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A Note on Potent Elements

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Abstract. In this paper, we prove that every exchange ring can be characterized by potent elements. Also we extend [10, Theorem 3.1 and Theorem 4.1] to quasi-clean rings in which every element is a sum of a potent element and a unit.

1. Introduction

Let R be an associative ring with an identity. An element $e \in R$ is potent in case there exists some integer $n \geq 2$ such that $e^n = e$. An element $e \in R$ is idempotent in case $e^2 = e$. An element $e \in R$ is periodic in case there exist positive integers $k, l(k \neq l)$ such that $x^k = x^l$. Clearly, every potent element is periodic. But the converse is not true. Since $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^3$, we know that $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{Z})$ is a periodic element, while it is not a potent element. Also we see that every idempotent is potent, while there exists a potent element which is not idempotent. For example, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in M_2(\mathbb{Z})$ is a potent element, while it is not idempotent. Thus we have proper inclusions { all idempotents } $\subset$ { all potent elements } $\subset$ { all periodic elements } in a ring R.

Recall that a ring R is an exchange ring if for every right R-module A and two decompositions $A = M \oplus N = \oplus_{i \in I} A_i$, where $M_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $A = M \oplus (\oplus_{i \in I} A_i')$. Clearly, regular rings, π-regular rings, semi-perfect rings, left or right continuous rings, clean rings and unit C^*-algebras of real rank zero (cf. [2, Theorem 7.2]) are all exchange rings. In this paper, we prove that every exchange ring can be characterized by potent elements.

A ring R is a clean ring if every element in R is a sum of an idempotent and a unit. Many author studied clean rings such as [4]-[5] and [8]-[9]. Following Y. Ye([10]), we say that R is a semi-clean ring if every element is a sum of a periodic element and a unit. He proved that if G is a cyclic group of order 3 then $\mathbb{Z}_p G$ is a semi-clean ring, while $\mathbb{Z}_7 G$ is not a clean ring. In this paper, we introduce the

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notion of quasi-clean rings as a natural generalization of semi-clean rings. We say that a ring R is a quasi-clean ring in case every element in R is a sum of a potent element and a unit. In fact, we observe that if G is a cyclic group of order 3 then $\mathbb{Z}_pG$ is a quasi-clean ring.

2. Exchange ring

It is well known that a ring R is an exchange ring if and only if for any $x \in R$ there exists an idempotent $e \in Rx$ such that $1-e \in R(1-x)$. In this paper, we prove that every exchange ring can be characterized by potent elements.

Lemma 2.1. *The following are equivalent*:

(1) *R is an exchange ring.*

(2) *For any $x \in R$, there exists a potent $e \in Rx$ such that $1-e \in R(1-x)$.*

Proof. (1) $\Rightarrow$ (2) is clear by [8, Theorem 2.1 and Proposition 1.1].

(2) $\Rightarrow$ (1). For any $x \in R$, there exists a potent $e \in Rx$ such that $1-e \in R(1-x)$. Assume that $e^n = e$ for some integer $n \geq 2$. Let $f = e^{n-1}$. Then $f^2 = e^{2n-2} = ee^{n-2} = f$. Furthermore, we have $f \in Re \subseteq Rx$ such that $1-f = 1-e^{n-1} = (1+e+\cdots+e^{n-2})(1-e) \in R(1-e) \subseteq R(1-x)$. Using [8, Theorem 2.1 and Proposition 1.1], R is an exchange ring. □

Recall that an element $u \in R$ is a square root of 1 if $u^2 = 1$. In [4, Proposition 10], it is shown that a ring R with $\frac{1}{2} \in R$ is a clean ring if and only if every element of R is a sum of a unit and a square root of 1.

Proposition 2.2. *The following are equivalent*:

(1) *R is an exchange ring.*

(2) *For any $x \in R$, there exist an idempotent $e \in Rx$ and a central square root u of 1 such that $u-e \in R(1-x)$.*

Proof. (1) $\Rightarrow$ (2) is obvious by [8, Theorem 2.1 and Proposition 1.1].

(2) $\Rightarrow$ (1). For any $x \in R$, there exist an idempotent $e \in Rx$ and a central square root u of 1 such that $u-e \in R(1-x)$. Let $f = ue$. Then $f \in Rx$ and $1-f \in R(1-x)$. It is easy to verify that $f^3 = f$. Therefore R is an exchange ring by Lemma 2.1. □

Lemma 2.3. *The following are equivalent*:

(1) *R is an exchange ring.*

(2) *For any $x \in R$, there exists a potent $e \in R$ such that $e-x \in R(x-x^2)$.*

Proof. (1) $\Rightarrow$ (2) is clear by [8, Theorem 2.1 and Proposition 1.1].

(2) $\Rightarrow$ (1). For any $x \in R$, there exists a potent $e \in R$ such that $e-x \in R(x-x^2)$. Assume that $e-x = r(x-x^2)$ for a $r \in R$. Hence we have $e = \big(1+r(1-x)\big)x \in Rx$

such that $1-e=(1-rx)(1-x)\in R(1-x)$. According to Lemma 2.1, R is an exchange ring. □

Let I be a left ideal of a ring R. We say that idempotents can be p-lifted modulo I provided that if $x-x^2\in I$ then there exists a potent $y\in R$ such that $x-y\in I$.

Theorem 2.3. *The following are equivalent*:

(1) *R is an exchange ring.*

(2) *Idempotents can be p-lifted modulo every left ideal.*

Proof. (1) ⇒ (2) is obvious by [8, Corollary 1.3].

(2) ⇒ (1). Let $x\in R$, and let $I=R(x-x^2)$. Clearly, $x-x^2\in I$. Thus we have a potent $e\in R$ such that $e-x\in I$. That is, $e-x\in R(x-x^2)$. In view of Lemma 2.3, we conclude that R is an exchange ring. □

We use $J(R)$ to denote the Jacobson radical of R. Furthermore, we can derive the following characterizations of exchange rings.

Proposition 2.4. *The following are equivalent*:

(1) *R is an exchange ring.*

(2) *For any $x\in R$, there exist a potent $e\in Rx$ and a $c\in R$ such that $(1-e)-c(1-x)\in J(R)$.*

(3) *For any $x\in R$, there exists a potent $e\in Rx$ such that $R=Re+R(1-x)$.*

(4) *$R/J(R)$ is an exchange ring and idempotents can be p-lifted modulo $J(R)$.*

Proof. (1) ⇒ (2) is clear by [8, Proposition 1.1].

(2) ⇒ (3). For any $x\in R$, we have a potent $e\in Rx$ and a $c\in R$ such that $(1-e)-c(1-x)\in J(R)$. Hence $e+c(1-x)=1+r$ for a $r\in J(R)$. Clearly, $1+r\in U(R)$; hence, $(1+r)^{-1}e+(1+r)^{-1}c(1-x)=1$. This means that $R=Re+R(1-x)$.

(3) ⇒ (1). For any $x\in R$, there exist a potent $e\in Rx$ such that $R=Re+R(1-x)$. So we have $r,s\in R$ such that $re+s(1-x)=1$. As $e\in R$ is a potent element, we can find an integer n such that $e^n=e$. Let $f=e^{n-1}+(1-e^{n-1})re$. Then we check that $f=f^2\in Rx$. Furthermore, we have $1-f=(1-e^{n-1})(1-re)=(1-e^{n-1})s(1-x)\in R(1-x)$, as required.

(1) ⇒ (4) is obvious by [8, Theorem 2.1 and Proposition 1.1].

(4) ⇒ (1). Let $e\in R$ be an idempotent. Then we have a potent $f\in R$ such that $e-f\in J(R)$. Since $f\in R$ is potent, we can find an integer $n\geq 2$ such that $f^n=f$. As $\overline{e}=\overline{f}$, we have $\overline{e}=\overline{e}^{n-1}=\overline{f}^{n-1}=\overline{f^{n-1}}$. Clearly, $f^{n-1}\in R$ is an idempotent. So idempotents can be lifted modulo $J(R)$. Therefore R is an exchange ring by [8, Proposition 1.5]. □

3. Quasi-clean ring

Let $e_1,\ e_2,\ \cdots,\ e_n \in R$ be idempotents. Then $\begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix} = \left\{ \begin{pmatrix} e_1r_{11}e_1 & \cdots & e_1r_{1n}e_n \\ \vdots & \ddots & \vdots \\ e_1r_{n1}e_1 & \cdots & e_1r_{nn}e_n \end{pmatrix} \ \middle|\ r_{ij} \in R \quad (1 \leq i,j \leq n) \right\}$ forms a ring with the identity diag$(e_1,\ \cdots,\ e_n)$.

Lemma 3.1. *Let $e_1,\ e_2,\ \cdots,\ e_n$ be idempotents of a ring R. If $e_1Re_1,\ e_2Re_2,\ \cdots,\ e_nRe_n$ are all quasi-clean rings, then so is the ring* $\begin{pmatrix} e_1Re_1 & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots \\ e_nRe_1 & \cdots & e_nRe_n \end{pmatrix}$.

Proof. The result holds for $n=1$. Assume that the result holds for $n=k\geq 1$. Let $n=k+1$. Set

$$B = \begin{pmatrix} e_2Re_2 & \cdots & e_2Re_{k+1} \\ \vdots & \ddots & \vdots \\ e_{k+1}Re_2 & \cdots & e_{k+1}Re_{k+1} \end{pmatrix}_{k\times k}, \quad M = \begin{pmatrix} e_2Re_1 \\ \vdots \\ e_{k+1}Re_1 \end{pmatrix}_{k\times 1},$$

$$N = \begin{pmatrix} e_1Re_2 & \cdots & e_1Re_{k+1} \end{pmatrix}_{1\times k} \quad \text{and} \quad T = \begin{pmatrix} e_1Re_1 & N \\ M & B \end{pmatrix}_{(k+1)\times(k+1)}.$$

Given any $\begin{pmatrix} a & n \\ m & b \end{pmatrix} \in T$, we can choose potent elements $e_1 \in e_1Re_1,\ e_2 \in B$ and invertible elements $u_1 \in U(e_1Re_1),\ u_2 \in U(B)$ such that $a = e_1 + u_1$ and $b - mu_1^{-1}n = e_2 + u_2$. Clearly, we have

$$\begin{pmatrix} a & n \\ m & b \end{pmatrix} = \begin{pmatrix} 0 & e_2 \end{pmatrix} + \begin{pmatrix} u_1 & n \\ m & u_2 + mu_1^{-1}n \end{pmatrix}.$$

One easily checks that

$$\begin{pmatrix} u_1 & n \\ m & u_2 + mu_1^{-1}n \end{pmatrix}^{-1} = \begin{pmatrix} u_1^{-1} + u_1^{-1}nu_2^{-1}mu_1^{-1} & -u_1^{-1}nu_2^{-1} \\ -u_2^{-1}mu_1^{-1} & u_2^{-1} \end{pmatrix}.$$

Clearly, $\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \in T$ is a potent matrix. By induction, we complete the proof. □

Theorem 3.2. *If R is a quasi-clean ring, then so is $M_n(R)$ for all positive integers n.*

Proof. In Lemma 3.1, we choose $e_1 = \cdots = e_n = 1$. Then we prove that $M_n(R)$ is a quasi-clean ring, as asserted. □

Lemma 3.3. *Let e be an idempotent of a ring R. If eRe and $(1-e)R(1-e)$ are quasi-clean rings, then so is R.*

Proof. Clearly, we have $R \cong \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$. Therefore we complete the proof by Lemma 3.1. □

Lemma 3.4. *Let $e_1,\ e_2,\ \cdots,\ e_n$ be idempotents of a ring R. Then the following are equivalent*:

(1) *$e_1Re_1,\ e_2Re_2,\ \cdots,\ e_nRe_n$ are quasi-clean rings.*

(2) *The ring*

$$\begin{pmatrix} e_1Re_1 & 0 & \cdots & 0 \\ e_2Re_1 & e_2Re_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_nRe_1 & e_nRe_2 & \cdots & e_nRe_n \end{pmatrix}$$

is a quasi-clean ring.

Proof. (1) $\Rightarrow$ (2). Suppose $e_1Re_1,\ e_2Re_2,\ \cdots,\ e_nRe_n$ are quasi-clean rings. Clearly, the result holds for $n=1$. Assume now that the result holds for $n=k\geq 1$. Let

$$B = \begin{pmatrix} e_2Re_2 & 0 & \cdots & 0 \\ e_3Re_2 & e_3Re_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{k+1}Re_2 & e_{k+1}Re_3 & \cdots & e_{k+1}Re_{k+1} \end{pmatrix}_{k\times k}, \quad M = \begin{pmatrix} e_2Re_1 \\ \vdots \\ e_{k+1}Re_1 \end{pmatrix}_{1\times k},$$

$$T = \begin{pmatrix} e_1Re_1 & 0 \\ M & B \end{pmatrix}_{(k+1)\times(k+1)} \quad \text{and} \quad e = \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}_{(k+1)\times(k+1)}.$$

Then B is a quasi-clean ring. Since $eTe \cong e_1Re_1$ and $\big(diag(e_1,\cdots,e_{k+1})-e\big)T\big(diag(e_1,\cdots,e_{k+1})-e\big) \cong B$ are both quasi-clean rings, by Lemma 3.3, T is a quasi-clean ring. By induction, we have $\begin{pmatrix} e_1Re_1 & 0 & \cdots & 0 \\ e_2Re_1 & e_2Re_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_nRe_1 & e_nRe_2 & \cdots & e_nRe_n \end{pmatrix}$ is a quasi-clean ring.

Conversely, assume that the condition (2) holds. Let

$$A = \begin{pmatrix} e_1Re_1 & 0 & \cdots & 0 \\ e_2Re_1 & e_2Re_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ e_{n-1}Re_1 & e_{n-1}Re_2 & \cdots & e_{n-1}Re_{n-1} \end{pmatrix}_{(n-1)\times(n-1)},$$

$$M = \begin{pmatrix} e_nRe_1 & e_nRe_2 & \cdots & e_nRe_{n-1} \end{pmatrix}_{(n-1)\times 1}.$$

Given any $a \in e_nRe_n$, then we check that $\begin{pmatrix} 0_{n-1} & 0 \\ 0 & a \end{pmatrix} \in \begin{pmatrix} A & 0 \\ M & e_nRe_n \end{pmatrix}_{n\times n}$. So we have a potent matrix $\begin{pmatrix} f & 0 \\ m & g \end{pmatrix}$ and an invertible matrix $\begin{pmatrix} u & 0 \\ n & v \end{pmatrix}$ such that $\begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} f & 0 \\ m & g \end{pmatrix} + \begin{pmatrix} u & 0 \\ n & v \end{pmatrix}$. Clearly, $a = g + v$ and $v \in U(e_nRe_n)$. As $\begin{pmatrix} f & 0 \\ m & g \end{pmatrix}$ is potent, we can find an integer $p \geq 2$ such that $\begin{pmatrix} f & 0 \\ m & g \end{pmatrix}^p = \begin{pmatrix} f & 0 \\ m & g \end{pmatrix}$; hence, $g^p = g$. This means that $g \in e_nRe_n$ is a potent element. Therefore we show that e_nRe_n is a quasi-clean ring. Likewise, $e_1Re_1, \cdots, e_{n-1}Re_{n-1}$ are quasi-clean rings. □

Let $LTM_n(R)$ denote the ring of all $n \times n$ lower triangular matrices over R and $UTM_n(R)$ denote the ring of all $n \times n$ upper triangular matrices over R. A natural problem is how to extend Theorem 3.2 to triangular matrix extensions. We now derive the following.

Theorem 3.5. *Let n be a positive integer. Then the following are equivalent*:

(1) *R is a quasi-clean ring.*

(2) *$LTM_n(R)$ is a quasi-clean ring.*

(3) *$UTM_n(R)$ is a quasi-clean ring.*

Proof. (1) ⇔ (2). In Lemma 3.4, we choose $e_1 = \cdots = e_n = 1$. Then R is a quasi-clean ring if and only if so is $LTM_n(R)$.

(1) ⇔ (3) is proved in the same manner. □

Corollary 3.6. *If G is a cyclic group of order 3, then the ring $M_n(\mathbb{Z}_pG)$, $LTM_n(R)$ and $UTM_n(\mathbb{Z}_pG)$ are quasi-clean rings for all positive integers.*

Proof. Let $G = \{1, a, a^2\}$ with $a^3 = 1$. Given any $\frac{m_1}{n_1} + \frac{m_2}{n_2}a + \frac{m_3}{n_3}a^2 = \frac{k+la+ma^2}{n} \in \mathbb{Z}_pG$, by [10, Theorem 3.1], we have

$$\frac{k+la+ma^2}{n} \in \mathbb{Z}_pG$$
$$= \begin{cases} -a^2 + \frac{k+la+(m+n)a^2}{n} & p \neq 2; \\ 1 + \frac{k-n+la+ma^2}{n} & p = 2,\ k,\ l,\ m \text{ are all even}; \\ \frac{2-a-a^2}{3} + \frac{3k-2n+(3l+n)a+(3m+n)a^2}{n} & p = 2,\ k,\ l,\ m \text{ are all odd}; \\ a + \frac{k+(l-n)a+ma^2}{n} & p = 2, \text{ one of } k,\ l,\ m \\ & \quad \text{is even, the other odd}; \\ 0 + \frac{k+la+ma^2}{n} & \text{otherwise}. \end{cases}$$

In the proof of [10, Theorem 3.1], the elements on the second columns at the right are all units. Clearly, the elements on the first columns at the right are all potent elements. So $\mathbb{Z}_pG$ is a quasi-clean ring. Therefore we complete the proof by Theorem 3.2 and Theorem 3.5. □

Corollary 3.7. *Let G be a cyclic group of order 3, $A_1 \in M_n(\mathbb{Z}_pG)$, $A_2 \in LTM_n(\mathbb{Z}_pG)$ and $A_3 \in UTM_n(\mathbb{Z}_pG)$. Then the following hold*:

(1) *There exist an idempotent matrix $E_1 \in M_n(R)$ and two invertible matrices $U_1,\ V_1 \in M_n(R)$ such that $A_1 = E_1U_1 + V_1$.*

(2) *There exist an idempotent matrix $E_2 \in LTM_n(R)$ and two invertible matrices $U_2,\ V_2 \in LTM_n(R)$ such that $A_2 = E_2U_2 + V_2$.*

(3) *There exist an idempotent matrix $E_3 \in UTM_n(R)$ and two invertible matrices $U_3,\ V_3 \in UTM_n(R)$ such that $A_3 = E_3U_3 + V_3$.*

Proof. (1) In view of Corollary 3.6, we have a potent $B_1 \in M_n(R)$ and an invertible matrix $V_1 \in M_n(R)$ such that $A_1 = B_1 + V_1$. As B_1 is potent, there exists an integer $m \geq 2$ such that $B_1 = B_1^m$. Set $W_1 = B_1^{m-2} - B_1^{m-1} + I_m$. Then $U^{-1} = B_1 - B_1^{-1} + I_m$. Furthermore, we have $B_1U = B_1^{m-1}$. Let $E_1 = B_1^{m-1}$ and $U_1 = W_1^{-1}$. Then $A_1 = E_1U_1 + V_1$, as asserted.

(2) and (3) are proved in the same manner. □

It is well known that every clean ring is an exchange ring. We note that there exists a quasi-clean ring which is not an exchange ring. Let $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0 \text{ and } 3 \nmid b \text{ and } 5 \nmid b\}$. By [1, Proposition 16], each element $x \in R$ can be written in the form $x = u + e$ or $x = u - e$ where $u \in U(R)$ and $e \in R$ is an idempotent. Clearly, $\pm e \in R$ is a potent element. Hence R is a quasi-clean ring. But R is not an exchange ring because it is indecomposable, and not quasilocal.

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