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Ω-BIFUZZY SUBSEMIGROUPS IN SEMIGROUPS

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Abstract. Given a set Ω, the notion of an Ω-bifuzzy subsemigroup in semigroups is given, and some properties are investigated. Homomorphic image and inverse image of an Ω-bifuzzy subsemigroup are considered.

1. Introduction

Hong et al. [3] and Kuroki [5, 6] have studied several properties of fuzzy left (right) ideals, fuzzy bi-ideals and fuzzy interior ideals in semigroups. For more other study on the fuzzy theory in semigroups, we refer to papers [7, 9, 10, 11]. In this paper, by using a set Ω, we define Ω-bifuzzy subsemigroups, and investigate some properties. We state how the homomorphic images and inverse images of Ω-bifuzzy subsemigroups become Ω-bifuzzy subsemigroups.

2. Preliminaries

By a *subsemigroup* of a semigroup S we mean a nonempty subset G of S such that $G^2 \subseteq G$.

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A *fuzzy set* in S is a function μ from S into the unit interval $[0, 1]$. A fuzzy set μ in S is called a *fuzzy subsemigroup* of S if it satisfies

$$(\forall x, y \in S)\,(\mu(xy) \geq \min\{\mu(x), \mu(y)\}).$$

Let X be a nonempty set and let μ_A and γ_A be two functions from X to $[0, 1]$ such that

$$(\forall x \in X)\,(0 \leq \mu_A(x) + \gamma_A(x) \leq 1).$$

By the original definition of Atanassov in [1], an *intuitionistic fuzzy set* is an object of the form: $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$. This object is also called a *bifuzzy set* (according to [2]). We consider it in a form of an ordered triple: $A = \langle X; \mu_A, \gamma_A\rangle$ where X, μ_A and γ_A are as above.

A bifuzzy set $A = \langle S; \mu_A, \gamma_A\rangle$ in a semigroup S is called *bifuzzy subsemigroup* of S (see [8]) if it satisfies:

$$(\forall x, y \in S)\,(\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\},\ \gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\}).$$

3. Ω-bifuzzy subsemigroups

In what follows let S and Ω denote a semigroup and a nonempty set, respectively, unless otherwise specified.

An *Ω-bifuzzy set* A_Ω in S is defined to be an object having the form

$$A_\Omega := \{\langle (x, \alpha); \mu_{A_\Omega}(x, \alpha), \gamma_{A_\Omega}(x, \alpha)\rangle \mid (x, \alpha) \in S \times \Omega\}$$

where the function $\mu_{A_\Omega} : S \times \Omega \to [0, 1]$ and $\gamma_{A_\Omega} : S \times \Omega \to [0, 1]$ define the degree of membership and the degree of nonmembership of the element $(x, \alpha) \in S \times \Omega$ to $A_\Omega \subset S \times \Omega$, respectively, such that

$$(\forall (x, \alpha) \in S \times \Omega)\,(0 \leq \mu_{A_\Omega}(x, \alpha) + \gamma_{A_\Omega}(x, \alpha) \leq 1).$$

We shall use the symbol $A_\Omega = \langle S \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ for the Ω-bifuzzy set

$$A_\Omega := \{\langle (x, \alpha); \mu_{A_\Omega}(x, \alpha), \gamma_{A_\Omega}(x, \alpha)\rangle \mid (x, \alpha) \in S \times \Omega\}.$$

Definition 3.1. An Ω-bifuzzy set $A_\Omega = \langle S \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ in S is called an *Ω-bifuzzy subsemigroup* of S if it satisfies

$$(\forall \alpha \in \Omega)\,(\forall x, y \in S)\,(\mu_{A_\Omega}(xy, \alpha) \geq \min\{\mu_{A_\Omega}(x, \alpha), \mu_{A_\Omega}(y, \alpha)\}),$$

$$(\forall \alpha \in \Omega)\,(\forall x, y \in S)\,(\gamma_{A_\Omega}(xy, \alpha) \leq \max\{\gamma_{A_\Omega}(x, \alpha), \gamma_{A_\Omega}(y, \alpha)\}).$$

Example 3.2. Consider a semigroup $S = \{a, b\}$ with the following Cayley table:

	a	b
a	a	b
b	b	a

Let $\Omega = \{1, 2\}$ and let $A_\Omega = \langle S \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ be an Ω-bifuzzy set in S defined by

$$A_\Omega = \big\langle S \times \Omega;, \big(\tfrac{(a,1)}{1}, \tfrac{(a,2)}{1}, \tfrac{(b,1)}{0.8}, \tfrac{(b,2)}{0.5}\big), \big(\tfrac{(a,1)}{0}, \tfrac{(a,2)}{0}, \tfrac{(b,1)}{0.1}, \tfrac{(b,2)}{0.3}\big)\big\rangle.$$

It is easy to verify that $A_\Omega = \langle S \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ is an Ω-bifuzzy subsemigroup of S.

Let $S^\Omega := \{u \mid u : \Omega \to S\}$. For any $u, v \in S^\Omega$, we define $(uv)(\alpha) = u(\alpha)v(\alpha)$ for all $\alpha \in \Omega$. Then S^Ω is a semigroup (see [4]).

Example 3.3. Let $A = \langle S; \mu_A, \gamma_A\rangle$ be a bifuzzy subsemigroup of S and let $A_\Omega = \langle S^\Omega \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ be an Ω-bifuzzy set in S^Ω defined by $\mu_{A_\Omega}(u, \alpha) = \mu_A(u(\alpha))$ and $\gamma_{A_\Omega}(u, \alpha) = \gamma_A(u(\alpha))$ for all $u \in S^\Omega$ and $\alpha \in \Omega$. Then $A_\Omega = \langle S^\Omega \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ is an Ω-bifuzzy subsemigroup of S^Ω.

Proposition 3.4. *Let $A_\Omega = \langle S \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ be an Ω-bifuzzy subsemigroup of S. For each $\omega \in \Omega$, the bifuzzy set $A_\Omega^\omega := \langle S; \mu_{A_\Omega^\omega}, \gamma_{A_\Omega^\omega}\rangle$ in S given by $\mu_{A_\Omega^\omega}(x) = \mu_{A_\Omega}(x, \omega)$ and $\gamma_{A_\Omega^\omega} = \gamma_{A_\Omega}(x, \omega)$ for all $x \in S$ is a bifuzzy subsemigroup of S.*

PROOF. Let $x, y \in S$. Then

$$\begin{aligned}\mu_{A_\Omega^\omega}(xy) &= \mu_{A_\Omega}(xy,\omega) \geq \min\{\mu_{A_\Omega}(x,\omega), \mu_{A_\Omega}(y,\omega)\}\\ &= \min\{\mu_{A_\Omega^\omega}(x), \mu_{A_\Omega^\omega}(y)\},\end{aligned}$$

$$\begin{aligned}\gamma_{A_\Omega^\omega}(xy) &= \gamma_{A_\Omega}(xy,\omega) \leq \max\{\gamma_{A_\Omega}(x,\omega), \gamma_{A_\Omega}(y,\omega)\}\\ &= \max\{\gamma_{A_\Omega^\omega}(x), \gamma_{A_\Omega^\omega}(y)\}.\end{aligned}$$

This completes the proof. □

Proposition 3.5. *For each $\omega \in \Omega$, let $A_\Omega^\omega = \langle S; \mu_{A_\Omega^\omega}, \gamma_{A_\Omega^\omega}\rangle$ be a bifuzzy subsemigroup of S. Then the Ω-bifuzzy set $A_\Omega = \langle S\times\Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ in S given by $\mu_{A_\Omega}(x,\omega) = \mu_{A_\Omega^\omega}(x)$ and $\gamma_{A_\Omega}(x,\omega) = \gamma_{A_\Omega^\omega}(x)$ is an Ω-bifuzzy subsemigroup of S.*

PROOF. For any $x, y \in S$, we have

$$\begin{aligned}\mu_{A_\Omega}(xy,\omega) &= \mu_{A_\Omega^\omega}(xy) \geq \min\{\mu_{A_\Omega^\omega}(x), \mu_{A_\Omega^\omega}(y)\}\\ &= \min\{\mu_{A_\Omega}(x,\omega), \mu_{A_\Omega}(y,\omega)\},\end{aligned}$$

$$\begin{aligned}\gamma_{A_\Omega}(xy,\omega) &= \gamma_{A_\Omega^\omega}(xy) \leq \max\{\gamma_{A_\Omega^\omega}(x), \gamma_{A_\Omega^\omega}(y)\}\\ &= \max\{\gamma_{A_\Omega}(x,\omega), \gamma_{A_\Omega}(y,\omega)\}.\end{aligned}$$

Hence $A_\Omega = \langle S\times\Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ is an Ω-bifuzzy subsemigroup of S. □

Theorem 3.6. *Let $\Phi = \langle S^\Omega; \mu_\Phi, \gamma_\Phi\rangle$ be a bifuzzy subsemigroup of S^Ω and let $A_\Phi = \langle S\times\Omega; \mu_{A_\Phi}, \gamma_{A_\Phi}\rangle$ be an Ω-bifuzzy set in S defined by*

$$\mu_{A_\Phi}(x,\alpha) := \sup\{\mu_\Phi(u) \mid u \in S^\Omega, u(\alpha) = x\},$$

$$\gamma_{A_\Phi}(x,\alpha) := \inf\{\gamma_\Phi(u) \mid u \in S^\Omega, u(\alpha) = x\}$$

for all $x \in S$ and $\alpha \in \Omega$. Then $A_\Phi = \langle S\times\Omega; \mu_{A_\Phi}, \gamma_{A_\Phi}\rangle$ is an Ω-bifuzzy subsemigroup of S.

PROOF. Let $x, y \in S$ and $\alpha \in \Omega$. Then

$$\begin{aligned}\mu_{A_\Phi}(xy,\alpha) &= \sup\{\mu_\Phi(u) \mid u \in S^\Omega,\ u(\alpha) = xy\}\\ &\geq \sup\{\mu_\Phi(uv) \mid u, v \in S^\Omega,\ u(\alpha) = x,\ v(\alpha) = y\}\\ &\geq \sup\{\min\{\mu_\Phi(u), \mu_\Phi(v)\} \mid u, v \in S^\Omega,\ u(\alpha) = x,\ v(\alpha) = y\}\\ &= \min\{\sup\{\mu_\Phi(u) \mid u \in S^\Omega,\ u(\alpha) = x\}, \sup\{\mu_\Phi(v) \mid v \in S^\Omega,\ v(\alpha) = y\}\}\\ &= \min\{\mu_{A_\Phi}(x,\alpha), \mu_{A_\Phi}(y,\alpha)\},\end{aligned}$$

$$\begin{aligned}\gamma_{A_\Phi}(xy,\alpha) &= \inf\{\gamma_\Phi(u) \mid u \in S^\Omega,\ u(\alpha) = xy\}\\ &\leq \inf\{\gamma_\Phi(uv) \mid u, v \in S^\Omega,\ u(\alpha) = x,\ v(\alpha) = y\}\\ &\leq \inf\{\max\{\gamma_\Phi(u), \gamma_\Phi(v)\} \mid u, v \in S^\Omega,\ u(\alpha) = x,\ v(\alpha) = y\}\\ &= \max\{\inf\{\gamma_\Phi(u) \mid u \in S^\Omega,\ u(\alpha) = x\}, \inf\{\gamma_\Phi(v) \mid v \in S^\Omega,\ v(\alpha) = y\}\}\\ &= \max\{\gamma_{A_\Phi}(x,\alpha), \gamma_{A_\Phi}(y,\alpha)\}.\end{aligned}$$

Hence $A_\Phi = \langle S \times \Omega; \mu_{A_\Phi}, \gamma_{A_\Phi}\rangle$ is an Ω-bifuzzy subsemigroup of S. □

Example 3.7. Let $S = \{a, b\}$ be a semigroup in Example 3.2 and let $\Omega := \{1, 2\}$. Then $S^\Omega := \{e, u, v, w\}$, where $e(1) = e(2) = v(1) = w(2) = a$ and $u(1) = u(2) = v(2) = w(1) = b$, is a semigroup (in fact, a commutative group) under the following Cayley table:

	e	u	v	w
e	e	u	v	w
u	u	e	w	v
v	v	w	e	u
w	w	v	u	e

Let $\Phi = \langle S^\Omega; \mu_\Phi, \gamma_\Phi\rangle$ be a bifuzzy set in S^Ω defined by

$$\Phi = \left\langle S^\Omega; \left(\tfrac{e}{0.9}, \tfrac{u}{0.2}, \tfrac{v}{0.2}, \tfrac{w}{0.7}\right), \left(\tfrac{e}{0.1}, \tfrac{u}{0.7}, \tfrac{v}{0.7}, \tfrac{w}{0.2}\right)\right\rangle.$$

Then $\Phi = \langle S^\Omega; \mu_\Phi, \gamma_\Phi\rangle$ is a bifuzzy subsemigroup of S^Ω. Thus we can obtain an Ω-bifuzzy subsemigroup $A_\Phi = \langle S \times \Omega; \mu_{A_\Phi}, \gamma_{A_\Phi}\rangle$ of S as follows:

$$\mu_{A_\Phi}(a, 1) = \sup\{\mu_\Phi(x) \mid x \in S^\Omega,\ x(1) = a\} = \sup\{\mu_\Phi(e), \mu_\Phi(v)\} = 0.9,$$

$$\gamma_{A_\Phi}(a,1) = \inf\{\gamma_\Phi(x) \mid x \in S^\Omega,\ x(1) = a\} = \inf\{\gamma_\Phi(e), \gamma_\Phi(v)\} = 0.1,$$

$$\mu_{A_\Phi}(a,2) = \sup\{\mu_\Phi(x) \mid x \in S^\Omega,\ x(2) = a\} = \sup\{\mu_\Phi(e), \mu_\Phi(w)\} = 0.9,$$

$$\gamma_{A_\Phi}(a,2) = \inf\{\gamma_\Phi(x) \mid x \in S^\Omega,\ x(2) = a\} = \inf\{\gamma_\Phi(e), \gamma_\Phi(w)\} = 0.1,$$

$$\mu_{A_\Phi}(b,1) = \sup\{\mu_\Phi(x) \mid x \in S^\Omega,\ x(1) = b\} = \sup\{\mu_\Phi(u), \mu_\Phi(w)\} = 0.7,$$

$$\gamma_{A_\Phi}(b,1) = \inf\{\gamma_\Phi(x) \mid x \in S^\Omega,\ x(1) = b\} = \inf\{\gamma_\Phi(u), \gamma_\Phi(w)\} = 0.2,$$

$$\mu_{A_\Phi}(b,2) = \sup\{\mu_\Phi(x) \mid x \in S^\Omega,\ x(2) = b\} = \sup\{\mu_\Phi(u), \mu_\Phi(v)\} = 0.2,$$

$$\gamma_{A_\Phi}(b,2) = \inf\{\gamma_\Phi(x) \mid x \in S^\Omega,\ x(2) = b\} = \inf\{\gamma_\Phi(u), \gamma_\Phi(v)\} = 0.7.$$

Theorem 3.8. *Let $A_\Omega = \langle S \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ be an Ω-bifuzzy subsemigroup of S and let $\Phi = \langle S^\Omega; \mu_\Phi, \gamma_\Phi\rangle$ be a bifuzzy set in S^Ω defined by*

$$\mu_\Phi(u) = \inf\{\mu_{A_\Omega}(u(\alpha), \alpha) \mid \alpha \in \Omega\},$$

$$\gamma_\Phi(u) = \sup\{\gamma_{A_\Omega}(u(\alpha), \alpha) \mid \alpha \in \Omega\}$$

for all $u \in S^\Omega$. Then $\Phi = \langle S^\Omega; \mu_\Phi, \gamma_\Phi\rangle$ is a bifuzzy subsemigroup of S^Ω.

PROOF. For any $u, v \in S^\Omega$, we have

$$\begin{aligned}
\mu_\Phi(uv) &= \inf\{\mu_{A_\Omega}((uv)(\alpha), \alpha) \mid \alpha \in \Omega\} \\
&= \inf\{\mu_{A_\Omega}(u(\alpha)v(\alpha), \alpha) \mid \alpha \in \Omega\} \\
&\geq \inf\{\min\{\mu_{A_\Omega}(u(\alpha), \alpha), \mu_{A_\Omega}(v(\alpha), \alpha) \mid \alpha \in \Omega\} \\
&= \min\{\inf\{\mu_{A_\Omega}(u(\alpha), \alpha) \mid \alpha \in \Omega\}, \inf\{\mu_{A_\Omega}(v(\alpha), \alpha) \mid \alpha \in \Omega\}\} \\
&= \min\{\mu_\Phi(u), \mu_\Phi(v)\},
\end{aligned}$$

$$\begin{aligned}
\gamma_\Phi(uv) &= \sup\{\gamma_{A_\Omega}((uv)(\alpha), \alpha) \mid \alpha \in \Omega\} \\
&= \sup\{\gamma_{A_\Omega}(u(\alpha)v(\alpha), \alpha) \mid \alpha \in \Omega\} \\
&\leq \sup\{\max\{\gamma_{A_\Omega}(u(\alpha), \alpha), \gamma_{A_\Omega}(v(\alpha), \alpha) \mid \alpha \in \Omega\} \\
&= \max\{\sup\{\gamma_{A_\Omega}(u(\alpha), \alpha) \mid \alpha \in \Omega\}, \sup\{\gamma_{A_\Omega}(v(\alpha), \alpha) \mid \alpha \in \Omega\}\} \\
&= \max\{\gamma_\Phi(u), \gamma_\Phi(v)\}.
\end{aligned}$$

Thus $\Phi = \langle S^\Omega; \mu_\Phi, \gamma_\Phi\rangle$ is a bifuzzy subsemigroup of S^Ω. □

Example 3.9. Let $A_\Omega = \langle S\times\Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ be the Ω-bifuzzy subsemigroup of S in Example 3.2 and let S^Ω be the commutative group in Example 3.7. Then we can induce a bifuzzy subsemigroup $\Phi = \langle S^\Omega; \mu_\Phi, \gamma_\Phi\rangle$ of S^Ω as follows:

$$\begin{aligned}
\mu_\Phi(e) &= \inf\{\mu_{A_\Omega}(e(\alpha),\alpha) \mid \alpha\in\Omega\} = \inf\{\mu_{A_\Omega}(e(1),1), \mu_{A_\Omega}(e(2),2)\}\\
&= \inf\{\mu_{A_\Omega}(a,1), \mu_{A_\Omega}(a,2)\} = 1,
\end{aligned}$$

$$\begin{aligned}
\gamma_\Phi(e) &= \sup\{\gamma_{A_\Omega}(e(\alpha),\alpha) \mid \alpha\in\Omega\} = \sup\{\gamma_{A_\Omega}(e(1),1), \gamma_{A_\Omega}(e(2),2)\}\\
&= \sup\{\gamma_{A_\Omega}(a,1), \gamma_{A_\Omega}(a,2)\} = 0,
\end{aligned}$$

$$\begin{aligned}
\mu_\Phi(u) &= \inf\{\mu_{A_\Omega}(u(\alpha),\alpha) \mid \alpha\in\Omega\} = \inf\{\mu_{A_\Omega}(u(1),1), \mu_{A_\Omega}(u(2),2)\}\\
&= \inf\{\mu_{A_\Omega}(b,1), \mu_{A_\Omega}(b,2)\} = 0.5,
\end{aligned}$$

$$\begin{aligned}
\gamma_\Phi(u) &= \sup\{\gamma_{A_\Omega}(u(\alpha),\alpha) \mid \alpha\in\Omega\} = \sup\{\gamma_{A_\Omega}(u(1),1), \gamma_{A_\Omega}(u(2),2)\}\\
&= \sup\{\gamma_{A_\Omega}(b,1), \gamma_{A_\Omega}(b,2)\} = 0.3,
\end{aligned}$$

$$\begin{aligned}
\mu_\Phi(v) &= \inf\{\mu_{A_\Omega}(v(\alpha),\alpha) \mid \alpha\in\Omega\} = \inf\{\mu_{A_\Omega}(v(1),1), \mu_{A_\Omega}(v(2),2)\}\\
&= \inf\{\mu_{A_\Omega}(a,1), \mu_{A_\Omega}(b,2)\} = 0.5,
\end{aligned}$$

$$\begin{aligned}
\gamma_\Phi(v) &= \sup\{\gamma_{A_\Omega}(v(\alpha),\alpha) \mid \alpha\in\Omega\} = \sup\{\gamma_{A_\Omega}(v(1),1), \gamma_{A_\Omega}(v(2),2)\}\\
&= \sup\{\gamma_{A_\Omega}(a,1), \gamma_{A_\Omega}(b,2)\} = 0.3,
\end{aligned}$$

$$\begin{aligned}
\mu_\Phi(w) &= \inf\{\mu_{A_\Omega}(w(\alpha),\alpha) \mid \alpha\in\Omega\} = \inf\{\mu_{A_\Omega}(w(1),1), \mu_{A_\Omega}(w(2),2)\}\\
&= \inf\{\mu_{A_\Omega}(b,1), \mu_{A_\Omega}(a,2)\} = 0.8,
\end{aligned}$$

$$\begin{aligned}
\gamma_\Phi(w) &= \sup\{\gamma_{A_\Omega}(w(\alpha),\alpha) \mid \alpha\in\Omega\} = \sup\{\gamma_{A_\Omega}(w(1),1), \gamma_{A_\Omega}(w(2),2)\}\\
&= \sup\{\gamma_{A_\Omega}(b,1), \gamma_{A_\Omega}(a,2)\} = 0.1.
\end{aligned}$$

Definition 3.10. Let $\varphi : S \to T$ be a homomorphism of semigroups and let $B_\Omega = \langle T\times\Omega; \mu_{B_\Omega}, \gamma_{B_\Omega}\rangle$ be an Ω-bifuzzy set in T. Then the *inverse image* of $B_\Omega = \langle T\times\Omega; \mu_{B_\Omega}, \gamma_{B_\Omega}\rangle$, denoted by $\varphi^{-1}[B_\Omega] = \langle S\times\Omega; \mu_{\varphi^{-1}[B_\Omega]}, \gamma_{\varphi^{-1}[B_\Omega]}\rangle$, is the Ω-bifuzzy set in S given by $\mu_{\varphi^{-1}[B_\Omega]}(x,\alpha) = \mu_{B_\Omega}(\varphi(x),\alpha)$, $\gamma_{\varphi^{-1}[B_\Omega]}(x,\alpha) = \gamma_{B_\Omega}(\varphi(x),\alpha)$ for all $x\in S$ and $\alpha\in\Omega$.

Conversely, let $A_\Omega = \langle S\times\Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ be an Ω-bifuzzy set in S. The *image* of $A_\Omega = \langle S\times\Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$, written as $\varphi[A_\Omega] = \langle T\times\Omega; \mu_{\varphi[A_\Omega]}, \gamma_{\varphi[A_\Omega]}\rangle$, is an Ω-bifuzzy set in T defined by

$$\mu_{\varphi[A_\Omega]}(y,\alpha) = \begin{cases} \sup\limits_{z\in\varphi^{-1}(y)} \mu_{A_\Omega}(z,\alpha) & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\gamma_{\varphi[A_\Omega]}(y,\alpha) = \begin{cases} \inf\limits_{z\in\varphi^{-1}(y)} \gamma_{A_\Omega}(z,\alpha) & \text{if } \varphi^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

for all $y \in T$ and $\alpha \in \Omega$, where $\varphi^{-1}(y) = \{x \mid \varphi(x) = y\}$.

Theorem 3.11. *Let $\varphi : S \to T$ be a homomorphism of semigroups. If $B_\Omega = \langle T \times \Omega; \mu_{B_\Omega}, \gamma_{B_\Omega}\rangle$ is an Ω-bifuzzy subsemigroup of T, then the inverse image $\varphi^{-1}[B_\Omega] = \langle S \times \Omega; \mu_{\varphi^{-1}[B_\Omega]}, \gamma_{\varphi^{-1}[B_\Omega]}\rangle$ of $B_\Omega = \langle T \times \Omega; \mu_{B_\Omega}, \gamma_{B_\Omega}\rangle$ is an Ω-bifuzzy subsemigroup of S.*

Proof. Let $x, y \in S$ and $\alpha \in \Omega$. Then

$$\begin{aligned} \mu_{\varphi^{-1}[B_\Omega]}(xy,\alpha) &= \mu_{B_\Omega}(\varphi(xy),\alpha) = \mu_{B_\Omega}(\varphi(x)\varphi(y),\alpha) \\ &\geq \min\{\mu_{B_\Omega}(\varphi(x),\alpha), \mu_{B_\Omega}(\varphi(y),\alpha)\} \\ &= \min\{\mu_{\varphi^{-1}[B_\Omega]}(x,\alpha), \mu_{\varphi^{-1}[B_\Omega]}(y,\alpha)\}, \end{aligned}$$

$$\begin{aligned} \gamma_{\varphi^{-1}[B_\Omega]}(xy,\alpha) &= \gamma_{B_\Omega}(\varphi(xy),\alpha) = \gamma_{B_\Omega}(\varphi(x)\varphi(y),\alpha) \\ &\leq \max\{\gamma_{B_\Omega}(\varphi(x),\alpha), \gamma_{B_\Omega}(\varphi(y),\alpha)\} \\ &= \max\{\gamma_{\varphi^{-1}[B_\Omega]}(x,\alpha), \gamma_{\varphi^{-1}[B_\Omega]}(y,\alpha)\}. \end{aligned}$$

Hence $\varphi^{-1}[B_\Omega] = \langle S \times \Omega; \mu_{\varphi^{-1}[B_\Omega]}, \gamma_{\varphi^{-1}[B_\Omega]}\rangle$ is an Ω-bifuzzy subsemigroup of S. □

Theorem 3.12. *Let $\varphi : S \to T$ be a homomorphism between semigroups S and T. If $A_\Omega = \langle S \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ is an Ω-bifuzzy subsemigroup of S, then the image $\varphi[A_\Omega] = \langle T \times \Omega; \mu_{\varphi[A_\Omega]}, \gamma_{\varphi[A_\Omega]}\rangle$ of $A_\Omega = \langle S \times \Omega; \mu_{A_\Omega}, \gamma_{A_\Omega}\rangle$ is an Ω-bifuzzy subsemigroup of T.*

PROOF. We first prove that

$$\varphi^{-1}(y_1)\varphi^{-1}(y_2) \subseteq \varphi^{-1}(y_1y_2) \tag{3.1}$$

for all $y_1, y_2 \in T$. For, if $x \in \varphi^{-1}(y_1)\varphi^{-1}(y_2)$, then $x = x_1x_2$ for some $x_1 \in \varphi^{-1}(y_1)$ and $x_2 \in \varphi^{-1}(y_2)$. Since φ is a homomorphism, it follows that $\varphi(x) = \varphi(x_1x_2) = \varphi(x_1)\varphi(x_2) = y_1y_2$ so that $x \in \varphi^{-1}(y_1y_2)$. Hence (3.1) holds. Now let $y_1, y_2 \in T$ and $\alpha \in \Omega$. Assume that $y_1y_2 \notin \mathrm{Im}(\varphi)$. Then $\mu_{\varphi[A_\Omega]}(y_1y_2, \alpha) = 0$ and $\gamma_{\varphi[A_\Omega]}(y_1y_2, \alpha) = 1$. But if $y_1y_2 \notin \mathrm{Im}(\varphi)$, i.e., $\varphi^{-1}(y_1y_2) = \emptyset$, then $\varphi^{-1}(y_1) = \emptyset$ or $\varphi^{-1}(y_2) = \emptyset$ by (3.1). Thus $\mu_{\varphi[A_\Omega]}(y_1, \alpha) = 0$ or $\mu_{\varphi[A_\Omega]}(y_2, \alpha) = 0$, and $\gamma_{\varphi[A_\Omega]}(y_1, \alpha) = 1$ or $\gamma_{\varphi[A_\Omega]}(y_2, \alpha) = 1$. It follows that

$$\mu_{\varphi[A_\Omega]}(y_1y_2, \alpha) = 0 = \min\{\mu_{\varphi[A_\Omega]}(y_1, \alpha), \mu_{\varphi[A_\Omega]}(y_2, \alpha)\},$$

$$\gamma_{\varphi[A_\Omega]}(y_1y_2, \alpha) = 1 = \max\{\gamma_{\varphi[A_\Omega]}(y_1, \alpha), \gamma_{\varphi[A_\Omega]}(y_2, \alpha)\}.$$

Suppose that $\varphi^{-1}(y_1y_2) \neq \emptyset$. Then we should consider two cases as follows:

$$\text{(i)}\ \ \varphi^{-1}(y_1) = \emptyset \quad \text{or} \quad \varphi^{-1}(y_2) = \emptyset,$$

$$\text{(ii)}\ \ \varphi^{-1}(y_1) \neq \emptyset \quad \text{and} \quad \varphi^{-1}(y_2) \neq \emptyset.$$

For the case (i), we have $\mu_{\varphi[A_\Omega]}(y_1, \alpha) = 0$ or $\mu_{\varphi[A_\Omega]}(y_2, \alpha) = 0$, and $\gamma_{\varphi[A_\Omega]}(y_1, \alpha) = 1$ or $\gamma_{\varphi[A_\Omega]}(y_2, \alpha) = 1$. Hence

$$\mu_{\varphi[A_\Omega]}(y_1y_2, \alpha) \geq \min\{\mu_{\varphi[A_\Omega]}(y_1, \alpha), \mu_{\varphi[A_\Omega]}(y_2, \alpha)\},$$

$$\gamma_{\varphi[A_\Omega]}(y_1y_2, \alpha) \leq \max\{\gamma_{\varphi[A_\Omega]}(y_1, \alpha), \gamma_{\varphi[A_\Omega]}(y_2, \alpha)\}.$$

Case (ii) implies from (3.1) that

$$\begin{aligned}
\mu_{\varphi[A_\Omega]}(y_1y_2, \alpha) &= \sup_{z \in \varphi^{-1}(y_1y_2)} \mu_{A_\Omega}(z, \alpha) \geq \sup_{z \in \varphi^{-1}(y_1)\varphi^{-1}(y_2)} \mu_{A_\Omega}(z, \alpha) \\
&= \sup_{x_1 \in \varphi^{-1}(y_1),\, x_2 \in \varphi^{-1}(y_2)} \mu_{A_\Omega}(x_1x_2, \alpha) \\
&\geq \sup_{x_1 \in \varphi^{-1}(y_1),\, x_2 \in \varphi^{-1}(y_2)} \min\{\mu_{A_\Omega}(x_1, \alpha), \mu_{A_\Omega}(x_2, \alpha)\} \\
&= \min\{\sup_{x_1 \in \varphi^{-1}(y_1)} \mu_{A_\Omega}(x_1, \alpha), \sup_{x_2 \in \varphi^{-1}(y_2)} \mu_{A_\Omega}(y_2, \alpha)\} \\
&= \min\{\mu_{\varphi[A_\Omega]}(y_1, \alpha), \mu_{\varphi[A_\Omega]}(y_2, \alpha)\},
\end{aligned}$$

$$\begin{aligned}\gamma_{\varphi[A_\Omega]}(y_1y_2,\alpha) &= \inf_{z\in\varphi^{-1}(y_1y_2)} \gamma_{A_\Omega}(z,\alpha) \le \inf_{z\in\varphi^{-1}(y_1)\varphi^{-1}(y_2)} \gamma_{A_\Omega}(z,\alpha)\\ &= \inf_{x_1\in\varphi^{-1}(y_1),\, x_2\in\varphi^{-1}(y_2)} \gamma_{A_\Omega}(x_1x_2,\alpha)\\ &\le \inf_{x_1\in\varphi^{-1}(y_1),\, x_2\in\varphi^{-1}(y_2)} \max\{\gamma_{A_\Omega}(x_1,\alpha),\gamma_{A_\Omega}(x_2,\alpha)\}\\ &= \max\{\inf_{x_1\in\varphi^{-1}(y_1)} \gamma_{A_\Omega}(x_1,\alpha), \inf_{x_2\in\varphi^{-1}(y_2)} \gamma_{A_\Omega}(y_2,\alpha)\}\\ &= \max\{\gamma_{\varphi[A_\Omega]}(y_1,\alpha),\gamma_{\varphi[A_\Omega]}(y_2,\alpha)\}.\end{aligned}$$

This completes the proof. □

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