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Mutifractal Analysis of Perturbed Cantor Sets

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Abstract. Let $\{K_\alpha\}_{\alpha\in\mathbb{R}}$ be the multifractal spectrums of a perturbed Cantor set K. We find the set of values α of nonempty set K_α by using the Birkhoff ergodic theorem. And we also show that such K_α is a fractal set in the sense of Taylor [12].

1. Introduction

Multifractal analysis [2], [3], [8], [10], [11] aims to quantify the singularity structure of measures defined on a fractal set K in $\mathbb{R}^N$ and provide a model for phenomena in which scaling occurs with a range of different power laws. Specially, if K has a fractal dimension s and supports a natural finite measure μ, we expect that

$$0 < \limsup_{r\to 0} \frac{\mu(B(x,r))}{(2r)^s} < \infty$$

or more generally,

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log(2r)} = s \quad \text{for all} \quad x \in K.$$

However, multifractal theory is much interesting when this does not happen. In other word, this is concerned about subsets

$$K_\alpha = \{x \in K : \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha\}$$

of K for the parameter α. The main problem in multifractal theory is to estimate the size of K_α by using the $f(\alpha)$-spectrum defined by their Hausdorff (or packing) dimension. In general, one does not use the box dimensions of K_α because, for many cases, K_α's are dense in K, so their box dimensions are equal to the box dimension of K itself.

There are many problems in mathematics which may readily be solved in linear cases but which have non-linear counterparts that are much harder to analysis. In

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this research we describes a procedure which allows many results and ideas from the linear or piecewise linear situation to be extended to non-linear cases.

Now, we analyze multifractal structure for non-linear Cantor sets in $\mathbb{R}^N$, so called, perturbed Cantor set with a Bernoulli measure. For this, we first define a perturbed Cantor set [1], [9].

Fix $m \geq 2$, let $\Sigma = \{1, 2, \cdots, m\}$, $\Sigma^k = \{1, 2, \cdots, m\}^k$ and $\Sigma^* = \bigcup_{k \geq 1} \Sigma^k$. Suppose that $\{S_{i_1 i_2 \cdots i_k} : (i_1, i_2, \cdots, i_k) \in \Sigma^*\}$ is a sequence of mappings on a compact subset X of $\mathbb{R}^N$ with $|X| = 1$ such that

$$S_{i_1 i_2 \cdots i_k} : X \to X, \ i_j \in \{1, 2, \cdots, m\}$$

$|S_{i_1 i_2 \cdots i_k}(x) - S_{i_1 i_2 \cdots i_k}(y)| = r_{i_k}|x - y|$ for $x, y \in X$, $0 < r_{i_k} < 1$, and there exists $0 < C < 1$ such that

$$C|X_{i_1, \cdots, i_k}| \leq \min_{1 \leq i \neq j \leq m} \text{dist}(X_{i_1 i_2 \cdots i_k i}, \ X_{i_1 i_2 \cdots i_k j}) \ \text{ for all } k \in \mathbb{N}. \tag{1.1}$$

Put $X_{i_1 i_2 \cdots i_k} = S_{i_1} \circ S_{i_1 i_2} \circ \cdots \circ S_{i_1 i_2 \cdots i_k}(X)$ and

$$K := \bigcap_{k=1}^{\infty} \bigcup_{(i_1, i_2, \cdots i_k) \in \Sigma^k} X_{i_1 i_2 \cdots i_k}.$$

This K is called *a perturbed Cantor set* generated by $\{S_{i_1 i_2 \cdots i_k}\}$.

Clearly the condition (1.1) implies K satisfies the open set condition.

Noting $\cap_{k=1}^{\infty} X_{i_1 i_2 \cdots i_k}$ is a singleton, we can define a bijective map

$$\pi : \Sigma \to K \ \text{ by } \ \pi(\mathbf{i}) = \bigcap_{k=1}^{\infty} X_{i_1 i_2 \cdots i_k},$$

where $\mathbf{i} = (i_1, i_2, \cdots) \in \Sigma$.

Fix a probability vector $(p_1, p_2, \cdots, p_m)$ with $\sum_{i=1}^m p_i = 1$ and $0 < p_i < 1$. Let ν be the corresponding infinite product measure on Σ. Define $\mu = \nu \circ \pi^{-1}$ which is the Borel probability measure on K such that

$$\mu(X_{i_1 i_2 \cdots i_k}) = \prod_{j=1}^{k} p_{i_j} \ \text{ for } (i_1, i_2, \cdots, i_k) \in \Sigma^k. \tag{1.2}$$

This μ is called *the* $(p_1, p_2, \cdots, p_m)$*-Bernoulli measure* on K.

For each $\alpha \in \mathbb{R}$, let

$$K_\alpha := \{x \in \mathbb{R}^N : \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r} = \alpha\},$$

where $B(x, r) = \{y \in \mathbb{R}^N : |x - y| < r\}$. We say $\{K_\alpha\}_{\alpha \in \mathbb{R}}$ is the *multifractal decomposition* of K, and $\{f_H(\alpha)\}_{\alpha \in \mathbb{R}}$, $\{f_p(\alpha)\}_{\alpha \in \mathbb{R}}$ *the multifractal spectrums* (*or the singularity spectrums*) of μ, where $f_H(\alpha)(f_p(\alpha))$ is the Hausdorff(packing) dimension of K_α (see [4], [5] for more information).

We will prove that such K_α is a fractal in the sense of Taylor [12] and the multifractal spectrums of μ are the Legendre transformation of a famous auxiliary function β satisfied with $\sum_{i=1}^m p_i^q r_i^\beta = 1$.

Now let us recall some of basic facts for the auxiliary function $\beta(q)$ from [2], [3], [10].

Given a real number q, we define $\beta = \beta(q)$ as the positive number satisfying

$$\sum_{i=1}^m p_i^q r_i^\beta = 1. \tag{1.3}$$

Then $\beta : \mathbb{R} \to \mathbb{R}$ is a decreasing real analytic function with $\lim_{q\to-\infty} \beta(q) = \infty$ and $\lim_{q\to\infty} \beta(q) = -\infty$. Clearly $\beta(1) = 0$. And let f be the Legendre transformation of β. Then $f : [\alpha_{\min}, \alpha_{\max}] \to \mathbb{R}$ is given by $f(\alpha) = \beta(q) + \alpha q$, where

$$\alpha_{\min} = \min_{1\le i\le m} \frac{\log p_i}{\log r_i}, \quad \alpha_{\max} = \max_{1\le i\le m} \frac{\log p_i}{\log r_i},$$

are the negative slope of the asymptotes of the function β. And $f(\alpha_{\min}) = \alpha_*$, $f(\alpha_{\max}) = \alpha^*$ with

$$\sum_{i \,\in\, \{i:\frac{\log p_i}{\log r_i} \,=\, \alpha_{\min}\}} r_i^{\alpha_*} = 1 \quad \text{and} \quad \sum_{i \,\in\, \{i:\frac{\log p_i}{\log r_i} \,=\, \alpha_{\max}\}} r_i^{\alpha^*} = 1 \tag{1.4}$$

In particular,

$$\lim_{q\to\infty} \alpha(q) = \alpha_{\min} \quad \text{and} \quad \lim_{q\to-\infty} \alpha(q) = \alpha_{\max}. \tag{1.5}$$

2. Main results

Let's list or prove some basic but useful facts before our main theorem.

Lemma 2.1 [5, Proposition 2.3]**.** *Let E be a Borel set and μ be a finite measure on $\mathbb{R}^N$ as in* (1.3).

(1) *If* $\liminf_{r\to0} \dfrac{\log \mu(B(x,r))}{\log r} \ge s$ *for all $x \in E$ and $\mu(E) > 0$, then* $\dim_H E \ge s$.

(2) *If* $\liminf_{r\to0} \dfrac{\log \mu(B(x,r))}{\log r} \le s$ *for all $x \in E$, then* $\dim_H E \le s$.

(3) *If* $\limsup_{r\to0} \dfrac{\log \mu(B(x,r))}{\log r} \ge s$ *for all $x \in E$ and $\mu(E) > 0$, then* $\dim_p E \ge s$.

(4) *If* $\limsup_{r\to 0} \dfrac{\log \mu(B(x,r))}{\log r} \le s$ *for all* $x \in E$, *then* $\dim_p E \le s$.

We can easily get the following Lemma from the definition of a perturbed Cantor set.

Lemma 2.2. *Let* $d = \min_{1\le i\ne j\le m} \mathrm{dist}(X_i,\ X_j)$ *and* C *as in* (1.1). *If* $x \in X_{i_1 i_2\cdots i_k} \cap K$ *and* $|X_{i_1 i_2\cdots i_k}| \le r < |X_{i_1 i_2\cdots i_k}| d^{-1}$, *then*

$$B(x, Cdr) \cap K \subset X_{i_1 i_2\cdots i_k} \cap K \subset B(x,r).$$

For $x \in K$, we denote $X_k(x)$ for the k-th level set $X_{i_1 i_2\cdots i_k}$ that contains x.
It is not hard to show the next Proposition with Lemma 2.2.

Proposition 2.3.

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha \ \text{ iff } \ \lim_{k\to 0} \frac{\log \mu(X_k(x))}{\log |X_k(x)|} = \alpha.$$

We now introduce a mass distribution measure λ supported on $K_{\alpha(q)}$ for fixed q. For given $q \in \mathbb{R}$ and $\beta = \beta(q)$, we define a probability measure λ on X by

$$\lambda(X_{i_1 i_2\cdots i_k}) = \mu(X_{i_1 i_2\cdots i_k})^q |X_{i_1 i_2\cdots i_k}|^{\beta(q)} \tag{2.1}$$

and extend this to a Borel measure on $\mathbb{R}^N$ in the usual way.

Theorem 2.4. *Let* $\alpha \in (\alpha_{\min}, \alpha_{\max})$. *Then*

(1) $\lambda(K_\alpha) = 1$.

(2) $\lim_{r\to 0} \frac{\log \lambda(B(x,r))}{\log r} = f(\alpha)$ *for all* $x \in K_\alpha$.

Proof. (1) Define $\phi(x) = \log \mu(X_{i_1})$ and $\psi(x) = \log |X_{i_1}| = \log r_{i_1}$ for $x = \pi(i_1, i_2, \cdots)$. Then

$$\int |\phi| d\lambda = \sum_{i=1}^m \lambda(X_i) |\log \mu(X_i)| = \sum_{i=1}^m p_i^q r_i^\beta |\log p_i| < \infty \ \text{ and}$$

$$\int |\psi| d\lambda = \sum_{i=1}^m p_i^q r_i^\beta |\log r_i| < \infty.$$

Define the shift transformation $T : K \to K$ by $T(x) = \pi(i_2, i_3, \cdots)$, where $x = \pi(i_1, i_2, \cdots)$. Since the shift map $\sigma : \Sigma \to \Sigma$ defined by $\sigma(i_1, i_2, \cdots) = (i_2, i_3, \cdots)$ is ergodic with respect to ν, so is T respect to μ. Hence we can easily show that

T is ergodic with respect to λ by replacing a probability vector $(p_1, \cdots, p_m)$ by $(p_1^q r_1^{\beta(q)}, \cdots, p_m^q r_m^{\beta(q)})$. By Birkhoff ergodic Theorem([13]),

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \phi(T^k(x)) &= \int \phi \, d\lambda \text{ for } \lambda - \text{ a.e. } x \\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \psi(T^k(x)) &= \int \psi \, d\lambda \text{ for } \lambda - \text{ a.e. } x. \end{aligned}$$

That is,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \log \mu(X_n(x)) &= \sum_{i=1}^{m} p_i^q r_i^\beta \log p_i \quad \text{for} \quad \lambda - \text{a.e. } x \\ \lim_{n\to\infty} \frac{1}{n} \log |X_n(x)| &= \sum_{i=1}^{m} p_i^q r_i^\beta \log r_i \quad \text{for} \quad \lambda - \text{a.e. } x. \end{aligned}$$

So,

$$\lim_{k\to\infty} \frac{\log \mu(X_k(x))}{\log |X_k(x)|} = \frac{\sum_{i=1}^m p_i^q r_i^\beta \log p_i}{\sum_{i=1}^m p_i^q r_i^\beta \log r_i}$$

for λ - a.e. x.

By differentiating of (1.3) and $f(\alpha)$ with respect to q, we get $\alpha = \beta'(q) = \frac{\sum_{i=1}^m p_i^q r_i^\beta \log p_i}{\sum_{i=1}^m p_i^q r_i^\beta \log r_i}$ and so $\lambda \left\{ x \in X : \lim_{k\to\infty} \frac{\log \mu(X_k(x))}{\log |X_k(x)|} = \alpha \right\} = 1$. We have thus $\lambda(K_\alpha) = 1$ by Proposition 2.3.

For (2)

$$\frac{\log \lambda(X_k(x))}{\log |X_k(x)|} = q\frac{\log \mu(X_k(x))}{\log |X_k(x)|} + \beta \frac{\log |X_k(x)|}{\log |X_k(x)|} \to q\alpha + \beta = f(\alpha) \text{ as } k \to \infty$$

for all $x \in K_\alpha$. Since Proposition 2.3 remains true with λ replacing μ, our proof is complete. □

Theorem 2.5.

(1) $K_\alpha = \emptyset$ *for* $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$.

(2) $f(\alpha) = f_H(\alpha) = f_p(\alpha)$ *for* $\alpha \in [\alpha_{\min}, \alpha_{\max}]$.

Proof. Let $c_i = \dfrac{\log p_i}{\log r_i}$. Then, from (2.1),

$$\frac{\log \mu(X_{i_1 i_2 \cdots i_k})}{\log |X_{i_1 i_2 \cdots i_k}|} = \frac{\sum_{j=1}^k c_{i_j} \log r_{i_j}}{\sum_{j=1}^k \log r_{i_j}} \in [\alpha_{\min}, \alpha_{\max}] \text{ for all } k.$$

We get Thus $\lim_{r\to 0}\dfrac{\log\mu(B(x,r))}{\log r}\in[\alpha_{\min},\alpha_{\max}]$ by Proposition 2.3. In particular, $K_\alpha=\emptyset$ if $\alpha\notin[\alpha_{\min},\,\alpha_{\max}]$.

If α be in $(\alpha_{\min},\alpha_{\max})$, then, by Theorem 2.4, there exists a mass distribution λ concentrated on K_α with

$$\lim_{r\to 0}\frac{\log\lambda(B(x,r))}{\log r}=f(\alpha)$$

for all $x\in K_\alpha$. It follows from Lemma 2.1 that $f(\alpha)=f_H(\alpha)=f_p(\alpha)$.

If $\alpha=\alpha_{\min}$ and

$$M=\{x=\pi(i_1,i_2,\cdots):\frac{\log p_{i_j}}{\log r_{i_j}}=\alpha\ \text{ for }\ \text{all } j\},$$

then we can easily see that $M\subset K_\alpha$. Since M is constructed with ratios given by r_i's for which $\dfrac{\log p_i}{\log r_i}=\alpha$, so $\dim K_\alpha\ge\dim M=\alpha^*$ with $\sum_{i\,\in\,\{i\,:\,\frac{\log p_i}{\log r_i}=\alpha\}}r_i^{\alpha_*}=1$, where the dim represents either $\dim_H$ or $\dim_p$.

Let

$$N_q=\{x:\limsup_{k\to\infty}\frac{\log\mu(X_k(x))}{\log|X_k(x)|}\le\alpha(q)\}.$$

Then, using (1.5) and $\alpha(q)$ is decreasing as q is increases, we get $K_\alpha\subset N_q$, for all $q>0$. But, we can easily prove that $\dim N_q\le f(\alpha(q))$ for all $q>0$ by Lemma 2.1(2), (4). Hence $\dim K_\alpha\le f(\alpha(q))$ for all $q>0$.

Thus we have $f(\alpha(q))\to f(\alpha)=\alpha_*$ as $q\to\infty$ by (1.4) and (1.5).

Similarly we have $\dim K_\alpha=\alpha^*$ if $\alpha=\alpha_{\max}$. □

Example 2.6. Put $X=[0,1]\times[0,1]$ and define S_i,T_j, and $\{S_{i_1i_2\cdots i_k}\}:X\to X$ $i=1,2$ and $j=1,2$ by

$$\begin{aligned}
S_1 &: (x,y)\to\left(\frac{1}{3}x,\ \frac{1}{3}y\right)\\
S_2 &: (x,y)\to\left(\frac{1}{3}x+\frac{2}{3},\ \frac{1}{3}y+\frac{2}{3}\right)\\
T_1 &= S_1,\\
T_2 &: (x,y)\to\left(\frac{1}{3}x+\frac{2}{3},\ \frac{1}{3}y\right)
\end{aligned}$$

and, for $k\ge 2$,

$$S_{i_1i_2\cdots i_k}=\begin{cases}T_{i_k}, & i_1=1\\ S_{i_k}, & i_1=2.\end{cases}$$

Then we get a perturbed Cantor set K generated by $\{S_{i_1i_2\cdots i_k}\}$(See figure 1). Consider the (p_1,p_2)-Bernoulli measure μ on K, and $1=p_1^q(\frac{1}{3})^\beta+p_2^q(\frac{1}{3})^\beta$, $q\in\mathbb{R}$. If $q=0$, then $\dim_H K=\dim_p K=f(\alpha(0))=\beta(0)=\dfrac{\log 2}{\log 3}\doteqdot 0.63083$.

Specially, if we take $p_1 = \dfrac{1}{4}$ and $p_2 = \frac{3}{4}$ then the graphs of $f(\alpha)$ and $\beta(q)$ can be drawn like figure 2.

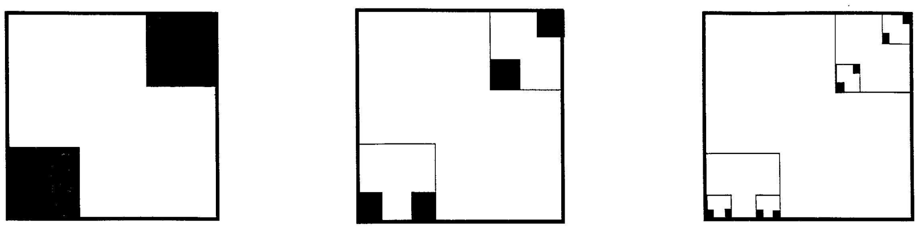

Figure 1: Step 1, 2 and 3 of a Perturbed Cantor set

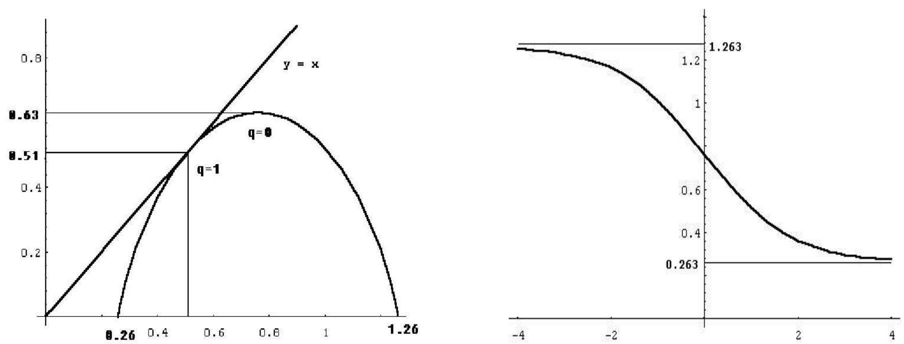

Figure 2: $f(\alpha)$ and $\beta(q)$

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