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Value Distribution of Meromorphic Derivatives

Indrajit Lahiri
Department of Mathematics, University of Kalyani, West Bengal 741235, India
e-mail : ilahiri@hotpop.com

Shyamali Dewan
Department of Mathematics, Bhairab Ganguly College, Kolkata 700056, India

Abstract. In this paper, we discuss the value distribution of the derivative of a meromorphic function.

1. Introduction, definitions and results

Let f be a transcendental meromorphic function defined in the open complex plane $\mathbb{C}$. Hayman ([9]) proved the following result.

Theorem A ([9]). *If $n(\geq 3)$ is an integer then $\psi = f^n f'$ assumes all finite values, except possibly zero, infinitely many times.*

He ([11]) also conjectured that Theorem A remains valid if $n = 1$ or 2. Mues ([13]) proved the result for $n = 2$ and the result for $n = 1$ was proved by Bergweiler and Eremenko ([2]) and independently by Chen and Fang ([5]).

In 1994 Yik-Man Chiang raised the question of the value distribution of $ff' - a$, where $a = a(z)$ is a meromorphic function which is not identically zero and satisfies $T(r, a) = S(r, f)$ (cf.[3]). To answer this question Bergweiler ([3]) proved the following theorem.

Theorem B ([3]). *Let f be a transcendental meromorphic function of finite order and let $c(\not\equiv 0)$ be a polynomial. Then $ff' - c$ has infinitely many zeros.*

Zhang ([16]) proved the following result which is also in the direction of the question of Chiang.

Theorem C ([16]). *Let f be a transcendental meromorphic function such that $\delta(\infty; f) > 7/9$. Then $ff' - a$ has infinitely many zeros, where $a(\not\equiv 0,\ \infty)$ is a meromorphic function satisfying $T(r, a) = S(r, f)$.*

Recently Yu ([15]) and the authors ([12]) treated the general case and proved the following theorem.

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Theorem D. *If f is a transcendental meromorphic function and $a(\not\equiv 0, \infty)$ is a meromorphic function satisfying $T(r,a) = S(r,f)$ then one of $ff' - a$ and $ff' + a$ has infinitely many zeros.*

However the result of Bergweiler ([3]) seems to be of interest as it imposes a restriction on the growth of the function f (in contrast to Theorem C where a restriction on the poles is imposed) and only one target function is involved (in contrast to Theorem D where two target functions are involved). In the paper we see that if instead of a polynomial we choose a monomial as the target function then in Theorem B the order restriction can be dropped. In fact we prove the following result.

Theorem 1. *Let f be a transcendental meromorphic function. Then $f^p f' - az^n$ has infinitely many zeros, where $a(\neq 0)$ is a constant and n is a nonnegative integer, p is a positive integer.*

Actually Theorem 1 follows as a consequence of the following result.

Theorem 2. *Let $Q(z)$ be a nonconstant polynomial having no simple zero and $P(z) = \frac{d}{dz}Q(z)$. If f is a transcendental meromorphic function then $P(f(z))f'(z) - az^n$ has infinitely many zeros, where $a(\neq 0, \infty)$ is a complex number and n is a nonnegative integer.*

Also Theorem 2 follows from the following theorem.

Theorem 3. *Let f be a transcendental meromorphic function having no simple zero and simple pole. Then $f'(P(z)) - \frac{az^n}{P'(z)}$ has infinitely many zeros, where $P(z)$ is a nonconstant polynomial, $a(\neq 0, \infty)$ is a complex number and n is a nonnegative integer.*

Fang ([7]) proved the following result.

Theorem E ([7]). *Let f be a transcendental meromorphic function of infinite order. If f and f' have the same zeros then $f'(z) - az^n$ has infinitely many zeros, where $a(\neq 0, \infty)$ is a complex number and n is a nonnegative integer.*

Considering $f(z) = \exp(\exp(z^2))$ we see that though f and f' have not the same set of zeros, $f'(z) - az^n$ has infinitely many zeros. In the following corollary to Theorem 3 we see that if we impose a minor restriction on the poles of f, the condition on the zeros of f can be relaxed.

Corollary 1. *Let f be a transcendental meromorphic function such that zeros of f are the zeros of f'. If f has no simple pole then $f'(z) - az^n$ has infinitely many zeros, where $a(\neq 0, \infty)$ is a complex number and n is a nonnegative integer.*

The Ahlfors-Shimizu characteristic function $T_0(r,f)$ of a meromorphic function f is defined as

$$T_0(r,f) = \int_0^r \frac{A(t,f)}{t} dt,$$

where $A(t,f)=\frac{1}{\pi}\iint\limits_{|z|\leq t}[f^{\#}(z)]^2dxdy$ and $f^{\#}(z)=\frac{|f'(z)|}{1+|f(z)|^2}$.

Also we know that $T(r,f)=T_0(r,f)+O(1)$. For the standard definitions and notations of the value distribution theory we refer to [10]. In the paper we adopt some techniques of Bergweiler ([3]) and Fang ([7]).

2. Lemmas

In this section we present the necessary lemmas.

Lemma 1. *Let f be a nonconstant rational function having no simple zero. If the number of poles of f, if there is any, is at least two (counted with multiplicity) then for any complex number $a(\neq 0,\ \infty)$, $f'+a$ has at least one zero.*

Proof. If f is a polynomial then the degree of f is at least two and so f' is a non-constant polynomial. Hence $f'+a$ has at least one zero.

Let $f=p/q$, where p, q are polynomials of degree m and $n(\geq 1)$ respectively and p, q have no common factor.

If possible we suppose that $f'+a$ has no zero. Now we consider the following cases.

Case 1. Let $m<n+1$. Then

$$f'+a=\frac{p'q-pq'+aq^2}{q^2}=\frac{R}{S},\ \text{say}.$$

Then R, S are non-constant polynomials such that degree of $R=$ degree of S. Since $f'+a$ has no zero, it follows that R and S share zeros (counting multiplicities). So $R=AS$, where A is a constant. Therefore $f'+a=A$ and so $f=(A-a)z+B$, where B is a constant. This is impossible because f is non-constant and has no simple zero.

Case 2. Let $m>n+1$. Then

$$f=r+\frac{p_1}{q},$$

where p_1 and r are polynomials with respective degrees m_1 and $t(\geq 2)$ such that $m=t+n$ and $m_1<n$. So

$$\begin{aligned} f'+a &= r'+\frac{p_1'q-p_1q'}{q^2}+a \\ &= \frac{(r'+a)q^2+p_1'q-p_1q'}{q^2} \\ &= \frac{R_1}{S},\ \text{say}. \end{aligned}$$

Let $p_1=a_{m_1}z^{m_1}+\cdots+a_1z+a_0$ and $q=b_nz^n+\cdots+b_1z+b_0$, where $a_{m_1}\neq 0$, $b_n\neq 0$. Since the coefficient of the leading term of $p_1'q-p_1q'$ is $(m_1-n)a_{m_1}b_n\neq 0$, the degree of $p_1'q-p_1q'$ is m_1+n-1. Since $m_1+n-1<2n-1<2n+t-1$, we see that degree of $R_1 \geq$ degree of S. So as the *Case 1* f becomes a linear polynomial, which is impossible.

Case 3. Let $m=n+1$. Then

$$f=\alpha z+\beta+\frac{p_1}{q}, \tag{1}$$

where α, β are constants and p_1 is a polynomial of degree $m_1<n$.

Let $a+\alpha\neq 0$. Then

$$f'+a=\frac{p_1'q-p_1q'+(a+\alpha)q^2}{q^2}=\frac{R_2}{S}, \text{ say.}$$

Since the degree of $p_1'q-p_1q'$ is $m_1+n-1<2n-1<2n$, it follows that degree of $R_2 =$ degree of S. So as the *Case 1* f becomes a linear polynomial, which is impossible.

Let $a+\alpha=0$. Then

$$f'+a=\frac{p_1'q-p_1q'}{q^2}, \tag{2}$$

where the degree of $p_1'q-p_1q'$ is m_1+n-1.

Now we consider the following two subcases.

Subcase 3.1. Let $p_1'q-p_1q'$ have no zero. Then $m_1+n=1$ and so $m_1=0$ and $n=1$ because $m_1<n$. So from (1) we see that

$$f=\alpha z+\beta+\frac{D}{\gamma z+\delta},$$

where $\gamma,\ \delta,\ D$ are constants.

This is impossible because f is non-constant and has no simple zero and has at least two poles (counted with multiplicity), if there is any.

Subcase 3.2. Let $p_1'q-p_1q'$ have some zero. Since $f'+a$ has no zero, it follows that all the factors of $p_1'q-p_1q'$ are factors of q^2. Hence we can write

$$q^2=(p_1'q-p_1q')q_1,$$

where q_1 is a polynomial of degree $n+1-m_1(>0)$.

From (2) we get

$$f'+a=\frac{1}{q_1}. \tag{3}$$

Let $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_l$ be distinct zeros of q_1 with respective multiplicities $k_1,\ k_2,\ \cdots,\ k_l$. Since q_1 is non-constant, we see that $l\geq 1$.

Since $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_l$ are the only poles of f' with respective multiplicities $k_1,\ k_2,\ \cdots,\ k_l$, it follows that $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_l$ are the only poles of f with respective multiplicities $k_1-1,\ k_2-1,\ \cdots,\ k_l-1$. Hence we get from (1) and (3)

$$k_1+k_2+\cdots+k_l=n+l \quad \text{and} \quad k_1+k_2+\cdots+k_l=n+1-m_1.$$

So $m_1+l=1$. Since $l\geq 1$, it follows that $m_1=0,\ l=1$ and $k_1=n+1$. Therefore from (1) we get

$$f = \alpha z+\beta+\frac{D}{(\gamma z+\delta)^n} = \frac{(\alpha z+\beta)(\gamma z+\delta)^n+D}{(\gamma z+\delta)^n}, \tag{4}$$

where γ, δ and D are constants. Since f is not a linear polynomial, it follows that $D\neq 0$. Also we see that $z=-\delta/\gamma$ is the only pole of f.

Let $Q=(\alpha z+\beta)(\gamma z+\delta)^n+D$. Then $Q'=(\gamma z+\delta)^{n-1}\left[\alpha(\gamma+n)z+\alpha\delta+n\beta\right]$. Since f has no simple zero, it follows that a zero of f is also a zero of Q' and so a zero of $\alpha(\gamma+n)z+\alpha\delta+n\beta$. So f and Q have only one double zero. Hence we get $n=1$. Therefore from (4) we get

$$f=\alpha z+\beta+\frac{D}{\gamma z+\delta},$$

which is impossible because f is non-constant and has no simple zero and and has at least two poles (counted with multiplicity), if there is any. This proves the lemma. □

Lemma 2 ([2]). *Let f be a meromorphic function of finite order. If f has infinitely many multiple zeros then f' assumes every finite nonzero value infinitely many times.*

Lemma 3 (p.60 [10]). *Let f be a transcendental meromorphic function. If f has only finitely many zeros then f' assumes every finite nonzero value infinitely many times.*

Combining the above lemmas we obtain the following lemma.

Lemma 4. *Let f be a nonconstant meromorphic function of finite order having no simple zero. If the number of poles of f, if there is any, is at least two (counted with multiplicity) then for every complex number $a(\neq 0,\ \infty)$, $f'+a$ has at least one zero.*

Lemma 5 ([4], [14]). *Let $\mathfrak{F}$ be a family of meromorphic functions defined in a domain $\mathfrak{D}$ such that every function $f\in\mathfrak{F}$ has zeros, if there is any, of multiplicities at least k. If $\mathfrak{F}$ is not normal at a point $z_0\in\mathfrak{D}$ then for $0\leq\alpha<k$, there exist a sequence of functions $f_j\in\mathfrak{F}$, a sequence of complex numbers $z_j\to z_0$ and a sequence*

of positive numbers $\rho_j \to 0$, such that $\rho_j^{-\alpha} f_j(z_j + \rho_j\zeta)$ converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$ on $\mathbb{C}$. Moreover the order of g is at most two and g has only zeros, if there is any, of multiplicities at least k.

Lemma 6. *Let $\mathfrak{F}$ be a family of meromorphic functions in a domain $\mathfrak{D}$ and $a = a(z)(\not\equiv 0)$ be a nonvanishing analytic function in $\mathfrak{D}$. If for every $f \in \mathfrak{F}$*

(i) *f has no simple zero and simple pole,*

(ii) *$f(z) = a(z)$ whenever $f'(z) = a(z)$,*

then $\mathfrak{F}$ is a normal family.

Proof. If possible, let $\mathfrak{F}$ be not normal at a point $z_0 \in \mathfrak{D}$. Then by Lemma 5 for $\alpha = 1$ there exist a sequence of functions $f_j \in \mathfrak{F}$, a sequence of complex numbers $z_j \to z_0$ and a sequence of positive numbers $\rho_j \to 0$ such that $g_j(\zeta) = \rho_j^{-1} f_j(z_j+\rho_j\zeta)$ converges spherically and locally uniformly to a nonconstant meromorphic function $g(\zeta)$. Moreover g is of finite order and has only multiple zeros (if there is any). Also we note that g has at least two poles (counted with multiplicity), if there is any.

Since $a(z_0) \neq 0, \infty$, by Lemma 4 there exists $\zeta_0 \in \mathbb{C}$ such that $g'(\zeta_0) = a(z_0)$. Hence g' and so g are analytic in some neighborhood of ζ_0. Therefore in some neighborhood of ζ_0 g_j's are analytic for all sufficiently large values of j and $g_j \to g$, $g_j' \to g'$ uniformly in that neighborhood of ζ_0.

First we suppose that $g_j'(\zeta) - a(z_j + \rho_j\zeta) \neq 0$ for infinitely many values of j and in some neighborhood of ζ_0. Since $g_j'(\zeta) - a(z_j + \rho_j\zeta)$ converges uniformly to $g'(\zeta) - a(z_0)$ in some neighborhood of ζ_0, by Hurwitz's theorem we get $g'(\zeta) - a(z_0) \equiv 0$ in some neighborhood of ζ_0. Since g is meromorphic, it follows that $g'(\zeta) \equiv a(z_0)$ in $\mathbb{C}$, which contradicts the fact that g has no simple zero.

Next we suppose that there exists a sequence $\zeta_j \to \zeta_0$ such that $g_j'(\zeta_j) = f'(z_j + \rho_j\zeta_j) = a(z_j + \rho_j\zeta_j)$. Since $f_j(z) = a(z)$ whenever $f_j'(z) = a(z)$, it follows that $f_j(z_j + \rho_j\zeta_j) = a(z_j + \rho_j\zeta_j)$ and so $\rho_j g_j(\zeta_j) = a(z_j + \rho_j\zeta_j)$. This implies that $g_j(\zeta_j) = \frac{1}{\rho_j} a(z_j + \rho_j\zeta_j) \to \infty$ as $j \to \infty$, which contradicts the fact that $g_j(\zeta_j) \to g(\zeta_0) \neq \infty$. This proves the lemma. □

Lemma 7 ([6]). *Let f be a nonconstant meromorphic function and $F = \sum_{j=1}^{q} \frac{1}{f-\phi_j}$, where ϕ_j's are meromorphic functions and $T(r,\phi_j) = o\{T(r,f)\}$ as $r \to \infty$. Then*

$$\sum_{j=1}^{q} m(r, \frac{1}{f-\phi_j}) \leq m(r,F) + S(r,f).$$

Lemma 8 ([8]). *Let f be a transcendental meromorphic function. Then for each positive number ε and each positive integer k we have*

$$k\overline{N}(r,f) \leq N(r,0;f^{(k)}) + N(r,f) + \varepsilon T(r,f) + S(r,f).$$

Lemma 9 ([2]). *Let f be a transcendental meromorphic function of finite order. If f has only finitely many critical values then f has only finitely many asymptotic values.*

Lemma 10 ([1]). *Let f be a transcendental meromorphic function such that $f(0) \neq \infty$. If the set of finite critical and asymptotic values f is bounded then there exists $R(>0)$ such that*

$$\mid f'(z) \mid \geq \frac{\mid f(z) \mid}{2\pi \mid z \mid} \log \frac{\mid f(z) \mid}{R},$$

for all $z \in \mathbb{C}\backslash\{0\}$ which are not the poles of f.

3. Proof of the Theorems

Proof of Theorem 3. We consider the following cases.

Case I. Let $f(P)$ be of infinite order. Then $f(P(z))/z^{n+1}$ is of infinite order. Hence

$$\limsup_{r\to\infty} \frac{T(r, \frac{f(P(z))}{z^{n+1}})}{(\log r)^2} = \infty \text{ and so } \limsup_{r\to\infty} \frac{A(r, \frac{f(P(z))}{z^{n+1}})}{\log r} = \infty.$$

Let $\mathfrak{F} = \{g_j(z) : g_j(z) = \frac{f(P(2^j z))}{2^{(n+1)j} z^{n+1}}, j = 1, 2, 3, \cdots ; \frac{1}{2} < |z| < \frac{5}{2}\}$. If possible we suppose that $\mathfrak{F}$ is a normal family. Then by Marty's criterion there exists $M > 0$ such that

$$g_j^{\#}(z) \leq M \text{ for } j = 1, 2, 3, \cdots ; 1 \leq |z| \leq 2.$$

Now

$$\begin{aligned} A(2^j, \frac{f(P(z))}{z^{n+1}}) &= \frac{1}{\pi} \iint_{|z|<2^j} \left(\left(\frac{f(P(z))}{z^{n+1}}\right)^{\#}\right)^2 dxdy \\ &= \frac{1}{\pi} \sum_{m=0}^{j-1} \int\int_{2^m \leq |z| \leq 2^{m+1}} \left(\left(\frac{f(P(z))}{z^{n+1}}\right)^{\#}\right)^2 dxdy + O(1) \\ &= \frac{1}{\pi} \sum_{m=0}^{j-1} \iint_{1\leq|z|\leq 2} \left(g_m^{\#}(w)\right)^2 d\xi d\eta + O(1) \\ &\leq 3jM^2 = Kj, \end{aligned}$$

where $w = \xi + i\eta$ and $K = 3M^2$.

So for any r $(2^{j-1} \leq r < 2^j)$ we have

$$A\left(r, \frac{f(P(z))}{z^{n+1}}\right) \leq A\left(2^j, \frac{f(P(z))}{z^{n+1}}\right) \leq Kj \leq K\left(\frac{\log r}{\log 2} + 1\right),$$

a contradiction. So $\mathfrak{F}$ is not normal in $\frac{1}{2} < |z| < \frac{5}{2}$. Hence the family $\mathfrak{F}_1 = \{h_j(z) : h_j(z) = z^{n+1} g_j(z), j = 1, 2, 3, \cdots\}$ is not normal in $\frac{1}{2} < |z| < \frac{5}{2}$.

Then for $a(z) = az^n$ we see by Lemma 6 that there exists infinitely many j and z_j such that $h'_j(z_j) = az_j^n$ and so $f'(P(2^j z_j)) = \frac{a(2^j z_j)^n}{P'(2^j z_j)}$. Hence $f'(P(z)) - \frac{az^n}{P'(z)}$ has infinitely many zeros.

Case II. Let $f(P)$ be of finite order. First we suppose that f and so $f(P)$ have finitely many zeros. Now in view of Lemmas 7 and 8 we get for $b = az^n$

$$\begin{aligned} m(r, \frac{1}{f}) + m(r, \frac{1}{f' - b}) &\leq m(r, \frac{1}{f^{(2)}}) + m(r, \frac{1}{f^{(2)} - b'}) \\ &\leq m(r, \frac{1}{f^{(2)}} + \frac{1}{f^{(2)} - b'}) + S(r, f) \\ &\leq m(r, \frac{1}{f^{(n+2)}}) + S(r, f) \\ &= T(r, f^{(n+2)}) - N(r, 0; f^{(n+2)}) + S(r, f) \\ &\leq m(r, f') + N(r, f') + (n+1)\overline{N}(r, f) - N(r, 0; f^{(n+2)}) + S(r, f) \\ &\leq T(r, f') + \frac{n+1}{n+2} N(r, 0; f^{(n+2)}) + \frac{n+1}{n+2} N(r, f) + \frac{1}{2n+4} T(r, f) \\ &\quad - N(r, 0; f^{(n+2)}) + S(r, f) \end{aligned}$$

i.e.,

$$T(r, f) + T(r, \frac{1}{f' - b}) \leq T(r, f') + \frac{1}{2n+4} T(r, f) + N(r, \frac{1}{f' - b}) + \frac{n+1}{n+2} N(r, f) + S(r, f)$$

and so

$$T(r, f) \leq (2n+4) N(r, \frac{1}{f' - b}) + S(r, f).$$

Replacing f by $f(P)$ in the above inequality we get

$$\begin{aligned} T(r, f(P)) &\leq (2n+4) N(r, \frac{1}{f'(P(z))P'(z) - b}) + S(r, f(P)) \\ &\leq (2n+4) N(r, \frac{1}{f'(P(z)) - \frac{az^n}{P'(z)}}) + S(r, f(P)), \end{aligned}$$

which shows that $f'(P(z)) - \frac{az^n}{P'(z)}$ has infinitely many zeros.

Next we suppose that f and so $f(P)$ have infinitely many zeros. Let $z_1, z_2, z_3, \cdots$ be the zeros of $f(P)$. We put $g(z) = f(P(z)) - az^{n+1}/(n+1)$. We now suppose that $g'(z)$ has finitely many zeros, then g has finitely many critical values. So by Lemma 9, g has finitely many asymptotic values. Without loss of generality we suppose that $g(0) = f(P(0)) \neq \infty$. Then by Lemma 10 we get

$$\frac{|z_j g'(z_j)|}{|g(z_j)|} \geq \frac{1}{2\pi} \log \frac{|g(z_j)|}{R}.$$

Since $\frac{1}{2\pi}\log\frac{|g(z_j)|}{R} \to \infty$ as $j \to \infty$, it follows that $\frac{|z_j g'(z_j)|}{|g(z_j)|} \to \infty$ as $j \to \infty$. On the other hand $\frac{|z_j g'(z_j)|}{|g(z_j)|} \to n+1$ as $j \to \infty$, a contradiction. Therefore $g'(z)$ and so $f'(P(z)) - \frac{az^n}{P'(z)}$ has infinitely many zeros. This proves the theorem. □

Note 1. If we suppose that f is a transcendental meromorphic function of finite order having no simple zero then similar to Case II of the above proof we can prove that $f'(P(z)) - \frac{Q(z)}{P'(z)}$ has infinitely many zeros, where $Q(z)(\not\equiv 0)$ is a polynomial.

Proof of Theorem 2. Since $Q(f(z))$ has no simple zero and simple pole, by Theorem 3

$$P(f(z))f'(z) - az^n = \frac{d}{dz}Q(f(z)) - az^n$$

has infinitely many zeros. □

Proof of Theorem 1. Theorem 1 follows from Theorem 2 if we choose $Q(z) = \frac{z^{p+1}}{p+1}$.□

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