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Generalizations of V-rings

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Abstract. In this paper, we introduce a new notion which we call a generalized weakly ideal. We also investigate characterizations of strongly regular rings with the condition that every maximal left ideal is a generalized weakly ideal. It is proved that R is a strongly regular ring if and only if R is a left GP-V-ring whose every maximal left (right) ideal is a generalized weakly ideal. Furthermore, if R is a left SGPF ring, and every maximal left (right) ideal is a generalized weakly ideal, it is shown that $R/J(R)$ is strongly regular. Several known results are improved and extended.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. For a nonempty subset X of R, the left (right) annihilator of X in R will be denoted by $l(X)$ $(r(X))$. If $X = \{a\}$, we always abbreviate it to $l(a)$ $(r(a))$. $J(R)$ denotes the Jacobson radical of R, and the ideal means a two-sided ideal of R. R is called strongly regular if, for any a in R, there exists b in R such that $a = ba^2$. This notion was introduced by Arens and Kaplansky ([1]). Since then, strongly regular rings have drawn the attention of many authors ([1], [4]-[6]). Recall that

(1) R is called reduced if it contains no non-zero nilpotent element.

(2) A left R-module M is called YJ-injective if, for any $0 \neq a \in R$, there exists a positive integer n with $a^n \neq 0$ such that any left R-homomorphism from Ra^n to M extends to one from R to M ([9]).

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(3) A left R-module M is called GP-injective if, for any $a \in R$, there exists a positive integer n such that any left R-homomorphism from Ra^n to M extends to one from R to M ([7]).

(4) R is called a left GP-V-ring if every simple left R-module is YJ-injective [13]. Note that GP-injectivity differs from YJ-injectivity in this paper.

Now, we introduce a new notion.

Definition 1.1. Let R be a ring, and L a left ideal of R. L is said to be a generalized weakly ideal (briefly GW-ideal) if, for any a in L, there exists a positive integer n such that $a^nR \subseteq L$. Similarly, the notion of GW-ideal for a right ideal K of R can be defined.

The following examples show that a GW-ideal of a ring need not to be an ideal and a left (or right) ideal of a ring need not to be a GW-ideal.

Example 1.2. Let $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & 0 \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{Z}_2 \right\}$. Then $\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in R and $R\alpha$ is a left nilpotent ideal of R. This yields that $R\alpha$ is a GW-ideal, but it is not an ideal of R.

Example 1.3. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \; a, b, c \in \mathbb{Z}_2 \right\}$. It is clear that $K = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \middle| \; a \in \mathbb{Z}_2 \right\}$ is a right ideal of R and $L = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \middle| \; a \in \mathbb{Z}_2 \right\}$ is a left ideal of R, but neither K nor L is a GW-ideal of R.

In this paper, we first prove that R is a strongly regular ring if and only if R is a left GP-V-ring whose every maximal left (right) ideal is a GW-ideal. This result extends Theorem 1 in [10] and Theorem 10 in [11]. Then we show that if R is a left SGPF ring whose every maximal left (right) ideal is a GW-ideal, then $R/J(R)$ is strongly regular. Therefore Theorem 2.3 of [8] is improved and extended.

2. Main results

We start with the following well known lemma ([10]).

Lemma 2.1. *Let R be a GP-V-ring, then $J(R) = 0$.*

Theorem 2.2. *The following are equivalent for a ring R:*

(1) *R is strongly regular.*

(2) *R is a left GP-V-ring whose every maximal left ideal of R is a GW-ideal.*

(3) *R is a left GP-V-ring whose every maximal right ideal of R is a GW-ideal.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are trivial.

(2) $\Rightarrow$ (1). First we show that R is reduced. If it is not the case, then there exists $0 \neq a \in R$ such that $a^2 = 0$. Hence $l(a)$ is contained in a maximal left ideal M of R. Since R is a left GP-V-ring, we may define an R-homomorphism $f : Ra \to R/M$ given by $f(ra) = r + M$ satisfying $1 + M = f(a) = ab + M$ for some $b \in R$. It is clear that $1 - ab \in M$. By the hypothesis, M is a GW-ideal and $ba \in M$, so there exists a positive integer n such that $(ba)^n b \in M$. Since M is a left ideal of R, $b - bab \in M$ implies that

$$(ba)^{n-1}b = (ba)^{n-1}(b - bab) + (ba)^n b \in M$$

Continuing in this process, we have $bab \in M$. Thus $b = (b - bab) + bab \in M$ and $ab \in M$. Therefore $1 \in M$, which contradicts $M \neq R$. This proves that R is reduced.

Now we prove that R is strongly regular. Indeed, if $l(a) + Ra \neq R$ for some $0 \neq a \in R$, then it must be contained in a maximal left ideal M of R. Thus R/M is YJ-injective, hence there exists a positive integer n such that $a^n \neq 0$ and any left R-homomorphism from Ra^n to R/M extends to an R-homomorphism from R to R/M. Now we define a map $f : Ra^n \to R/M$ by $f(ra^n) = r + M$ for any r in R . Since R is reduced, $l(a^n) = l(a)$. It yields f is well defined. Thus there exists $b \in R$ such that $1 - a^n b \in M$ and hence $b - ba^n b \in M$, $ba^n \in M$. By hypothesis, M is a GW-ideal. As the proof in the first part, we have $ba^n b \in M$. Furthermore, $b = (b - ba^n b) + ba^n b \in M$, and $a^n b \in M$, whence $1 \in M$. This contradiction shows that

$$l(a) + Ra = R$$

for any $0 \neq a \in R$. Therefore R is a strongly regular ring.

(3) $\Rightarrow$ (1). By Lemma 2.1, we have $J(R) = 0$. Suppose R is not reduced, then there exists $0 \neq a \in R$ such that $a^2 = 0$. Since $a \notin J(R)$, it follows that $a \notin K$ for some maximal right ideal K of R, and $K + aR = R$. Moreover, $a = ak$ for some k in K since $a^2 = 0$. By hypothesis, K is a GW-ideal. Then there exists a positive integer n such that $ak^n \in K$. It follows that

$$a = ak = (ak)k = ak^2 = \cdots = ak^n \in K$$

which is a contradiction. Therefore R is reduced, and hence $l(b) = r(b)$ is an ideal for any $b \in R$. If $l(a) + aR \neq R$ for some $a \in R$, then it must be contained in a maximal right ideal K of R. Since K is a GW-ideal, $Ra^n \subseteq K$ for some positive integer n. Moreover,

$$l(a) + Ra^n R \subseteq K \subsetneq R$$

Then there exists a maximal left ideal M such that $l(a) + Ra^n R \subseteq M \subsetneq R$. Since R/M is YJ-injective, there is a positive integer m such that $(a^n)^m \neq 0$ and any left R-homomorphism $R(a^n)^m \to R/M$ extends to an R-homomorphism $R \to R/M$. Define $g : R(a^n)^m \to R/M$ by $g(r(a^n)^m) = r + M$. Since R is reduced, $l((a^n)^m) =$

$l(a)$. It is clear that g is well defined and $1 + M = (a^n)^m b + M$ for some $b \in R$. But $(a^n)^m b \in Ra^n R \subseteq M$, then $1 \in M$, a contradiction. Therefore

$$l(a) + aR = R$$

for any $0 \neq a \in R$. It follows that $a = ada$ for some $d \in R$. Thus R is a von Neumann regular ring. Hence R is strongly regular. □

Corollary 2.3 ([10]). *The following conditions are equivalent.*

(1) *R is strongly regular.*

(2) *R is a left quasi-duo ring whose simple right modules are YJ-injective.*

(3) *R is a left quasi-duo ring whose simple left modules are YJ-injective.*

Recall that R is a left SPF ring [8] if every simple left R-module is either P-injective or flat; R is a left SGPF ring [12] if every simple left R-module is GP-injective or flat. It is well known that $R/J(R)$ plays an important role in ring theory ([3], [7]). Now, we study the strongly regularity of $R/J(R)$ for a left SGPF ring R.

The next lemma is easy, so we omit the proof.

Lemma 2.4. *Let L be a left (right) ideal of R which contains an ideal I. If L is a GW-ideal , then L/I is a GW-ideal of R/I.*

Lemma 2.5. *Let R be a semiprimitive ring. If any maximal left (right) ideal of R is a GW-ideal, then R is reduced.*

Proof. Suppose there is $0 \neq a \in R$ such that $a^2 = 0$. Since R is semiprimitive, $a \notin J(R)$, there exists a maximal left ideal M such that $a \notin M$. It yields $M + Ra = R$, whence $Ma = Ra$, and $a = ba$ for some $b \in M$. By hypothesis, M is a GW-ideal. Then we have $b^n a \in M$ for some positive n. Now

$$b^n a = b^{(n-1)} ba = b^{(n-1)} a = b^{(n-2)} ba = \cdots = ba = a \in M$$

a contradiction. Therefore R is reduced. Similarly, if any maximal right ideal of a semiprimitive ring R is a GW-ideal, we also obtain that R is reduced. □

According to Lemma 2.5 and Theorem 5 in [12], we have

Corollary 2.6. *If R is a semiprimitive left SGPF ring whose every maximal left ideal is a GW-ideal, then R is fully left and right idempotent.*

Lemma 2.7. *Let R be a SGPF ring. If I is an ideal of R, then R/I is a SGPF ring.*

Proof. Suppose $\overline{R} = R/I$ and $\overline{L}$ is a simple left $\overline{R}$-module. Then $\overline{L}$ is a simple left R-module. Since R is a SGPF ring, $\overline{L}$ is flat or GP-injective as left R-module. If $\overline{L}$ is a flat left R-module. It is easy to see that $\overline{L}$ is flat. If $\overline{L}$ is a GP-injective left R-module. For any $\overline{a} \in \overline{L}$, there exists a positive integer n such that any left R-homomorphism from Ra^n to $\overline{L}$ extends to one from R to $\overline{L}$. If $\overline{f}$ is any $\overline{R}$-homomorphism from $\overline{R}\overline{a}^n$ to $\overline{L}$, $\overline{f}$ can also be viewed as R-homomorphism. Let $\pi : Ra^n \to \overline{R}\overline{a}^n$ be a canonical R-homomorphism. It yields that $f = \overline{f}\pi : Ra^n \to \overline{L}$ is a left R- homomorphism. Hence $f(a^n) = a^n\overline{b}$ for some $\overline{b} \in \overline{L}$. One has that $\overline{f}(\overline{a}^n) = \overline{f}\pi(a^n) = f(a^n) = a^n\overline{b} = \overline{a}^n\overline{b}$ which implies $\overline{L}$ is GP-injective. Therefore $R/I = \overline{R}$ is a SGPF ring. □

Theorem 2.8. *Let R be a left SGPF ring. If every maximal left ideal of R is a GW-ideal, then $R/J(R)$ is strongly regular.*

Proof. Let $B = R/J(R)$, then $J(B) = 0$. For any maximal left ideal L of B, there exists a maximal left ideal M such that $M \supseteq J(R)$ and $L = M/J(R)$. Since M is a left GW-ideal of R, L ia a GW-ideal of B by Lemma 2.4. It follows that B is a reduced left SGPF ring by Lemma 2.5 and Lemma 2.7. Suppose $Bc + l_B(c) \neq B$ for some $c \in B$, then it must be contained in a maximal left ideal L which implies B/L is simple. Hence B/L is either GP-injective or flat. If B/L is GP-injective, $l(c^n) = l(c)$ since B is reduced. It follows that $f : Bc^n \to B/L$ given by $f(bc^n) = b + L$ for any $b \in B$ is well-defined. Hence $1 + L = f(c^n) = c^n d + L$ for some $d \in B$ which implies $1 - c^n d \in L$. Since L is a GW-ideal of B. Following the proof in Theorem 2.2, we have $1 \in L$, which is a contradiction. If B/L is flat, then $c = cu$ for some $u \in L$ ([2]). It follows that $1 - u = r_B(c) = l_B(c) \subseteq L$, whence $1 \in L$, which is also a contradiction. This shows that

$$Bc + l_B(c) = B$$

for any $c \in B$. Thus there exists $b \in B$ such that $c = bc^2$ for any $c \in B$. Therefore $R/J(R) = B$ is strongly regular. □

Corollary 2.9 ([8]). *If R is a left quasi-duo SPF ring, then $R/J(R)$ is strongly regular.*

Theorem 2.10. *Let R be a left SGPF ring. If every maximal right ideal of R is a GW-ideal, then $R/J(R)$ is strongly regular.*

Proof. Let $B = R/J(R)$. As the proof in Theorem 2.8, we have B is reduced. Moreover B is a left SGPF ring by Lemma 2.7. If $cB + l_B(c) \neq B$ for some $c \in B$. Following the process of the proof in Theorem 2.2, then there exist a maximal left ideal L and a positive integer n such that

$$l_B(c) + Bc^nB \subseteq L \subsetneq B$$

It follows B/L is either GP-injective or flat. If B/L is GP-injective. we define map $g : B(c^n)^m \to B/L$ by $g(b(c^n)^m) = b + L$. Since B is reduced, $l((c^n)^m) = l(c)$. It follows that map g is well-defined. Then we have $1 - (c^n)^m d \in L$. Hence $1 = (1 - (c^n)^m d) + (c^n)^m d \in L$ since $(c^n)^m d \in Bc^nB \subseteq L$, a contradiction. If B/L is flat. As the proof in Theorem 2.8, one has $1 \in L$, a contradiction. This proves that

$$cB + l_B(c) = B$$

for any $c \in B$. So for any $c \in B$, there exists $b \in B$ such that $c = cbc$. This means that B is a von Neumann regular ring. Therefore $R/J(R) = B$ is strongly regular. □

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