KYUNGPOOK Math. J. 45(2005), 405-411

Uniform Topology on Hilbert Algebras

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Abstract. We use a congruence relation on deductive systems of a Hilbert algebra H, to define a uniform structure on H and investigate the corresponding topology.

1. Introduction

The notion of a Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implicative in intuicionstic and other non-classical logics. In 60-ties, these algebras were studied especially by A. Horn and A. Diego from algebraic point of view. I. Chajda and R. Halas [2] and W. A. Dudek [3] introduce and study deductive systems (ideals) and congruence relations in Hilbert algebra. In this paper we consider a collection of deductive systems and use congruence relation with respect to deductive systems to define a uniformity and make the Hilbert algebra into a uniform topological space with the desired subset as the open sets.

Towards our goal, we renew some needed algebraic notions in section 2. Then consider the uniformity based on congruence relations with respect to given collection of deductive systems and construct the induced topology by this uniformity in section 3. In the last section we study the properties of these topology.

2. Preliminaries

Definition 2.1 ([3]). A Hilbert algebra is an algebra $(H, *, 1)$ where H is a nonempty set, $*$ is a binary operation and 1 is a constant such that the following axioms hold for each $x, y, z \in H$:

(H1) $x * (y * x) = 1$,

(H2) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,

(H3) $x * y = 1$ and $y * x = 1$ imply $x = y$.

Received March 30, 2004, and, in revised form, May 28, 2004.
2000 Mathematics Subject Classification: 54E15, 03G25.
Key words and phrases: uniformity, Hilbert algebra.

Lemma 2.2 ([3]). *In each Hilbert algebra H, the following relation hold for all $x, y, z \in H$:*

(1) $x * x = 1$,

(2) $1 * x = x$,

(3) $x * 1 = 1$,

(4) $x * (y * z) = y * (x * z)$.

It is easily checked that in Hilbert algebra H the relation $\leq$ defined by

$$x \leq y \Leftrightarrow x * y = 1$$

is a partial order on H with 1 as the largest element.

Definition 2.3 ([2]). A nonempty subset I of a Hilbert algebra H is called an ideal of H if

(1) $1 \in I$,

(2) $x * y \in I$ for all $x \in H, y \in I$,

(3) $(y_2 * (y_1 * x)) * x \in I$ for all $x \in H, y_1, y_2 \in I$.

Definition 2.4 ([3]). A deductive system of Hilbert algebra H is a nonempty set $D \subseteq H$ such that

(1) $1 \in D$,

(2) $x \in D$ and $x * y \in D$ imply $y \in D$.

Theorem 2.5 ([3]). *A nonempty subset A of Hilbert algebra H is an ideal if and only if it is a deductive system of H.*

Theorem 2.6 ([3]). *If D is a deductive system of a Hilbert algebra H, then the relation Θ_D defined by*

$$(a, b) \in \Theta_D \Leftrightarrow a * b \in D \text{ and } b * a \in D$$

is a congruence relation on H.

3. Uniformity in Hilbert algebra

From now on H is a Hilbert algebra and $D \subseteq H$ is a deductive system of Hilbert algebra H.

Let X be a nonempty set and U, V be any subset of $X \times X$. Define:

$$\begin{aligned} U \circ V &= \{(x, y) \in X \times X \mid \text{ for some } z \in X,\ (z, y) \in U \text{ and } (x, z) \in V\}, \\ U^{-1} &= \{(x, y) \in X \times X \mid (y, x) \in U\}, \\ \Delta &= \{(x, x) \in X \times X \mid x \in X\}. \end{aligned}$$

Definition 3.1 ([4]). By a uniformity on X we shall mean a nonempty collection $\mathcal{K}$ of subsets of $X \times X$ which satisfies the following conditions:

(U_1) $\Delta \subseteq U$ for any $U \in \mathcal{K}$,

(U_2) if $U \in \mathcal{K}$, then $U^{-1} \in \mathcal{K}$,

(U_3) if $U \in \mathcal{K}$, then there exists a $V \in \mathcal{K}$, such that $V \circ V \subseteq U$,

(U_4) if $U, V \in \mathcal{K}$, then $U \cap V \in \mathcal{K}$,

(U_5) if $U \in \mathcal{K}$, and $U \subseteq V \subseteq X \times X$ then $V \in \mathcal{K}$.

The pair $(X, \mathcal{K})$ is called a *uniform structure* (uniform space).

Theorem 3.2. *Let H be a Hilbert algebra and Λ be an arbitrary family of deductive systems of the Hilbert algebra H such that it is closed under intersection. If $U_D = \{(x,y) \in H \times H \mid x\Theta_D y\}$ and $\mathcal{K}^* = \{U_D \mid D \in \Lambda\}$, then $\mathcal{K}^*$ satisfies the conditions (U_1)-(U_4).*

Proof. (U_1): Since D is a deductive system of H, $x\Theta_D x$ for any $x \in H$, hence $\Delta \subseteq U_D$, for all $U_D \in \mathcal{K}^*$.
(U_2): For any $U_D \in \mathcal{K}^*$ we have

$$(x,y) \in (U_D)^{-1} \Leftrightarrow (y,x) \in U_D \Leftrightarrow y\Theta_D x \Leftrightarrow x\Theta_D y \Leftrightarrow (x,y) \in U_D.$$

Therefore $(U_D)^{-1} = U_D \in \mathcal{K}^*$
(U_3): For any $U_D \in \mathcal{K}^*$, the transitivity of Θ_D implies that $U_D \circ U_D \subseteq U_D$.
(U_4): For any $U_D, U_J \in \mathcal{K}^*$, we claim that $U_D \cap U_J = U_{D\cap J}$. Let $(x,y) \in U_D \cap U_J$. Then $x\Theta_D y$ and $x\Theta_J y$. Hence $x * y \in D$, $y * x \in D$, $x * y \in J$ and $y * x \in J$. Then $x\Theta_{(D\cap J)}y$ and hence $(x,y) \in U_{D\cap J}$.

Conversely, let $(x,y) \in U_{D\cap J}$. Then $x\Theta_{(D\cap J)}y$, hence $x * y \in D \cap J$ and $y * x \in D \cap J$. Then $x * y \in D$, $y * x \in D$, $x * y \in J$ and $y * x \in J$. Therefore $x\Theta_D y$ and $x\Theta_J y$. Then $(x,y) \in U_D \cap U_J$. So $U_D \cap U_J = U_{D\cap J}$. Since D and J are in Λ, $D \cap J \in \Lambda$, thus $U_D \cap U_J \in \mathcal{K}^*$. □

Theorem 3.3. *Let $\mathcal{K} = \{U \subseteq H \times H \mid U_D \subseteq U$ for some $U_D \in \mathcal{K}^*\}$. Then $\mathcal{K}$ is a uniformity on H and the pair $(H, \mathcal{K})$ is a uniform structure.*

Proof. By Theorem 3.2, the collection $\mathcal{K}$ satisfies the conditions (U_1)-(U_4). It suffices to show that $\mathcal{K}$ satisfies (U_5). Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq H \times H$. Then there exists a $U_D \subseteq U \subseteq V$, which implies that $V \in \mathcal{K}$. □

Let $x \in H$ and $U \in \mathcal{K}$. Define:

$$U[x] := \{y \in H \mid (x,y) \in U\}.$$

Theorem 3.4. *Given a Hilbert algebra H, then*

$$T = \{G \subseteq H \mid \forall\ x \in G,\ \exists\ U \in \mathcal{K},\ U[x] \subseteq G\}$$

is a topology on H.

Proof. It is clear that $\emptyset \in T$ and $H \in T$. Also from the very nature of that definition, it is clear that T is closed under arbitrary union. Finally to show that T is closed under finite intersection, let $G, K \in T$ and suppose $x \in G \cap K$. Then there exist U and $V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq K$. Let $W = U \cap V$, then $W \in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ hence $W[x] \subseteq G \cap K$ and then $G \cap K \in T$. Thus T is topology on H. □

Note that for any x in H, $U[x]$ is an open neighborhood of x.

Lemma 3.5. *Let* H *be a Hilbert algebra. If* $D \neq \{1\}$ *then* $U_D \neq U_{\{1\}}$.

Proof. Since $D \neq \{1\}$, there exists $z \in D$ such that $z \neq 1$. We have $z * 1 = 1 \in D$ and $1 * z = z \in D$, since D is a deductive system. Hence $1 \in U_D[z]$ and then $U_D \neq U_{\{1\}}$. □

Corollary 3.6. *Let* H *be a Hilbert algebra. Then* $D = \{1\} \in \Lambda$, *if and only if* T *is discrete topology.*

Proof. Let $D = \{1\} \in \Lambda$. Then $U_D \in \mathcal{K}^*$. Hence $U_D[x] \in T$, for all $x \in H$. We have

$$U_D[x] = \{y \mid y\Theta_D x\} = \{y \mid x * y = 1, y * x = 1\} = \{x\}$$

Hence T is discrete topology on H.

Conversely, let T be a discrete topology on H, hence $\{x\}$ is open, for all $x \in H$. Given $x \in H$, there exists $D \in \Lambda$ such that $U_D[x] \subseteq \{x\}$. Since $x \in U_D[x]$, $U_D[x] = \{x\}$. We denote $D := D_x$. Now if $J = \bigcap\{D_x \mid x \in H\}$, then $U_J[x] = \{x\}$ for arbitrary $x \in H$, indeed if $y \in U_J[x]$ then $y\Theta_J x$. Hence $x * y \in D_x$, $y * x \in D_x$, for all $x \in H$. Then $y\Theta_D x$, for all $x \in H$. It is follows that $y \in U_D[x] = \{x\}$, then $y = x$. Hence $U_J[x] = \{x\} = U_{\{1\}}[x]$, for all $x \in H$. Then $U_J = U_{\{1\}}$, and by Lemma 3.5 we have $J = \{1\}$. Since Λ is closed under intersection, we conclude that $J = \{1\} \in \Lambda$. □

Definition 3.7. Let $(H, \mathcal{K})$ be a uniform structure, where H is a Hilbert algebra. Then the topology T is called the uniform topology on H induced by $\mathcal{K}$.

Proposition 3.8. *Topological space* (H, T) *is completely regular.*

Proof. See Theorem 14.2.9, [4]. □

4. Topological property of space (H, T)

Let H be a Hilbert algebra and B, C subsets of H. Then we define $B * C$ as follows:

$$B * C = \{x * y \mid x \in B, y \in C\}.$$

Let H be a Hilbert algebra and T be a topology defined on H. We say that the pair (H, T) is a topological Hilbert algebra, if $*$ is continuous with respect to T. The continuity of $*$ is equivalent to the following property:

(I): Let O be an open set and $a, b \in H$ such that $a * b \in O$. Then there exist open sets O_1 and O_2 such that $a \in O_1$, $b \in O_2$ and $O_1 * O_2 \subseteq O$.

Let H be a Hilbert algebra and T defined as in Theorem 3.4. Then with the above notations we have the following:

Theorem 4.1. *The pair (H, T) is a topological Hilbert algebra.*

Proof. Assume that $x * y \in G$, with $x, y \in H$ and G an open subset of H. Then there exists $U \in \mathcal{K}$, $U[x * y] \subseteq G$ and a deductive system D such that $U_D \subseteq U$. We claim that the following relation holds:

$$U_D[x] * U_D[y] \subseteq U[x * y]$$

Indeed for $h \in U_D[x]$ and $k \in U_D[y]$ we get $x\Theta_D h$ and $y\Theta_D k$. It follows that $x * y\Theta_D h * k$, hence $(x * y, h * k) \in U_D \subseteq U$. Thus $h * k \in U_D[x * y] \subseteq U[x * y]$ and then $h * k \in G$. □

Theorem 4.2 ([4]). *Let X be a set and $\mathcal{S} \subset \mathcal{P}(X \times X)$ be a family such that for every $U \in \mathcal{S}$ the following conditions hold*:

(a) $\Delta \subseteq U$,

(b) U^{-1} *contains a member of* $\mathcal{S}$, *and*

(c) *there exists a* $V \in \mathcal{S}$, *such that* $V \circ V \subseteq U$. *Then there exists a unique uniformity* $\mathcal{U}$, *for which* $\mathcal{S}$ *is a sub-base.*

Theorem 4.3. *Let $\mathcal{B} = \{U_D \mid D$ is a deductive system of $H\}$. Then $\mathcal{B}$ is a sub-base for a uniformity of H. We denote this topology by* S.

Proof. Since Θ_D is congruence relation, then it is clear that $\mathcal{B}$ satisfies axioms of Theorem 4.2. □

Corollary 4.4. *Topology T is weaker than* S.

Proposition 4.5. *If we let $\mathcal{M} = \{U_M \mid M$ is a maximal deductive system of $H\}$. Then $\mathcal{M}$ is a sub-base for a uniformity of H. We denote this topology by* Max.

Corollary 4.6. *Topology T is weaker than* Max.

Theorem 4.7. *Any deductive system in the collection Λ is a clopen subset of H.*

Proof. Let D be a deductive system of H in Λ and $y \in D^c$. Then $y \in U_D[y]$ and we get $D^c \subseteq \bigcup\{U_D[y] \mid y \in D^c\}$. We claim that, $U_D[y] \subseteq D^c$, for all $y \in D^c$. Let $z \in U_D[y]$, then $z\Theta_D y$. Hence $z * y \in D$. If $z \in D$ then $y \in D$, a contradiction. So $z \in D^c$ and we get $\bigcup\{U_D[y] \mid y \in D^c\} \subseteq D^c$. Hence $D^c = \bigcup\{U_D[y] \mid y \in D^c\}$ and since $U_D[y]$ is open for all $y \in H$, D is a closed subset. We show that $D = \bigcup\{U_D[y] \mid y \in D\}$. If $y \in D$ then $y \in U_D[y]$ and we get $D \subseteq \bigcup\{U_D[y] \mid y \in D\}$. Let $y \in D$, if $z \in U_D[y]$ then $z\Theta_D y$ and so $y * z \in D$. Since $y \in D$ hence $z \in D$ and we get $\bigcup\{U_D[y] \mid y \in D\} \subseteq D$. So D is also an open subset of H. □

Theorem 4.8. *For any $x \in H$ and $D \in \Lambda$, $U_D[x]$ is a clopen subset of H.*

Proof. We show that $(U_D[x])^c$ is open. Let $y \in (U_D[x])^c$, then $x * y \in D^c$ or $y * x \in D^c$. Without loss of generality $y * x \in D^c$. Since D^c is open then there exists $U \in \mathcal{K}$ such that $U[y * x] \subseteq D^c$. From $y * x \in D^c$ we conclude that $U_D[y * x] \subseteq D^c$. Therefore $U_D[y] * U_D[x] \subseteq U_D[y * x] \subseteq D^c$. We claim that $U_D[y] \subseteq (U_D[x])^c$. Let $z \in U_D[y]$, then $z * x \in (U_D[y] * U_D[x])$. So $z * x \in D^c$ then we get $z \in (U_D[x])^c$. Hence $U_D[x]$ is closed. It is clear that $U_D[x]$ is open . So $U_D[x]$ is clopen subset of H. □

Corollary 4.9. *The topological space (H, T) is not connected space.*

Notation. We denote the uniform topology obtained by an arbitrary family Λ, by T_Λ and if $\Lambda = \{D\}$, we denote T_Λ by T_D.

Theorem 4.10. *$T_\Lambda = T_J$, where $J = \bigcap\{D \mid D \in \Lambda\}$.*

Proof. $\mathcal{K}$ and $\mathcal{K}^*$ be as in Theorem 3.2 and 3.3. Now let $\Lambda_0 = \{J\}$, define: $(K_0)^* = \{U_J\}$ and $K_0 = \{U \mid U_J \subseteq U\}$.

Let $G \in T_\Lambda$. So for all $x \in G$, there exists $U \in \mathcal{K}$ such that $U[x] \subseteq G$. From $J \subseteq D$ we get that $U_J \subseteq U_D$, for all deductive systems D of H. Since $U \in \mathcal{K}$, there exists $D \in \Lambda$ such that $U_D \subseteq U$. Hence $U_J[x] \subseteq U_D[x] \subseteq G$. Since $U_J \in K_0$, $G \in T_J$. So $T_\Lambda \subseteq T_J$.

Conversely, let $I \in T_J$ then for all $x \in I$, there exist $U \in K_0$ such that $U[x] \subseteq I$. So $U_J[x] \subseteq I$ and sine Λ is closed under intersection, $J \in \Lambda$. Then we get $U_J \in \mathcal{K}$ and so $I \in T_\Lambda$. Thus $T_J \subseteq T_\Lambda$. □

Corollary 4.11. *Let D and J are deductive systems of Hilbert algebra H and $D \subseteq J$ then J is clopen in topological space (H, T_D).*

Proof. Let $\Lambda = \{D, J\}$. Then by Theorem 4.10, $T_\Lambda = T_D$ and therefore J is clopen in topological space (H, T_D). □

Theorem 4.12. *Let D and J be deductive systems of Hilbert algebra H. Then $T_D \subseteq T_J$ if and only if $J \subseteq D$.*

Proof. Let $J \subseteq D$. Consider: $\Lambda_1 = \{D\}$, ${K_1}^* = \{U_D\}$, $K_1 = \{U \mid U_D \subseteq U\}$ and $\Lambda_2 = \{J\}$, ${K_2}^* = \{U_J\}$, $K_2 = \{U \mid U_J \subseteq U\}$.

Let $G \in T_D$. Then for all $x \in G$, there exist $U \in K_1$ such that $U[x] \subseteq G$. Since $J \subseteq D$, then $U_J \subseteq U_D$ and since $U_D[x] \subseteq G$, we get $U_J[x] \subseteq G$. $U_J \in K_2$ and so $G \in T_J$.

Conversely, let $T_D \subseteq T_J$. Assuming the contrary let $a \in J \setminus D$. Since $D \in T_D$, we obtain that $D \in T_J$. Then for all $x \in D$, there exists $U \in K_2$ such that $U[x] \subseteq D$, and so $U_J[x] \subseteq D$. Then $U_J[1] \subseteq D$. Since J is deductive system $a \in J$ implies, $1 * a \in J$. Then $a \Theta_J 1$, so $a \in U_J[1]$, thus $a \in D$, which is a contradiction. □

Corollary 4.13. *Let D, J be deductive systems of H. Then $D = J$ if and only if $T_D = T_J$.*

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