_x^1 at the point x by using (5.4). We choose an orthonormal basis $\{e_1,\cdots,e_n,e_{n+1}=Je_1,\cdots,e_{2n}=Je_n\}$ of $T_xM^n(4c;\mathbb{C})$. Equation (5.3) shows that $\langle R(e_i,e_j)e_j,e_i\rangle=c$ for $1\leqq i\neq j\leqq n$. So, we may apply Lemma 3 to the linear subspace of $T_xM^n(4c;\mathbb{C})$, which is generated by $\{e_1,\cdots,e_n\}$. Thus either the case (ii) or the case (iii) of Lemma 3 must hold at x.

Straightforward computation, by virtue of (5.4), yields the orthogonal relations.

$$(5.5)\qquad \langle\sigma(e_i,Je_j),\sigma(e_k,Je_\ell)\rangle=\frac{\lambda^2-(c-\tilde{c})}{3}\cdot\delta_{ik}\delta_{j\ell}$$

for $1\leqq i<j\leqq n$ and $1\leqq k<\ell\leqq n$.

$$(5.6)\qquad \langle\sigma(e_i,e_j),\sigma(e_k,Je_\ell)\rangle=0\quad\text{for } 1\leqq i\leqq j\leqq n\quad\text{and}\quad 1\leqq k<\ell\leqq n.$$

Then, in consideration of Lemma 3, (5.5) and (5.6), the codimension p satisfies

$$p\geqq n(n+1)/2-1+n(n-1)/2=n^2-1$$

at the fixed point x. We note that x is not an umbilic point, since $\sigma(e_i,Je_j)\neq 0$ for $1\leqq i<j\leqq n$. Here we take n vectors $\sigma(e_i,Je_i)$ $(i=1,\cdots,n)$.

Similar computation shows the following orthogonal relations.

$$(5.7)\qquad \langle\sigma(e_i,Je_i),\sigma(e_j,Je_j)\rangle=\frac{\lambda^2-(4c-\tilde{c})}{3}\cdot\delta_{ij}\quad\text{for}\quad i,j=1,\cdots,n.$$

$$(5.8)\qquad \langle\sigma(e_i,e_j),\sigma(e_k,Je_k)\rangle=0\quad\text{for}\quad 1\leqq i\leqq j\leqq n\quad\text{and}\quad 1\leqq k\leqq n.$$

$$(5.9)\qquad \langle\sigma(e_i, Je_j), \sigma(e_k, Je_k)\rangle = 0 \quad \text{for} \quad 1 \leqq i < j \leqq n \quad \text{and} \quad 1 \leqq k \leqq n.$$

Now suppose that $\lambda^2 \neq 4c - \tilde{c}$. Then, in view of (5.7), (5.8) and (5.9), we find that $p \geqq (n^2 - 1) + n$, which contradicts our assumption $p \leqq n^2 + n - 2$. And hence we have

$$(5.10)\qquad \lambda^2 = 4c - \tilde{c}.$$

Substituting (5.10) into the right-hand side of (5.4), we obtain

$$\begin{aligned}(5.11)\qquad & \langle\sigma(X,Y), \sigma(Z,W)\rangle \\ = \ & (2c - \tilde{c})\langle X,Y\rangle\langle Z,W\rangle + c\{\langle X,W\rangle\langle Y,Z\rangle \\ & + \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle + \langle JX,Z\rangle\langle JY,W\rangle\}\end{aligned}$$

for all vector fields $X,\ Y,\ Z$ and W tangent to $M^n(4c;\mathbb{C})$.

Equation (5.11) implies the following.

$$\begin{aligned}(5.12)\qquad \langle\sigma(X,Y), \sigma(X,Y)\rangle &= \langle\sigma(JX,JY), \sigma(JX,JY)\rangle \\ &= (3c - \tilde{c})\langle X,Y\rangle^2 + c\{\|X\|^2\|Y\|^2 - \langle JX,Y\rangle^2\}. \\ (5.13)\quad \langle\sigma(X,Y), \sigma(JX,JY)\rangle &= (3c - \tilde{c})\langle X,Y\rangle^2 + c\{\|X\|^2\|Y\|^2 - \langle JX,Y\rangle^2\}.\end{aligned}$$

Thus, in view of (5.12) and (5.13), we can get $\sigma(X,Y) = \sigma(JX,JY)$ for all $X,\ Y$. And hence, from Lemma 5, we find that the second fundamental form of our immersion is parallel on U. Therefore, due to the works of [8], [9], [21], there occurs the case (3) and (4) of our Theorem 5.

(ii) Lastly, we consider the case of $\lambda^2(x_0) = c - \tilde{c}$. The above discussion asserts that the continuous function λ on U is $\lambda^2 = 4c - \tilde{c}$ or $\lambda^2 = c - \tilde{c}$. And hence, we have only to consider the case that $\lambda^2 = c - \tilde{c}$ on U. Let ι be a totally real totally geodesic immersion of a real space form $M^n(c;\mathbb{R})$ into $M^n(4c;\mathbb{C})$. It follows (i) of Lemma 3 that our manifold $(M^n(c;\mathbb{R}), f \circ \iota)$ is totally umbilic in $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$.

Here, we take an arbitrary geodesic γ in $M^n(4c;\mathbb{C})$. Since $M^n(4c;\mathbb{C})$ is a Euclidean space or a Riemannian symmetric space of rank one, we may think that the curve γ is a geodesic in $M^n(c;\mathbb{R})$. Hence the curve $(f \circ \iota) \circ \gamma$ is a circle in $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$, so that the curve $f \circ \gamma$ is a circle in the ambient space $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$, which implies that the immersion f has parallel second fundamental form (see [19]). Thus we know that our immersion f is one of (2), (3) and (4) of our Theorem. However there occurs only the case (2). In fact, in both of cases (3) and (4), we know that $\lambda^2 = 4c - \tilde{c}$ (see Lemma 6). On the other hand, in our case, $\lambda^2 = c - \tilde{c}$. This is a contradiction, because $c > 0$. Therefore, we can get the conclusion. □

A quaternionic n-dimensional quaternionic space form $M^n(c;\mathbb{Q})$ is a quaternionic Kähler manifold of constant quaternionic sectional curvature c, which is locally congruent to either a quaternionic projective space $\mathbb{Q}P^n(c)$, a quaternionic Euclidean space $\mathbb{Q}^n (= \mathbb{R}^{4n})$ or a quaternionic hyperbolic space $\mathbb{Q}H^n(c)$, according as c is positive, zero or negative.

We next investigate isotropic immersions of quaternionic space forms into real space forms with low codimension. Our aim here is to prove the following theorem ([4], [13]).

Theorem 6. *Let f be a λ-isotropic immersion of a quaternionic space form $M^n(4c;\mathbb{Q})$ $(n \geqq 2)$ of constant quaternionic sectional curvature $4c$ into a real space form $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$. If $p \leqq 2n^2+2n-2$, then f is a parallel immersion and locally equivalent to one of the following*:

(1) *f is a totally geodesic immersion of $\mathbb{Q}^n(=\mathbb{R}^{4n})$ into $\mathbb{R}^{4n+p}$, where $p \leqq 2n^2+2n-2$,*

(2) *f is a totally umbilic immersion of $\mathbb{Q}^n(=\mathbb{R}^{4n})$ into $\mathbb{R}H^{4n+p}(\tilde{c})$, where $p \leqq 2n^2+2n-2$,*

(3) *f is the first standard minimal immersion of $\mathbb{Q}P^n(4c)$ into $S^{4n+p}(\tilde{c})$, where $p = 2n^2-n-1$ and $\tilde{c} = 2(n+1)c/n$,*

(4) *f is a parallel immersion defined by*

$$f = f_2 \circ f_1 : \mathbb{Q}P^n(4c) \xrightarrow{f_1} S^{2n^2+3n-1}(2(n+1)c/n) \xrightarrow{f_2} \widetilde{M}^{4n+p}(\tilde{c};\mathbb{R}),$$

where f_1 is the first standard minimal immersion, f_2 is a totally umbilic immersion, $2n^2-n \leqq p \leqq 2n^2+2n-2$ and $2(n+1)c/n \geqq \tilde{c}$.

Here, we prepare the following similar lemmas in order to prove Theorem 6.

Lemma 7 ([8], [13], [21]). *Let M be a quaternionic n-dimensional connected quaternionic Kähler manifold with canonical local basis $\{I,J,K\}$ which is isometrically immersed into a real space form $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$. Then the following two conditions are equivalent*:

(i) *The second fundamental form σ of M in $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$ is parallel,*

(ii) *$\sigma(IX,IY) = \sigma(JX,JY) = \sigma(KX,KY) = \sigma(X,Y)$ for all $X, Y \in TM$.*

Lemma 8. *The value of λ in our Theorem 6 is the following*:
(1) $\lambda = 0$, (2) $\lambda^2 = -\tilde{c}$; (3), (4) $\lambda^2 = 4c - \tilde{c}$.

Proof. Let ι be a totally real totally geodesic immersion of a real projective space $\mathbb{R}P^n(c)$ into $\mathbb{Q}P^n(4c)$. The rest of the proof is similar to that of Lemma 6. □

Now we shall prove Theorem 6:

Proof of Theorem 6. Let $\{I,\ J,\ K\}$ be the canonical local basis on $M^n(4c;\mathbb{Q})$. Then the curvature tensor R of $M^n(4c;\mathbb{Q})$ is given by

$$\begin{aligned} &R(X,Y)Z \\ = \ & c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle IY,Z\rangle IX - \langle IX,Z\rangle IY \\ &+\langle JY,Z\rangle JX - \langle JX,Z\rangle JY + \langle KY,Z\rangle KX - \langle KX,Z\rangle KY \\ &+2\langle X,IY\rangle IZ + 2\langle X,JY\rangle JZ + 2\langle X,KY\rangle KZ\} \end{aligned} \tag{5.14}$$

for all vector fields X, Y and Z tangent to $M^n(4c;\mathbb{Q})$.

It follows from (4.1) and (5.14) that

$$\begin{aligned}(5.15)\quad &\langle\sigma(X,Y),\sigma(Z,W)\rangle \\ = &\frac{\lambda^2+2(c-\tilde{c})}{3}\langle X,Y\rangle\langle Z,W\rangle+\frac{\lambda^2-(c-\tilde{c})}{3}\{\langle X,W\rangle\langle Y,Z\rangle+\langle X,Z\rangle\langle Y,W\rangle\} \\ &+c\{\langle IX,W\rangle\langle IY,Z\rangle+\langle IX,Z\rangle\langle IY,W\rangle \\ &+\langle JX,W\rangle\langle JY,Z\rangle+\langle JX,Z\rangle\langle JY,W\rangle \\ &+\langle KX,W\rangle\langle KY,Z\rangle+\langle KX,Z\rangle\langle KY,W\rangle\}\end{aligned}$$

for all vector fields X, Y, Z and W tangent to $M^n(4c;\mathbb{Q})$.

It suffices to study on the open dense subset $U(=\{x\in M^n(4c;\mathbb{Q})|\lambda(x)>0\})$. Our discussion is divided into the two cases: (i) $\lambda^2(x)\neq c-\tilde{c}$, (ii) $\lambda^2(x)=c-\tilde{c}$.

(i) In the following, we study at an arbitrary fixed point x of U. Note that $\lambda^2(x)\neq c-\tilde{c}$. Now we investigate the first normal space N^1_x at the point x by using (5.15). We choose an orthonormal basis $\{e_1,\cdots,e_n,e_{n+1}=Ie_1,\cdots,e_{2n}=Ie_n,e_{2n+1}=Je_1,\cdots,e_{3n}=Je_n,e_{3n+1}=Ke_1,\cdots,e_{4n}=Ke_n\}$ of $T_xM^n(4c;\mathbb{Q})$. Equation (5.14) shows that $\langle R(e_i,e_j)e_j,e_i\rangle=c$ for $1\leqq i\neq j\leqq n$. So, we may apply Lemma 3 to the linear subspace of $T_xM^n(4c;\mathbb{Q})$, which is generated by $\{e_1,\cdots,e_n\}$. Thus either the case (ii) or the case (iii) of Lemma 3 must hold at x.

Straightforward computation, by virtue of (5.15), yields the orthogonal relations.

$$\begin{aligned}(5.16)\quad \langle\sigma(e_i,e_j),\sigma(e_k,Ie_\ell)\rangle &= \langle\sigma(e_i,e_j),\sigma(e_k,Je_\ell)\rangle \\ &= \langle\sigma(e_i,e_j),\sigma(e_k,Ke_\ell)\rangle = 0 \\ &\quad \text{for } 1\leqq i\leqq j\leqq n \text{ and } 1\leqq k<\ell\leqq n.\end{aligned}$$

$$\begin{aligned}(5.17)\quad \langle\sigma(e_i,Ie_j),\sigma(e_k,Je_\ell)\rangle &= \langle\sigma(e_i,Je_j),\sigma(e_k,Ke_\ell)\rangle \\ &= \langle\sigma(e_i,Ke_j),\sigma(e_k,Ie_\ell)\rangle = 0 \\ &\quad \text{for } 1\leqq i<j\leqq n \text{ and } 1\leqq k<\ell\leqq n.\end{aligned}$$

$$\begin{aligned}(5.18)\quad \langle\sigma(e_i,Ie_j),\sigma(e_k,Ie_\ell)\rangle &= \langle\sigma(e_i,Je_j),\sigma(e_k,Je_\ell)\rangle \\ &= \langle\sigma(e_i,Ke_j),\sigma(e_k,Ke_\ell)\rangle \\ &= \frac{\lambda^2-(c-\tilde{c})}{3}\cdot\delta_{ik}\delta_{j\ell} \\ &\quad \text{for } 1\leqq i<j\leqq n \text{ and } 1\leqq k<\ell\leqq n.\end{aligned}$$

Then, in consideration of Lemma 3, (5.16), (5.17) and (5.18), the codimension p satisfies

$$p\geqq n(n+1)/2-1+3n(n-1)/2=2n^2-n-1$$

at the fixed point x. We note that x is not an umbilic point, since $\sigma(e_i, Je_j) \neq 0$ for $1 \leqq i < j \leqq n$. Here we take $3n$ vectors $\sigma(e_i, Ie_i), \sigma(e_i, Je_i)$ and $\sigma(e_i, Ke_i)$ $(i = 1, \cdots, n)$. Similar computation shows the following orthogonal relations.

$$
\begin{aligned}
(5.19) \qquad \langle \sigma(e_i, e_j), \sigma(e_k, Ie_k)\rangle &= \langle \sigma(e_i, e_j), \sigma(e_k, Je_k)\rangle \\
&= \langle \sigma(e_i, e_j), \sigma(e_k, Ke_k)\rangle = 0 \\
&\quad \text{for } 1 \leqq i \leqq j \leqq n \text{ and } 1 \leqq k \leqq n. \\
(5.20) \qquad \langle \sigma(e_i, Ie_j), \sigma(e_k, Ie_k)\rangle &= \langle \sigma(e_i, Je_j), \sigma(e_k, Je_k)\rangle \\
&= \langle \sigma(e_i, Ke_j), \sigma(e_k, Ke_k)\rangle = 0 \\
&\quad \text{for } 1 \leqq i < j \leqq n \text{ and } 1 \leqq k \leqq n. \\
(5.21) \qquad \langle \sigma(e_i, Ie_i), \sigma(e_j, Je_j)\rangle &= \langle \sigma(e_i, Je_i), \sigma(e_j, Ke_j)\rangle \\
&= \langle \sigma(e_i, Ke_i), \sigma(e_j, Ie_j)\rangle = 0 \\
&\quad \text{for } i, j = 1, \cdots, n.
\end{aligned}
$$

$$
\begin{aligned}
(5.22) \qquad & \langle \sigma(e_i, Ie_i), \sigma(e_j, Je_k)\rangle = \langle \sigma(e_i, Ie_i), \sigma(e_j, Ke_k)\rangle \\
= \; & \langle \sigma(e_i, Je_i), \sigma(e_j, Ie_k)\rangle = \langle \sigma(e_i, Je_i), \sigma(e_j, Ke_k)\rangle \\
= \; & \langle \sigma(e_i, Ke_i), \sigma(e_j, Ie_k)\rangle = \langle \sigma(e_i, Ke_i), \sigma(e_j, Je_k)\rangle = 0 \\
& \text{for } 1 \leqq i \leqq n \text{ and } 1 \leqq j < k \leqq n.
\end{aligned}
$$

$$
\begin{aligned}
(5.23) \langle \sigma(e_i, Ie_i), \sigma(e_j, Ie_j)\rangle &= \langle \sigma(e_i, Je_i), \sigma(e_j, Je_j)\rangle \\
&= \langle \sigma(e_i, Ke_i), \sigma(e_j, Ke_j)\rangle = \frac{\lambda^2 - (4c - \tilde{c})}{3} \cdot \delta_{ij} \\
&\quad \text{for } i, j = 1, \cdots, n.
\end{aligned}
$$

Now suppose that $\lambda^2 \neq 4c - \tilde{c}$. Then, in view of (5.19), (5.20), (5.21), (5.22) and (5.23), we find that $p \geqq (2n^2 - n - 1) + 3n = 2n^2 + 2n - 1$, which contradicts our assumption $p \leqq 2n^2 + 2n - 2$. And hence we have

$$
(5.24) \qquad \lambda^2 = 4c - \tilde{c}.
$$

Substituting (5.24) into the right-hand side of (5.15), we obtain

$$
\begin{aligned}
(5.25) \quad \langle \sigma(X, Y), \sigma(Z, W)\rangle = \; & (2c - \tilde{c})\langle X, Y\rangle\langle Z, W\rangle \\
& + c\{\langle X, W\rangle\langle Y, Z\rangle + \langle X, Z\rangle\langle Y, W\rangle \\
& + \langle IX, W\rangle\langle IY, Z\rangle + \langle IX, Z\rangle\langle IY, W\rangle \\
& + \langle JX, W\rangle\langle JY, Z\rangle + \langle JX, Z\rangle\langle JY, W\rangle \\
& + \langle KX, W\rangle\langle KY, Z\rangle + \langle KX, Z\rangle\langle KY, W\rangle\}
\end{aligned}
$$

for all vector fields $X,\ Y,\ Z$ and W tangent to $M^n(4c; \mathbb{Q})$.

Equation (5.25) implies the following.

$$\begin{aligned}(5.26)\qquad &\langle\sigma(X,Y),\sigma(X,Y)\rangle=\langle\sigma(IX,IY),\sigma(IX,IY)\rangle\\ =\ &\langle\sigma(JX,JY),\sigma(JX,JY)\rangle=\langle\sigma(KX,KY),\sigma(KX,KY)\rangle\\ =\ &(3c-\tilde{c})\langle X,Y\rangle^2+c\{\|X\|^2\|Y\|^2-\langle IX,Y\rangle^2-\langle JX,Y\rangle^2-\langle KX,Y\rangle^2\}.\end{aligned}$$

$$\begin{aligned}(5.27)\quad &\langle\sigma(X,Y),\sigma(IX,IY)\rangle=\langle\sigma(X,Y),\sigma(JX,JY)\rangle=\langle\sigma(X,Y),\sigma(KX,KY)\rangle\\ =\ &(3c-\tilde{c})\langle X,Y\rangle^2+c\{\|X\|^2\|Y\|^2-\langle IX,Y\rangle^2-\langle JX,Y\rangle^2-\langle KX,Y\rangle^2\}.\end{aligned}$$

Thus, in consideration of (5.26) and (5.27), we can get $\sigma(X,Y)=\sigma(IX,IY)=\sigma(JX,JY)=\sigma(KX,KY)$ for all X, Y. And hence, from Lemma 7, we find that the second fundamental form of our immersion is parallel on U. Therefore there occurs the case (3) and (4).

(ii) Lastly, we consider the case of $\lambda^2(x)=c-\tilde{c}$. ¿From Lemma 6 and the same discussion as in the proof of Theorem 5, we can get the conclusion. □

Here we denote the Cayley numbers by $\mathbb{C}ay$, which is an 8-dimensional non-associative division algebra over the real numbers. It has a multiplicative identity and a positive definite symmetric bilinear form $\langle\ ,\ \rangle$. The tangent space of Cayley projective plane $\mathbb{C}ayP^2(c)$ of maximal sectional curvature c may be identified with $V=\mathbb{C}ay\oplus\mathbb{C}ay$. The vector space V has a positive definite symmetric bilinear form $\langle\ ,\ \rangle$ given by $\langle(a,c),(b,d)\rangle=\langle a,b\rangle+\langle c,d\rangle$. The curvature tensor R of $\mathbb{C}ayP^2(c)$ is given by

$$\begin{aligned}(5.28)\qquad &\langle R((a,b),(c,d))(e,f),(g,h)\rangle\\ =\ &\frac{c}{4}\Big(4\langle c,e\rangle\langle a,g\rangle-4\langle a,e\rangle\langle c,g\rangle+\langle ed,gb\rangle-\langle eb,gd\rangle\\ &+\langle ad-cb,gf\rangle+\langle cf,ah\rangle-\langle af,ch\rangle-4\langle b,f\rangle\langle d,h\rangle\\ &+4\langle d,f\rangle\langle b,h\rangle-\langle ad-cb,eh\rangle\Big),\end{aligned}$$

where c is a positive constant (for details, see [7]). $\mathbb{C}ayH^2(c)$ of minimal sectional curvature $c(<0)$ is the noncompact dual of $\mathbb{C}ayP^2(|c|)$.

The following gives a characterization of parallel immersions of the Cayley projective plane into a real space form with low codimension ([16]):

Theorem 7. *We denote by M a connected open submanifold of either the Cayley projective plane or its noncompact dual. Let f be a λ-isotropic immersion of M into a real space form $\widetilde{M}^{16+p}(\tilde{c};\mathbb{R})$. If $p\leqq 10$, then f is a parallel immersion and locally equivalent to one of the following*:

1. *f is the first standard minimal immersion of $\mathbb{C}ayP^2(c)$ into $S^{25}(\tilde{c})$, where $c=4\tilde{c}/3$ and $p=9$,*
2. *f is a parallel immersion defined by*

$$f=f_2\circ f_1:\mathbb{C}ayP^2(c)\xrightarrow{f_1}S^{25}(3c/4)\xrightarrow{f_2}\widetilde{M}^{26}(\tilde{c}:\mathbb{R}),$$

where f_1 *is the first standard minimal immersion,* f_2 *is a totally umbilic immersion,* $(3c/4) \geqq \tilde{c}$ *and* $p = 10$.

Proof. We immediately find from (5.28) that

$$K((a,0),(b,0)) = \langle R((a,0),(b,0))(b,0),(a,0)\rangle = c \quad \text{if } (a,0) \wedge (b,0) \neq 0.$$

So we may apply to the linear subspace $\mathbb{C}ay \oplus \{0\}$ of T_xM. Then Lemma 3 asserts that

$$\lambda^2 = c - \tilde{c}, \tag{5.29}$$

since $p < 35 = (8 \times 9)/2 - 1$. It follows from (5.1), (5.28) and (5.29) that

$$\begin{aligned} &\langle \sigma((a,b),(c,d)), \sigma((e,f),(g,h))\rangle \\ &= (c-\tilde{c})(\langle a,c\rangle\langle e,g\rangle + \langle b,d\rangle\langle f,h\rangle) - \tilde{c}(\langle b,d\rangle\langle e,g\rangle + \langle a,c\rangle\langle f,h\rangle) \\ &\quad + \frac{c}{3}\{\langle b,d\rangle\langle e,g\rangle + \langle a,c\rangle\langle f,h\rangle + \langle a,g\rangle\langle d,f\rangle \\ &\quad + \langle b,h\rangle\langle c,e\rangle + \langle b,f\rangle\langle c,g\rangle + \langle a,e\rangle\langle d,h\rangle\} \\ &\quad + \frac{c}{12}\{\langle cb,eh\rangle + \langle ah,cf\rangle + \langle ad,gf\rangle + \langle ed,gb\rangle + \langle cb,gf\rangle + \langle af,ch\rangle \\ &\quad + \langle ad,eh\rangle + \langle eb,gd\rangle - 2\langle ch,eb\rangle - 2\langle af,gd\rangle - 2\langle cf,gb\rangle - 2\langle ah,ed\rangle\}. \end{aligned} \tag{5.30}$$

For simplicity, we put $X_i = (e_i, 0)$ and $Y_i = (0, e_i)$ for $0 \leqq i \leqq 7$, where $\{e_0 = 1, e_1, \cdots, e_7\}$ is a basis of $\mathbb{C}ay$. By using (5.30), we see that the vectors $\sigma(X_0,X_0), \sigma(X_0,Y_0), \sigma(X_0,Y_1), \cdots, \sigma(X_0,Y_7)$ are nonzero and mutually orthogonal. Then we have $\dim N_x^1 \geqq 9$ for all $x \in M$. We shall prove in the following that f has parallel second fundamental form.

Case (I): $p \leqq 9$. Note that the first normal space coincides with the normal space at each point of M. This, combined with the fact that λ is constant, implies that f has parallel second fundamental form (cf. Lemma 4). Therefore we see that M is locally congruent to a connected open submanifold of $\mathbb{C}ayP^2(c)$ and the immersion is locally equivalent to the first standard minimal immersion (see [8], [21]).

Case (II): $p = 10$. Our discussion is divided into the following two subcases:

Case (II a): $\sigma(X_0,X_0)$ and $\sigma(Y_0,Y_0)$ are linearly independent at the fixed point $x \in M$. So there exists a sufficiently small neighborhood U such that these two vectors are linearly independent at each point of U. We see from (5.30) that $\sigma(Y_0,Y_0), \sigma(X_0,Y_0), \sigma(X_0,Y_1), \cdots, \sigma(X_0,Y_7)$ are mutually orthogonal nonzero vectors so that $\dim N_x^1 \geqq 10$ for each $x \in U$. Therefore the same argument as in case (I) asserts that the manifold M is a parallel submanifold. Hence, we find that M is locally congruent to a connected open submanifold of $\mathbb{C}ayP^2(c)$ which is immersed into some totally umbilic (but not totally geodesic) hypersurface of $\widetilde{M}^{26}(\tilde{c};\mathbb{R})$ through the first standard minimal immersion.

Case (II b): $\sigma(X_0, X_0)$ and $\sigma(Y_0, Y_0)$ are linearly dependent at the fixed point $x \in M$. We have

$$|\langle \sigma(X_0, X_0), \sigma(Y_0, Y_0)\rangle| = \|\sigma(X_0, X_0)\|\|\sigma(Y_0, Y_0)\|$$

which, together with (5.30), implies that $|(c/2) - \tilde{c}| = |c - \tilde{c}|$ so that $c = 4\tilde{c}/3$. Substituting $c = 4\tilde{c}/3$ into (5.30), we get an equation, say, (5.30)', which implies that our immersion is minimal. We note that our immersion is minimal on some sufficiently small neighborhood of the point x, because in Case (II a) our immersion has nonzero constant mean curvature $H = \sqrt{(3c/4) - \tilde{c}}$.

On the other hand, the first standard minimal immersion f of $\mathbb{C}ayP^2(c)$ into $S^{25}(3c/4)$ is a parallel immersion. Moreover, the immersion f also satisfies (5.30)'.

We here recall the following which is so-called Simons equation ([20]):

$$\begin{aligned}(5.31) \qquad (1/2)\Delta\|\sigma\|^2 \;=\; & \|\bar{\nabla}\sigma\|^2 - 2\sum(h^{\alpha}_{ij}h^{\alpha}_{jk}h^{\beta}_{kl}h^{\beta}_{\ell i} - h^{\alpha}_{ij}h^{\beta}_{jk}h^{\alpha}_{k\ell}h^{\beta}_{\ell i}) \\ & - \sum h^{\alpha}_{ij}h^{\beta}_{ij}h^{\alpha}_{k\ell}h^{\beta}_{k\ell} + n\tilde{c}\|\sigma\|^2,\end{aligned}$$

where σ is the second fundamental form of a minimal submanifold M^n of a real space form $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$, h^{α}_{ij} denote the components of σ with respect to a local field of orthonormal frames and Δ denotes the Laplacian.

The above discussion in Case (II b), together with (5.31), implies that the immersion in our case has parallel second fundamental form. Hence we can see that M is locally congruent to a connected open submanifold of $\mathbb{C}ayP^2(c)$ which is immersed into some totally geodesic hypersurface of $S^{26}(3c/4)$ through the first standard minimal immersion (see [8], [21]). □

Remark. Let f be the first standard minimal immersion of $\mathbb{C}ayP^2(c)$ into $S^{25}(3c/4)$. we see from (5.30)' that $(2/\sqrt{c})\sigma(X_0, X_0)$, $(2/\sqrt{c})\sigma(X_0, Y_0)$, $(2/\sqrt{c})$, $\sigma(X_0, Y_1)$, $\cdots$, $\sigma(X_0, Y_7)$ is an orthonormal basis for the first normal space of f. Moreover, again by using (5.30)' we are able to investigate the first normal space of f in detail. Direct computation yields the following: For simplicity, we put ① $= \sigma(X_0, X_0)$, ② $= \sigma(X_0, Y_0)$, ③ $= \sigma(X_0, Y_1), \cdots,$ ⑨ $= \sigma(X_0, Y_7)$.

$$\text{Let } \sigma = \begin{pmatrix} (\sigma(X_i, X_j)) & (\sigma(X_i, Y_j)) \\ (\sigma(Y_i, X_j)) & (\sigma(Y_i, Y_j)) \end{pmatrix}_{0 \leqq i,j \leqq 7.}$$

Then

$$(\sigma(X_i, X_j))_{0\leqq i,j\leqq 7} = -(\sigma(Y_i, Y_j))_{0\leqq i,j\leqq 7} = \begin{pmatrix} ① & 0 & \cdots & 0 \\ 0 & ① & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & ① \end{pmatrix}$$

and

$$(\sigma(X_i,Y_j))_{0\leqq i,j\leqq 7}=\begin{pmatrix} \textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} & \textcircled{6} & \textcircled{7} & \textcircled{8} & \textcircled{9} \\ \textcircled{3} & -\textcircled{2} & \textcircled{5} & -\textcircled{4} & \textcircled{7} & -\textcircled{6} & -\textcircled{9} & \textcircled{8} \\ \textcircled{4} & -\textcircled{5} & -\textcircled{2} & \textcircled{3} & \textcircled{8} & \textcircled{9} & -\textcircled{6} & -\textcircled{7} \\ \textcircled{5} & \textcircled{4} & -\textcircled{3} & -\textcircled{2} & \textcircled{9} & -\textcircled{8} & \textcircled{7} & -\textcircled{6} \\ \textcircled{6} & -\textcircled{7} & -\textcircled{8} & -\textcircled{9} & -\textcircled{2} & \textcircled{3} & \textcircled{4} & \textcircled{5} \\ \textcircled{7} & \textcircled{6} & -\textcircled{9} & \textcircled{8} & -\textcircled{3} & -\textcircled{2} & -\textcircled{5} & \textcircled{4} \\ \textcircled{8} & \textcircled{9} & \textcircled{6} & -\textcircled{7} & -\textcircled{4} & \textcircled{5} & -\textcircled{2} & -\textcircled{3} \\ \textcircled{9} & -\textcircled{8} & \textcircled{7} & \textcircled{6} & -\textcircled{5} & -\textcircled{4} & \textcircled{3} & -\textcircled{2} \end{pmatrix}.$$

We here note that the above matrix $(\sigma(X_i,Y_j))_{0\leqq i,j\leqq 7}$ coincides with the table of the Cayley numbers.

6. Characterization of parallel immersions of rank one symmetric spaces (II)

In this section, we study parallel immersions of every rank one symmetric space M^n into a real space form $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$ from another point of view. We characterize these parallel immersions by using inequalities related to mean curvatures H.

We first prove the following ([3]):

Theorem 8. *Let f be an isotropic immersion of an $n(\geqq 2)$-dimensional compact oriented real space form $M^n(c;\mathbb{R})$ of curvature c into an $(n+p)$-dimensional space form $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$ of curvature $\tilde{c}$. Let Δ denote the Laplacian on $M^n(c;\mathbb{R})$. Suppose that*

(i) $H^2 \leqq \frac{2(n+1)}{n}c-\tilde{c}$,

(ii) $0 \leqq (1-n)\Delta H^2 + n\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle$.

Then $M^n(c;\mathbb{R})$ is a parallel submanifold of $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$ and f is locally equivalent to one of the following.

(1) *f is a totally umbilic immersion of $M^n(c;\mathbb{R})$ into $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$, $c \geqq \tilde{c}$. Here $H^2 \equiv c-\tilde{c}$.*

(2) $f = f_2\circ f_1 :\ S^n(c) \xrightarrow{f_1} S^{n+\frac{n(n+1)}{2}-1}\left(\frac{2(n+1)}{n}c\right) \xrightarrow{f_2} \widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$,
where f_1 is the second standard minimal immersion, f_2 is a totally umbilic immersion and $\frac{2(n+1)}{n}c \geqq \tilde{c}$. Here $H^2 \equiv \frac{2(n+1)}{n}c-\tilde{c}$.

Proof. We first generalize Simons equation (5.31) for every Riemannian submanifold M^n of $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$ as follows:

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\bar{\nabla}\sigma\|^2 - \tilde{c}n^2H^2 + \tilde{c}n\|\sigma\|^2 + \sum_{i,j,k=1}^{n}\langle D_{e_i}\Big(D_{e_j}\big(\sigma(e_k,e_k)\big)\Big),\sigma(e_i,e_j)\rangle \tag{6.1}$$

$$
\begin{aligned}
&+\sum_{i,j,k,\ell=1}^{n}\Big[2\langle\sigma(e_k,e_j),\sigma(e_i,e_\ell)\rangle\langle\sigma(e_l,e_k),\sigma(e_i,e_j)\rangle\\
&-2\langle\sigma(e_k,e_j),\sigma(e_k,e_l)\rangle\langle\sigma(e_l,e_i),\sigma(e_i,e_j)\rangle\\
&+\langle\sigma(e_k,e_k),\sigma(e_i,e_l)\rangle\langle\sigma(e_l,e_j),\sigma(e_i,e_j)\rangle\\
&-\langle\sigma(e_i,e_j),\sigma(e_l,e_k)\rangle\langle\sigma(e_l,e_k),\sigma(e_i,e_j)\rangle\Big].
\end{aligned}
$$

In the following we study a λ-isotropic immersion $f : M^n(c;\mathbb{R}) \to \widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$. Our computation shows that the second fundamental form of the immersion f satisfies

$$
\begin{aligned}
\langle\sigma(X,Y),\sigma(Z,W)\rangle \;=\; &\frac{c-\tilde{c}}{3}\big(2\langle X,Y\rangle\langle Z,W\rangle-\langle X,Z\rangle\langle Y,W\rangle\\
&-\langle X,W\rangle\langle Y,Z\rangle\big)+\frac{\lambda^2}{3}\big(\langle X,Y\rangle\langle Z,W\rangle\\
&+\langle X,Z\rangle\langle Y,W\rangle+\langle X,W\rangle\langle Y,Z\rangle\big),
\end{aligned}
\tag{6.2}
$$

where $X,\ Y,\ Z$ and W are vector fields tangent to $M^n(c;\mathbb{R})$.

Equation (6.2) yields the following:

$$
3\|\sigma(X,Y)\|^2+c-\tilde{c} \;=\; \lambda^2, \tag{6.3}
$$

where X and Y are orthonormal vector fields tangent to $M^n(c;\mathbb{R})$.

$$
H^2=\frac{2(n-1)(c-\tilde{c})+\lambda^2(n+2)}{3n}. \tag{6.4}
$$

$$
\|\sigma\|^2=n^2H^2-n(n-1)(c-\tilde{c}). \tag{6.5}
$$

Here we compute the fourth term of the right-hand side of (6.1). In order to compute this term easily without loss of generality we use the condition that $\nabla e_i=0$ at the point $x,\ i\in\{1,\cdots,n\}$. It follows from the Codazzi equation $(\bar{\nabla}_X\sigma)(Y,Z)=(\bar{\nabla}_Y\sigma)(X,Z)$, (6.2) and (6.4) that

$$
\begin{aligned}
&\sum_{i,j,k=1}^{n}\langle D_{e_i}\Big(D_{e_j}\big(\sigma(e_k,e_k)\big)\Big),\sigma(e_i,e_j)\rangle\\
=\;&\sum_{i,j,k=1}^{n}\Big[e_i\Big(\langle D_{e_j}\big(\sigma(e_k,e_k)\big),\sigma(e_i,e_j)\rangle\Big)-\langle D_{e_j}\big(\sigma(e_k,e_k)\big),D_{e_i}\big(\sigma(e_i,e_j)\big)\rangle\Big]\\
=\;&\sum_{i,j,k=1}^{n}\Big[e_i\Big(e_j\big(\langle\sigma(e_k,e_k),\sigma(e_i,e_j)\rangle\big)\Big)-e_i\Big(\langle\sigma(e_k,e_k),D_{e_j}\big(\sigma(e_i,e_j)\big)\rangle\Big)\\
&-\langle D_{e_j}\big(\sigma(e_k,e_k)\big),D_{e_j}\big(\sigma(e_i,e_i)\big)\rangle\Big]
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{i,j,k=1}^{n} \Big[e_i\Big(e_j\big(\langle\sigma(e_k,e_k),\sigma(e_i,e_j)\rangle\big)\Big) - \langle\sigma(e_k,e_k), D_{e_i}\Big(D_{e_i}\Big(\sigma(e_j,e_j)\Big)\Big)\rangle \\
&\quad - 2\langle D_{e_j}\big(\sigma(e_k,e_k)\big), D_{e_j}\big(\sigma(e_i,e_i)\big)\rangle\Big] \\
&= \sum_{i,j,k=1}^{n} \Big[e_i\Big(e_j\big[\frac{c-\tilde{c}}{3}(2\delta_{ij}-2\delta_{ki}\delta_{kj}) + \frac{3nH^2-2(n-1)(c-\tilde{c})}{3(n+2)}(\delta_{ij}+2\delta_{ki}\delta_{kj})\big]\Big) \\
&\quad - \langle\sigma(e_k,e_k), D_{e_i}\Big(D_{e_i}\Big(\sigma(e_j,e_j)\Big)\Big)\rangle - 2\langle D_{e_j}\big(\sigma(e_k,e_k)\big), D_{e_j}\big(\sigma(e_i,e_i)\big)\rangle\Big] \\
&= n\Delta H^2 - n^2\langle\mathfrak{h},\Delta\mathfrak{h}\rangle - 2n^2\|D\mathfrak{h}\|^2 \\
&= n\Delta H^2 + n^2\langle\mathfrak{h},\Delta\mathfrak{h}\rangle - n^2(2\|D\mathfrak{h}\|^2 + 2\langle\mathfrak{h},\Delta\mathfrak{h}\rangle) \\
&= n(1-n)\Delta H^2 + n^2\langle\mathfrak{h},\Delta\mathfrak{h}\rangle,
\end{aligned}$$

so that

$$\sum_{i,j,k=1}^{n} \langle D_{e_i}\Big(D_{e_j}\big(\sigma(e_k,e_k)\big)\Big), \sigma(e_i,e_j)\rangle = n(1-n)\Delta H^2 + n^2\langle\mathfrak{h},\Delta\mathfrak{h}\rangle. \tag{6.6}$$

Using (6.1), (6.2), (6.4), (6.5) and (6.6), we obtain the following equation:

$$\begin{aligned}
\frac{1}{2}\Delta\|\sigma\|^2 &= \|\bar{\nabla}\sigma\|^2 - \frac{n^3(n-1)}{n+2}\Big(H^2-c+\tilde{c}\Big)\Big(H^2-\frac{2(n+1)}{n}c+\tilde{c}\Big) \\
&\quad + n\Big((1-n)\Delta H^2 + n\langle\mathfrak{h},\Delta\mathfrak{h}\rangle\Big).
\end{aligned}$$

Here it follows from (6.3) and (6.4) that

$$H^2 - c + \tilde{c} = \frac{n+2}{n}(\lambda^2 - c + \tilde{c}) = \frac{n+2}{n}\|\sigma(X,Y)\|^2 \geqq 0$$

for each orthonormal pair of vectors X and Y.

This, together with the inequalities (i), (ii) in the assumption of Theorem 8 and a well-known Hopf's lemma, implies that $\bar{\nabla}\sigma = 0$. Moreover we have $H^2 \equiv c - \tilde{c}$ or $H^2 \equiv \frac{2(n+1)}{n}c - \tilde{c}$. Therefore we get the conclusion. □

Remarks 1.

(I) We comment on the inequality (ii): $0 \leqq (1-n)\Delta H^2 + n\langle\mathfrak{h},\Delta\mathfrak{h}\rangle$. By easy computation we know that this inequality means that the mean curvature vector $\mathfrak{h}$ is parallel when the mean curvature H is constant.

In fact, when H is constant, from this inequality we see that $0 \leqq \langle\mathfrak{h},\Delta\mathfrak{h}\rangle$. Again, by using the condition that H is constant, we get $\|D\mathfrak{h}\|^2 = -\langle\mathfrak{h},\Delta\mathfrak{h}\rangle$. It follows from these two inequalities that $D\mathfrak{h} = 0$.

(II) As an immediate consequence of Theorem 8 we obtain the following:

Corollary. *Let f be an isotropic immersion of an $n(\geqq 2)$-dimensional compact oriented real space form $M^n(c;\mathbb{R})$ into an $(n+p)$-dimensional real space form $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$. Suppose that*
(i) $H^2 \leqq \frac{2(n+1)}{n}c - \tilde{c}$,
(ii)′ *the mean curvature vector $\mathfrak{h}$ of f is parallel.*

Then $M^n(c;\mathbb{R})$ is a parallel submanifold of $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$ and f is locally equivalent to one of the following.

(1) *f is a totally umbilic immersion of $M^n(c;\mathbb{R})$ into $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$, $c \geqq \tilde{c}$. Here $H^2 \equiv c - \tilde{c}$.*

(2) $f = f_2 \circ f_1 : \ S^n(c) \xrightarrow{f_1} S^{n+\frac{n(n+1)}{2}-1}\left(\frac{2(n+1)}{n}c\right) \xrightarrow{f_2} \widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$,
where f_1 is the second standard minimal immersion, f_2 is a totally umbilic immersion and $\frac{2(n+1)}{n}c \geqq \tilde{c}$. Here $H^2 \equiv \frac{2(n+1)}{n}c - \tilde{c}$.

(III) We show that Theorem 8 is no longer true if we omit the condition (ii) in the hypothesis of Theorem 8. We recall the following example([14]):

Example. Let $\chi_1 : S^n\big(n/\big(2(n+1)\big)\big) \to S^{n+(n(n+1)/2)-1}(1)$ be the second standard minimal immersion and $\chi_2 : S^n\big(n/\big(2(n+1)\big)\big) \to S^n\big(n/\big(2(n+1)\big)\big)$ the identity mapping. Using these minimal immersions, for $t \in (0,\pi/2)$ we define the following minimal immersion

(6.7)
$$\chi_t(=(\chi_1,\chi_2)) : S^n\Big(\frac{n}{2(n+1)}\Big) \to S^{n+\frac{n(n+1)}{2}-1}\Big(\frac{1}{\cos^2 t}\Big) \times S^n\Big(\frac{n}{2(n+1)\sin^2 t}\Big).$$

Here the differential mapping $(\chi_t)_*$ of χ_t is given by $(\chi_t)_*X = (\cos t \cdot (\chi_1)_*X, \sin t \cdot (\chi_2)_*X)$ for each $X \in TS^n\big(n/\big(2(n+1)\big)\big)$. The product space of spheres in (6.7) can be imbedded into a sphere as a Clifford hypersurface:

$$\text{(6.8)}\quad S^{n+\frac{n(n+1)}{2}-1}\Big(\frac{1}{\cos^2 t}\Big) \times S^n\Big(\frac{n}{2(n+1)\sin^2 t}\Big) \to S^{n+\frac{n(n+3)}{2}}\Big(\frac{n}{n+(n+2)\sin^2 t}\Big).$$

Combining (6.7) with (6.8), we obtain the following isometric immersion f_t:

$$\text{(6.9)}\qquad f_t : S^n\Big(\frac{n}{2(n+1)}\Big) \to S^{n+\frac{n(n+3)}{2}}\Big(\frac{n}{n+(n+2)\sin^2 t}\Big).$$

By virtue of the result in [14] we obtain the following properties of f_t for each $t \in (0,\pi/2)$.

(a) The mean curvature H_t of f_t is given by

$$H_t = \|\mathfrak{h}_t\| = \frac{(n+2)\sin t\cos t}{\sqrt{2(n+1)(n+(n+2)\sin^2 t)}} \neq 0.$$

(b) The mean curvature vector $\mathfrak{h}_t$ of f_t is not parallel. The length of the derivative of $\mathfrak{h}_t$ is given by:

$$\|D\mathfrak{h}_t\|^2 = \frac{n(n+2)^2}{4(n+1)^2}\sin^2 t\cos^2 t \neq 0.$$

(c) f_t is constant λ_t-isotropic. λ_t is given by

$$\lambda_t = \sqrt{\cos^4 t\frac{n-1}{n+1} + \frac{(\tilde{c}_1\cos^2 t - \tilde{c}_2\sin^2 t)^2}{\tilde{c}_1 + \tilde{c}_2}} \neq 0,$$

where $\tilde{c}_1 = \frac{1}{\cos^2 t}$ and $\tilde{c}_2 = \frac{n}{2(n+1)\sin^2 t}$.

Now, in particular we set $\cos t = 1/\sqrt{n+1}$ and $\sin t = \sqrt{n/(n+1)}$. Then we have the following isometric immersion f:

$$\text{(6.10)}\quad f : S^n\Big(\frac{n}{2(n+1)}\Big) \to S^{n+\frac{n(n+1)}{2}-1}(n+1) \times S^n\Big(\frac{1}{2}\Big) \to S^{n+\frac{n(n+3)}{2}}\Big(\frac{n+1}{2n+3}\Big).$$

We shall show that the isometric immersion f given by (6.10) satisfies the inequality (i) but not the inequality (ii) in the statement of Theorem 8.

In fact, we have

$$\text{(i)}\qquad H^2 - \frac{2(n+1)}{n}c + \tilde{c} = \frac{(n+2)^2}{2(2n+3)(n+1)^2} - 1 + \frac{n+1}{2n+3} = -\frac{n(n+2)}{2(n+1)^2} < 0,$$

$$\text{(ii)}\qquad (1-n)\Delta H^2 + n\langle\mathfrak{h}, \Delta\mathfrak{h}\rangle = n\langle\mathfrak{h}, \Delta\mathfrak{h}\rangle = -n\|D\mathfrak{h}\|^2 = -\frac{n^3(n+2)^2}{4(n+1)^4} < 0.$$

This shows that Theorem 8 does not hold without the inequality (ii).

We next prove the following ([5]):

Theorem 9. *Let f be a λ-isotropic immersion of a complex $n(\geqq 2)$-dimensional complex space form $M^n(4c;\mathbb{C})$ of constant holomorphic sectional curvature $4c$ into a $(2n+p)$-dimensional real space form $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$. Suppose that $H^2 \leqq 2(n+1)c/n - \tilde{c}$.*
Then the immersion f is a parallel immersion and locally equivalent to one of the following:

(1) *f is a totally geodesic immersion of $\mathbb{C}^n(=\mathbb{R}^{2n})$ into $\mathbb{R}^{2n+p}$, where $H \equiv 0$,*

(2) *f is a totally umbilic immersion of $\mathbb{C}^n(=\mathbb{R}^{2n})$ into $\mathbb{R}H^{2n+p}(\tilde{c})$, where $H^2 \equiv -\tilde{c}$,*

(3) $f = f_2 \circ f_1 : \mathbb{C}P^n(4c) \xrightarrow{f_1} S^{n^2+2n-1}(2(n+1)c/n) \xrightarrow{f_2} \widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$, *where* f_1 *is the first standard minimal immersion,* f_2 *is a totally umbilic immersion,* $H^2 \equiv (2(n+1)c/n) - \tilde{c}$ *and* $2(n+1)c/n \geqq \tilde{c}$.

Proof. Equation (5.4) yields the following.

$$H^2 = \frac{(n+1)\lambda^2 + 2(n+1)c - (2n-1)\tilde{c}}{3n}. \tag{6.11}$$

$$\|\sigma(X, JX)\|^2 = \frac{\lambda^2 - 4c + \tilde{c}}{3}, \tag{6.12}$$

where X is an arbitrary unit vector field tangent to $M^n(4c;\mathbb{C})$.

It follows from (6.11) and (6.12) that

$$H^2 - \frac{2(n+1)}{n}c + \tilde{c} = \frac{n+1}{n}\|\sigma(X, JX)\|^2 \geqq 0,$$

where X is an arbitrary unit vector field tangent to $M^n(4c;\mathbb{C})$.

Therefore, by hypothesis we can find $H^2 \equiv \frac{2(n+1)}{n}c - \tilde{c}$ and get the following equation

$$\sigma(X, JX) = 0 \quad \text{for each } X. \tag{6.13}$$

Equation (6.13) is equivalent to the condition (ii) in Lemma 5. Thus we can get the conclusion. □

Remark. It is interesting to compare Theorem 8 with Theorem 9. Note that the statement of Theorem 9 is a local version. Moreover, the assumption of Theorem 9 does not need an inequality like the condition (ii) in Theorem 8.

The following is similar to Theorem 9 ([5]):

Theorem 10. *Let* f *be a* λ*-isotropic immersion of a quaternionic* $n(\geqq 2)$*- dimensional quaternionic space form* $M^n(4c;\mathbb{Q})$ *of constant quaternionic sectional curvature* $4c$ *into a* $(4n+p)$*-dimensional real space form* $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$ *of constant sectional curvature* $\tilde{c}$*. Suppose that* $H^2 \leqq 2(n+1)c/n - \tilde{c}$.
Then the immersion f *is a parallel immersion and locally equivalent to one of the following*:

(1) f *is a totally geodesic immersion of* $\mathbb{Q}^n(=\mathbb{R}^{4n})$ *into* $\mathbb{R}^{4n+p}$*, where* $H \equiv 0$,

(2) f *is a totally umbilic immersion of* $\mathbb{Q}^n(=\mathbb{R}^{4n})$ *into* $\mathbb{R}H^{4n+p}(\tilde{c})$*, where* $H^2 \equiv -\tilde{c}$,

(3) $f = f_2 \circ f_1 : \mathbb{Q}P^n(4c) \xrightarrow{f_1} S^{2n^2+3n-1}(2(n+1)c/n) \xrightarrow{f_2} \widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$, *where* f_1 *is the first standard minimal immersion,* f_2 *is a totally umbilic immersion,* $H^2 \equiv (2(n+1)c/n) - \tilde{c}$ *and* $2(n+1)c/n \geqq \tilde{c}$.

Proof. Equation (5.15) yields the following.

$$H^2 = \frac{(2n+1)\lambda^2 + 4(n+2)c - (4n-1)\tilde{c}}{6n}. \tag{6.14}$$

$$\|\sigma(X, IX)\|^2 = \|\sigma(X, JX)\|^2 = \|\sigma(X, KX)\|^2 = \frac{\lambda^2 - 4c + \tilde{c}}{3}, \tag{6.15}$$

where X is an arbitrary unit vector field tangent to $M^n(4c; \mathbb{Q})$.

It follows from (6.14) and (6.15) that

$$\begin{aligned} H^2 - \frac{2(n+1)}{n}c + \tilde{c} &= \frac{2n+1}{2n}\|\sigma(X, IX)\|^2 \\ &= \frac{2n+1}{2n}\|\sigma(X, JX)\|^2 = \frac{2n+1}{2n}\|\sigma(X, KX)\|^2 \geqq 0, \end{aligned}$$

where X is an arbitrary unit vector field tangent to $M^n(4c; \mathbb{Q})$.

Therefore, by hypothesis we can find $H^2 \equiv \frac{2(n+1)}{n}c - \tilde{c}$ and get the following equation

$$\sigma(X, IX) = \sigma(X, JX) = \sigma(X, KX) = 0 \quad \text{for each } X. \tag{6.16}$$

Equation (6.16) is equivalent to the condition (ii) in Lemma 7. Therefore we can get the desirable result. □

Finally we prove the following ([6]):

Theorem 11. *Let f be a λ-isotropic immersion of $\mathbb{C}ayP^2(c)$ of maximal sectional curvature c into $\widetilde{M}^m(\tilde{c}; \mathbb{R})$ of constant sectional curvature $\tilde{c}$. Suppose that*

(i) $8H^2 \leqq 9c - 8\tilde{c}$,

(ii) $0 \leqq 16\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle - 15\Delta H^2$, *where Δ denotes the Laplacian on $\mathbb{C}ayP^2(c)$.*

Then $\mathbb{C}ayP^2(c)$ is a parallel submanifold of $\widetilde{M}^m(\tilde{c})$ and the immersion f is decomposed as:

$$f = f_2 \circ f_1 : \ \mathbb{C}ayP^2(c) \xrightarrow{f_1} S^{25}(3c/4) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c}; \mathbb{R}),$$

where f_1 is the first standard minimal immersion, f_2 is a totally umbilic immersion and $3c/4 \geqq \tilde{c}$. Moreover, the mean curvature H of f is expressed as: $8H^2 \equiv 6c - 8\tilde{c} (< 9c - 8\tilde{c})$.

Proof. For the sake of simplicity, we put $E_i = (e_i, 0)$ and $E_{8+i} = (0, e_i)$ for $i = 1, \cdots, 8$, where $\{e_1 = 1, e_2, \cdots, e_8\}$ is a basis of $\mathbb{C}ay$.

Equation (5.30) yields the following.

$$\|\sigma\|^2 = -48c + 96\lambda^2 + 80\tilde{c}. \tag{6.17}$$

$$8H^2 = 3c + 3\lambda^2 - 5\tilde{c}. \tag{6.18}$$

$$4H^2 - 3c + 4\tilde{c} = \frac{9}{2}\|\sigma(E_1, E_2)\|^2 (\geqq 0). \tag{6.19}$$

Here we compute the fourth term of the right-hand side in (6.1). In order to compute this term easily we use again the condition that $\nabla E_k = 0$ at the point x, $k \in \{1, \cdots, 16\}$, where we choose a local field of orthonormal frames $\{E_1, \cdots, E_8, E_9, \cdots, E_{16}\}$ on $\mathbb{C}ayP^2(c)$. It follows from the Codazzi equation $(\bar{\nabla}_X \sigma)(Y, Z) = (\bar{\nabla}_Y \sigma)(X, Z)$, (5.30) and (6.18) that

$$\begin{aligned}
&\sum_{i,j,k=1}^{16} \langle D_{E_i}\Big(D_{E_j}\big(\sigma(E_k, E_k)\big)\Big), \sigma(E_i, E_j)\rangle \\
= &\sum_{i,j,k=1}^{16} \Big[E_i\Big(\langle D_{E_j}\big(\sigma(E_k, E_k)\big), \sigma(E_i, E_j)\rangle\Big) - \langle D_{E_j}\big(\sigma(E_k, E_k)\big), D_{E_i}\big(\sigma(E_i, E_j)\big)\rangle\Big] \\
= &\sum_{i,j,k=1}^{16} \Big[E_i\Big(E_j\big(\langle\sigma(E_k, E_k), \sigma(E_i, E_j)\rangle\big)\Big) - E_i\Big(\langle\sigma(E_k, E_k), D_{E_j}\big(\sigma(E_i, E_j)\big)\rangle\Big) \\
&- \langle D_{E_j}\big(\sigma(E_k, E_k)\big), D_{E_j}\big(\sigma(E_i, E_i)\big)\rangle\Big] \\
= &\sum_{i,j,k=1}^{16} \Big[E_i\Big(E_j\big(\langle\sigma(E_k, E_k), \sigma(E_i, E_j)\rangle\big)\Big) - \langle\sigma(E_k, E_k), D_{E_i}\Big(D_{E_i}\Big(\sigma(E_j, E_j)\Big)\Big)\rangle \\
&- 2\langle D_{E_j}\big(\sigma(E_k, E_k)\big), D_{E_j}\big(\sigma(E_i, E_i)\big)\rangle\Big] \\
= &\sum_{i,j=1}^{16} E_i\Big(E_j\big((6c + 6\lambda^2 - 10\tilde{c})\delta_{ij}\big)\Big) - (16)^2\langle\mathfrak{h}, \Delta\mathfrak{h}\rangle - 2(16)^2\|D\mathfrak{h}\|^2 \\
= \ &16\Delta H^2 - (16)^2\langle\mathfrak{h}, \Delta\mathfrak{h}\rangle - (16)^2(\Delta H^2 - 2\langle\mathfrak{h}, \Delta\mathfrak{h}\rangle) \\
= \ &16(16\langle\mathfrak{h}, \Delta\mathfrak{h}\rangle - 15\Delta H^2).
\end{aligned}$$

Therefore, we can get the following equation

$$\sum_{i,j,k=1}^{16} \langle D_{E_i}\Big(D_{E_j}\big(\sigma(E_k, E_k)\big)\Big), \sigma(E_i, E_j)\rangle = 16\big(16\langle\mathfrak{h}, \Delta\mathfrak{h}\rangle - 15\Delta H^2\big). \tag{6.20}$$

Using (5.30), (6.1), (6.17), (6.18) and (6.20), we obtain the following equation:

$$\begin{aligned}
\frac{1}{2}\Delta\|\sigma\|^2 \ = \ &\|\bar{\nabla}\sigma\|^2 + 16\big(16\langle\mathfrak{h}, \Delta\mathfrak{h}\rangle - 15\Delta H^2\big) \\
&- \frac{320}{3}\big(4H^2 - 3c + 4\tilde{c}\big)\big(8H^2 - 9c + 8\tilde{c}\big).
\end{aligned}$$

This, together with (6.19), the assumption of our Theorem and a well-known Hopf's lemma, implies that $\bar{\nabla}\sigma = 0$. Therefore we obtain the conclusion, so that the mean curvature H of our immersion f satisfies $H^2 = (3c/4) - \tilde{c}$. We here remark that $8H^2 - 9c + 8\tilde{c} = -3c < 0$. $\square$

Remarks 2.

(I) By the same reason as in (I) of Remarks 1 we see that the inequality (ii) in Theorem 11 implies that the mean curvature vector $\mathfrak{h}$ is parallel when the mean curvature H is constant.

(II) As an immediate consequence of Theorem 11 we have the following:

Corollary. *Let f be a λ-isotropic immersion of $\mathbb{C}ayP^2(c)$ of maximal sectional curvature c into $\widetilde{M}^m(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$. Suppose that*

(i) $8H^2 \leqq 9c - 8\tilde{c}$,

(ii)$'$ *the mean curvature vector $\mathfrak{h}$ of f is parallel.*

Then $\mathbb{C}ayP^2(c)$ is a parallel submanifold of $\widetilde{M}^m(\tilde{c})$ and the immersion f is decomposed as:

$$f = f_2 \circ f_1 : \ \mathbb{C}ayP^2(c) \xrightarrow{f_1} S^{25}(3c/4) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c};\mathbb{R}),$$

where f_1 is the first standard minimal immersion, f_2 is a totally umbilic immersion and $3c/4 \geqq \tilde{c}$. Moreover, the mean curvature H of f is expressed as: $8H^2 \equiv 6c - 8\tilde{c}(< 9c - 8\tilde{c})$.

(III) Theorem 11 is not true without the condition (ii) in the hypothesis of Theorem 11. We recall the following example similar to that of Remarks 1.

Example. Let $\chi_1 : \mathbb{C}ayP^2(c) \to S^{25}(3c/4)$ be the first standard minimal immersion and $\chi_2 : \mathbb{C}ayP^2(c) \to S^{323}(13c/8)$ the second standard minimal immersion. Using these minimal immersions, for $t \in (0, \pi/2)$ we define the following minimal immersion

$$\chi_t(= (\chi_1, \chi_2)) : \mathbb{C}ayP^2(c) \to S^{25}\Big(\frac{3c}{4\cos^2 t}\Big) \times S^{323}\Big(\frac{13c}{8\sin^2 t}\Big). \tag{6.21}$$

The product of spheres in (6.21) can be imbedded into a sphere as a Clifford hypersurface:

$$S^{25}\Big(\frac{3c}{4\cos^2 t}\Big) \times S^{323}\Big(\frac{13c}{8\sin^2 t}\Big) \to S^{349}\Big(\frac{39c}{4(6\sin^2 t + 13\cos^2 t)}\Big). \tag{6.22}$$

Combining (6.21) with (6.22), we obtain the following isometric immersion f_t:

$$f_t : \mathbb{C}ayP^2(c) \to S^{349}\Big(\frac{39c}{4(6\sin^2 t + 13\cos^2 t)}\Big).$$

Then we obtain the following properties of f_t for each $t \in (0, \pi/2)$.

(a) The mean curvature H_t of f_t is given by

$$H_t = \|\mathfrak{h}_t\| = \frac{7\cos t \sin t\sqrt{c}}{\sqrt{8(6\sin^2 t + 13\cos^2 t)}} > 0.$$

(b) The mean curvature vector $\mathfrak{h}_t$ of f_t is not parallel. The length of the derivative of $\mathfrak{h}_t$ is given by

$$\|D\mathfrak{h}_t\|^2 = \frac{49}{4}c^2 \sin^2 t \cos^2 t > 0.$$

(c) f_t is constant λ_t-isotropic. λ_t is given by

$$\lambda_t = \sqrt{\frac{c}{4}\cos^4 t + \frac{41c}{24}\sin^4 t + \frac{49c\cos^2 t \sin^2 t}{8(6\sin^2 t + 13\cos^2 t)}} > 0.$$

Now, in particular we set $\cos t = \sqrt{10/11}$ and $\sin t = \sqrt{1/11}$. Then we have the following isometric immersion f.

$$f : \mathbb{C}ayP^2(c) \to S^{25}\Big(\frac{33c}{40}\Big) \times S^{323}\Big(\frac{143c}{8}\Big) \to S^{349}\Big(\frac{429c}{544}\Big). \tag{6.23}$$

We shall show that the isometric immersion f given by (6.23) satisfies the inequality (i) but not the inequality (ii) in the statement of Theorem 11.

In fact, we have

(i) $8H^2 - 9c + 8\tilde{c} = -\frac{26}{11}c < 0$,

(ii) $16\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle - 15\Delta H^2 = 16\langle \mathfrak{h}, \Delta\mathfrak{h}\rangle = -16\|D\mathfrak{h}\|^2 = -\frac{1960}{121}c^2 < 0$.

7. Problems

We first pose the following problem related to Theorem 4:

Problem A. *Let f be a λ-isotropic immersion of a real space form $M^n(c;\mathbb{R})$ into a real space form $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$. If $p \leqq n(n+1)/2$, is f locally equivalent to one of the following?*

(1) *f is a totally umbilic immersion of $M^n(c;\mathbb{R})$ into $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$, where $p \leqq n(n+1)/2$ and $c \geqq \tilde{c}$.*

(2) *f is the second standard minimal immersion of $S^n(c)$ into $S^{n+p}(\tilde{c})$, where $p = n(n+1)/2 - 1$ and $\tilde{c} = 2(n+1)c/n$.*

(3) *f is a parallel immersion defined by*

$$f = f_2 \circ f_1 : S^n(c) \xrightarrow{f_1} S^{n(n+3)/2-1}(2(n+1)c/n) \xrightarrow{f_2} \widetilde{M}^{n+p}(\tilde{c};\mathbb{R}),$$

where f_1 is the second standard minimal immersion, f_2 is a totally umbilic immersion, $p = n(n+1)/2$ and $2(n+1)c/n \geqq \tilde{c}$.

Needless to say every (λ-) isotropic submanifold M^n of a standard sphere $S^{n+p}(c)$ is also ($\sqrt{\lambda^2+c}$-) isotropic in Euclidean space $\mathbb{R}^{n+p+1}$. This submanifold of Euclidean space is said to be *spherical*. In study of isotropic submanifolds until now we have no nonzero isotropic submanifolds of Euclidean space $\mathbb{R}^N$, which are *not spherical*. So we pose the following problem:

Problem B. *Find examples of non-spherical nonzero isotropic submanifolds in Euclidean space.*

The following problem is related to Theorems 4, 5, 6 and 7:

Problem C. *Let M^n be an n-dimensional Riemannian symmetric space of rank one which is isotropically immersed into an $(n+p)$-dimensional standard sphere $S^{n+p}(c)$. Give a sufficient condition that M^n has parallel second fundamental form in $S^{n+p}(c)$ by using an inequality related to the codimension p.*

In Theorems 4, 5, 6 and 7, we solve Problem 3 one by one for each rank one symmetric space. The following problem is a generalization of Problem C:

Problem D. *Let M^n be an n-dimensional locally symmetric space which is isotropically immersed into an $(n+p)$-dimensional standard sphere $S^{n+p}(c)$. Give a sufficient condition that M^n has parallel second fundamental form in $S^{n+p}(c)$ by using an inequality related to the codimension p.*

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